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**L^p-ESTIMATES FOR THE ROUGH SINGULAR INTEGRALS ASSOCIATED TO SURFACES**

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**Abstract**

In this paper we obtain the $L^p$-boundedness for the maximal functions and the singular integrals associated to surfaces $(y, \phi(|y|))$ with rough kernels, $1 < p < \infty$. The analogue estimate is also established for the corresponding maximal singular integrals.

**1. Introduction**

Let $K : \mathbb{R}^n \to \mathbb{R}$ be a Calderón–Zygmund standard kernel in $\mathbb{R}^n$ ($n \geq 2$), that is, $K(y) = \Omega(y)/|y|^n$ with $y \neq 0$, where $\Omega(y)$ satisfies

$$\Omega(y) \in C^\infty(S^{n-1}),$$

$$\Omega(\lambda y) = \Omega(y), \quad \lambda > 0,$$

and

$$\int_{S^{n-1}} \Omega(y) \, d\sigma(y) = 0. \quad (1.1)$$

Let $\Gamma : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map. Then, we define the singular integrals $T$ associated with $\Gamma$ by the principal-value integral

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - \Gamma(y))K(y) \, dy, \quad (1.2)$$

where $x \in \mathbb{R}^m$ and $f \in \mathcal{S}(\mathbb{R}^m)$. Similar to the case of classical singular integrals theory, one can define the corresponding maximal functions as

$$\mathcal{M}f(x) = \sup_{h > 0} \frac{1}{h^n} \int_{|y| \leq h} |f(x - \Gamma(y))| \, dy.$$

The boundedness of the two operators $T$ and $\mathcal{M}$ above on $L^p(\mathbb{R}^m)$ has been well studied. We begin with the classical results by Stein, which can be found in [15].

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Theorem A (See [15]). If $\Gamma$ is any polynomial map from $\mathbb{R}^n$ to $\mathbb{R}^m$, then the operators $T$ and $M$ are both bounded on $L^p(\mathbb{R}^m)$ for $1 < p < \infty$.

Moreover, if $\Gamma$ is a smooth mapping from the unit ball in $\mathbb{R}^n$ to $\mathbb{R}^m$, and of finite type at the origin, then $T$ and $M$ are bounded operators on $L^p(\mathbb{R}^m)$ for $1 < p < \infty$.

Later, the theorem above was extended. That is, even in the case $\Omega$ is rough, the two results above still holds (see [9] and [10]). Furthermore, $T$ is bounded on $L^{p,q}(\mathbb{R}^m)$ for $1 < p < \infty$ and $q < \infty$. Moreover, if $\Gamma(y) = (y, \phi(|y|))$, $y \in \mathbb{R}^n$ and $\phi \in C(\mathbb{R}^+)$, Kim, Wainger, Wright and Ziesler proved the following result in [11].

Theorem B (See [11]). Let $\phi(t)$ be a $C^2$ function on $[0, \infty)$, and assume that $\phi$ is convex and increasing on $[0,\infty)$, and $\phi(0) = 0$. Then, for $1 < p < \infty$, there exists a positive constant $A_p$ such that

$$\| T f \|_{L^p} \leq A_p \| f \|_{L^p} \quad \text{and} \quad \| M f \|_{L^p} \leq A_p \| f \|_{L^p} \quad (f \in L^p).$$

In this case, the $L^p$-boundedness for the singular integrals in (1.2) with rough kernel is studied by Chen–Fan [5] and Lu–Pan–Yang [13].

Let $P(t)$ be a real-valued polynomial of $t$ in $\mathbb{R}$, and assume that $\gamma$ satisfies

$$\gamma \in C^2[0, \infty), \quad \text{convex on} \quad [0, \infty) \quad \text{and} \quad \gamma(0) = 0.$$ 

In this paper, we consider the hypersurface parameterized by $\Gamma: \mathbb{R}^n \to \mathbb{R}^{n+1}$, where $\Gamma$ is given by

$$\Gamma(y) = (y, P(\gamma(|y|))), \quad y \in \mathbb{R}^n.$$ 

Then, the operators $T$ and $M$ above take the form

(1.3) \quad \quad T f(u) = p.v. \int_{\mathbb{R}^n} f(x - y, s - P(\gamma(|y|))) K(y) \, dy

and

(1.4) \quad \quad M f(u) = \sup_{h > 0} \frac{1}{h^n} \int_{|t| \leq h} |f(x - y, s - P(\gamma(|y|))| \Omega(y) \, dy,

where $x \in \mathbb{R}^n$, $s \in \mathbb{R}$ and $u = (x, s)$, $K$ is the Calderón–Zygmund standard kernel as before.

For the $L^p$-boundedness of the singular integrals $T$ in (1.3) and the maximal functions $M$ in (1.4), Bez proved the following theorem in [1].
Theorem C (See [1]). For $T$ in (1.3) and $M$ in (1.4), if $\gamma'(0) \geq 0$, $\Omega \in C^\infty(S^{n-1})$, then, for $1 < p < \infty$, there exists a positive constant $C$ only dependent on $p$, $n$, $\gamma$ and the degree of $P$ such that

$$\|Tf\|_{L^p} \leq C\|f\|_{L^p} \quad \text{and} \quad \|Mf\|_{L^p} \leq C\|f\|_{L^p} \quad (f \in L^p).$$

Remark 1.1. One may notice that there is a little difference between the maximal function in (1.4) and that in Bez’s paper [1], we represent the maximal function in this form just for convenient. But Bez’s results still hold, since $C^\infty(S^{n-1}) \subset L^\infty(S^{n-1})$.

Besides the operators $T$ and $M$ above, we also consider the corresponding maximal singular integrals

$$T^s f(u) = \sup_{\varepsilon > 0} \left| \int_{|y| \geq \varepsilon} f(x - y, s - P(\gamma(|y|)))K(y) \, dy \right|. \quad (1.5)$$

Appropriate estimates for the maximal singular integrals give the pointwise existence of the principle value singular integrals.

Remark 1.2. For $n = 1$, if $\Gamma$ satisfies a ‘finite type condition’ at origin in $\mathbb{R}^m$, the $L^p$-estimates for the Hilbert transform, the maximal function and the maximal Hilbert transform can be found in the survey [14] of results through 1978. For other one-dimensional curves $\Gamma$, there are considerable results about the $L^p$-estimates for the Hilbert transform and the maximal function, see [2], [7] and [8] for example. Specially, the maximal Hilbert transform has been discussed in detail in [8].

The purpose of this note is to study the $L^p$-boundedness for $T$ in (1.3) and $M$ in (1.4), also, the analogue estimate for the maximal singular integrals $T^s$ in (1.5) is considered. Main results are presented as follows.

Theorem 1.3. Let $T$ and $M$ be given as in (1.3) and (1.4), respectively. If $\gamma'(0) \geq 0$ and $\Omega \in L^q(S^{n-1})$ for some $1 < q \leq \infty$, then $T$ and $M$ are bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$.

Remark 1.4. Note that $C^\infty(S^{n-1}) \subset L^q(S^{n-1})$ for $1 < q \leq \infty$, so, Theorem 1.3 improves and extends Theorem C. Also, Theorem B is a special case of Theorem 1.3 for $P(t) = t$. Further, the $L^p$-boundedness for $M$ can be proved by using Calderón–Zygmund’s rotation method with $\Omega \in L^1(S^{n-1})$, if either (1) $P'(0) = 0$, or (2) $P'(0) \neq 0$ and $\gamma'(\lambda t) \geq 2\lambda'(t)$ for some $\lambda > 1$.

Theorem 1.5. Let $T^s$ be given as in (1.5). If $\gamma'(0) \geq 0$ and $\Omega \in L^q(S^{n-1})$ for some $1 < q \leq \infty$, then $T^s$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$. 

This paper is organized as follows. In Section 2 we list some key properties concerning polynomials of one variable and give some fundamental lemmas for the proof of main results. The $L^p$-boundedness of $\mathcal{M}$ and $\mathcal{T}$ is proved following the arguments of Bez [1] and Carbery et al. [2] in Section 3 and Section 4, respectively. The last section contains the proof of Theorem 1.5, where we use the ideas of Córdoba and Rubio de Francia [8].

2. Preliminaries

Without loss of generality, we suppose that $P(t) = \sum_{k=1}^d p_k t^k$, where $d \geq 2$. Let $z_1, z_2, \ldots, z_d$ be the $d$ complex roots of $P$ ordered as

$$0 = |z_1| \leq |z_2| \leq \cdots \leq |z_d|.$$

Let $A > 1$, whose value we fix in Lemma 2.1. Define $G_j = (A|z_j|, A^{-1}|z_{j+1}|]$ if it is nonempty for $1 \leq j < d$ and $G_d = (A|z_d|, \infty)$. Let $\mathcal{J} = \{j : G_j \neq \emptyset\}$, then, $(0, \infty) \setminus \bigcup_{j \in \mathcal{J}} G_j$ can be decomposed as $\bigcup_{k \in \mathcal{K}} D_k$, where $D_k$ is the interval between $G_k$ and adjacent $G_{k+l}$ for some $l \geq 1$, it obvious that $D_k$’s are disjoint. Then, we can split $(0, \infty)$ as

$$(0, \infty) = \bigcup_{j \in \mathcal{J}} \gamma^{-1}(G_j) \cup \bigcup_{k \in \mathcal{K}} \gamma^{-1}(D_k),$$

where $\gamma^{-1}(I) = \{t \in (0, \infty) : \gamma(t) \in I\}$.

The properties of $P$ on $D_k$ and $G_j$ are important for our proof, the following related lemma can be found in [1] and [3].

Lemma 2.1. There exists a constant $C_d > 1$ such that for any $A \geq C_d$ and any $j \in \mathcal{J}$,

1. $|P(t)| \sim |p_j| |t|^j$ for $|t| \in G_j$;
2. $P'(t)/P(t) > 0$ for $t \in G_j$, $P'(t)/P(t) < 0$ for $-t \in G_j$;
3. $|P'(t)/P(t)| \sim 1/|t|$ for $|t| \in G_j$;
4. $P''(t)/P(t) > 0$ and $P''(t)/P(t) \sim 1/t^2$ for $|t| \in G_j$, $j \in \mathcal{J} \setminus \{1\}$.

The following trivial fact follows the proof of Lemma 2.1 (see [1]), that is, we can choose $A > 0$ such that for $|t| \in G_j$,

$$(2.1) \quad |P(t)| \leq 2|p_j| |t|^j \quad \text{and} \quad \frac{1}{2} j|p_j| |t|^{j-1} \leq |P'(t)| \leq 2j|p_j| |t|^{j-1}.$$

Let $\rho = n + 2$, for $I \subset (0, \infty)$, $\mathcal{M}_I$ and $\mathcal{T}_I$ are given by

$$\mathcal{M}_I f(u) = \sup_{k \in \mathbb{Z}} \frac{1}{\rho^{\rho k}} \int_{|t| \in \gamma^{-1}(I) \cap (\rho^k, \rho^{k+1})} |f(x - y, s - P(\gamma(|y|)))| |\Omega(y)| \, dy,$$
and
\[
T_{I}f(u) = \int_{y \in y^{-1}(l)} f(x - y, s - P(y(|y|))) K(y) \, dy.
\]

For \( k \in \mathbb{Z} \) and \( j \in \mathcal{J} \), let
\[
A_{k,j} = \begin{pmatrix}
\rho^k & 0 & \cdots & 0 \\
0 & \rho^k & 0 & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \cdots & |p_j|y^j(\rho^k)
\end{pmatrix}_{(n+1) \times (n+1)},
\]
then, \( A_{k,j} \) satisfies Rivière condition, that is \( \|A_{k+1,j}^{-1} A_{k,j}\| \leq \alpha < 1 \). In fact,
\[
A_{k+1,j}^{-1} A_{k,j} = \begin{pmatrix}
\rho^{-1}I_n & 0 \\
0 & \left( \frac{\gamma(\rho^k)}{\gamma(\rho^{k+1})} \right)^j
\end{pmatrix}
\]
Note that \( \gamma \) is convex, \( \gamma(t)/t \leq \gamma(s)/s \) for \( 0 < t \leq s \), therefore,
\[
\left( \frac{\gamma(\rho^k)}{\gamma(\rho^{k+1})} \right) \leq \frac{1}{\rho} < 1.
\]

We choose \( \phi \in C^\infty(\mathbb{R}^{n+1}) \) such that \( \hat{\phi}(\xi) = 1 \) for \( |\xi| \leq 1 \) and \( \hat{\phi}(\xi) = 0 \) for \( |\xi| \geq 2 \). For \( k \in \mathbb{Z} \) and \( j \in \mathcal{J} \), the multiplier \( m_{k,j} \) is defined by
\[
m_{k,j}(\xi) = \hat{\phi}(A_{k,j}^* \xi) - \hat{\phi}(A_{k+1,j}^* \xi),
\]
where \( A_{k,j}^* \) is the adjoint of \( A_{k,j} \). Then, we define the operator \( S_{k,j} \) by
\[
(S_{k,j} f)^\wedge(\xi) = m_{k,j}(\xi) \hat{f}(\xi).
\]
In the next proposition, we state a useful result for future reference.

**Proposition 2.2.** For any \( j \in \mathcal{J} \), if \( m_{l+k,j}(\xi) \neq 0 \) for some \( k, l \in \mathbb{Z} \), then

(2.2) \[ |A_{k,j}^* \xi| \geq C \rho^{-l}, \quad l < 0; \]

and

(2.3) \[ |A_{k+1,j}^* \xi| \leq C \rho^{-l}, \quad l > 0. \]
Proof. If $m_{l+k,j}(\xi) \neq 0$, then $|A^*_l+1, j_1| \leq 2$ and $|A^*_l+1, j_2| > 1$. For $l < 0$, by the convexity of $\gamma$,

$$1 < |A^*_l+1, j_1| \leq \rho^{l+1} |A^*_k,j|,$$

that is (2.2). When $l > 0$,

$$2 \geq |A^*_l+1, j_1| \geq \rho^{l-1} |A^*_k+1, j_1|,$$

then, (2.3) is obtained. \qed

We need the following Littlewood–Paley theorem, which can be found in [2] and [4].

**Lemma 2.3.** For $m_{k,j}$ and $S_{k,j}$ above, we have the following properties:

(i) for each $\xi$ at most $C_0$ of the $m_{k,j}(\xi)$ are not zero;

(ii) for each $\xi \neq 0$, $\sum_{k \in Z} m_{k,j}(\xi) = 1$;

(iii) $\left(\sum_{k \in Z} |S_{k,j} f|^2\right)^{1/2} \|_{L^p} \leq C_p \| f \|_{L^p}$, $1 < p < \infty$;

(iv) $\left(\sum_{k \in Z} |S_{k,j} f_k|^2\right)^{1/2} \|_{L^p} \leq C_p \left(\sum_{k \in Z} |S_{k,j} f|^2\right)^{1/2} \|_{L^p}$, $1 < p < \infty$.

3. The $L^p$-boundedness for $\mathcal{M}$

It is trivial that

$$\mathcal{M} f(u) \leq C \left[ \sum_{k \in \mathcal{K}} \mathcal{M}_{D_k} f(u) + \sum_{j \in \mathcal{J}} \mathcal{M}_{G_j} f(u) \right].$$

Note that the cardinalities of $\mathcal{K}$ and $\mathcal{J}$ are less than $d$, so we just need to verify that $\mathcal{M}_{D_k}$ and $\mathcal{M}_{G_j}$ are $L^p$-bounded for each $k \in \mathcal{K}$ and $j \in \mathcal{J}$.

3.1. The $L^p$-boundedness for $\mathcal{M}_{D_k}$. For any $u \in \mathbb{R}^{n+1}$, there exists an integer $j(u)$ such that

$$\mathcal{M}_{D_k} f(u) \leq \frac{2}{\rho^{n(u)}} \int_{|y| < \gamma'^{(1)}(D_k) \cap \{\rho^{(a)}, \rho^{(a+1)}\}} |f(x - y, s - P(\gamma(|y|)))| \Omega(y) \, dy.$$

Then, by Minkowski’s inequality, the $L^p$-norm of $\mathcal{M}_{D_k} f$ can be dominated by

$$\left( \int_{\mathbb{R}^{n+1}} \left[ \frac{1}{\rho^{n(u)}} \int_{|y| < \gamma'^{(1)}(D_k) \cap \{\rho^{(a)}, \rho^{(a+1)}\}} |f(x - y, s - P(\gamma(|y|)))| \Omega(y) \, dy \right]^p \, du \right)^{1/p} \leq \int_{|y| < \gamma'^{(1)}(D_k)} \frac{\Omega(y)}{|y|^n} \left( \int_{\mathbb{R}^{n+1}} |f(x - y, s - P(\gamma(|y|)))|^p \, du \right)^{1/p} \, dy \leq C \| f \|_{L^p} \| \Omega \|_{L^1(S^{n-1})} \int_{r \in \gamma'^{(1)}(D_k)} \frac{1}{r} \, dr.$$
Let $D_k = (A^{-1}|z_j|, A|z_{j+l}|)$ for some $2 \leq j \leq d$ and $0 \leq l \leq d - j$, then
\[
A^{-1}|z_j| \leq A^{-1}|z_{j+l}| \leq A|z_j| \leq \cdots \leq A|z_{j+l}| < A^{-1}|z_{j+l+1}|
\]
and
\[
A^2 \leq \frac{A|z_{j+l}|}{A^{-1}|z_j|} \leq \frac{A|z_{j+l}|}{A^{-2-1}|z_{j+l}|} \leq A^{2+l}.
\]

Notice that $\gamma$ is convex and $\gamma(0) = 0$, so, $\gamma(t) \leq t\gamma'(t)$ for $t > 0$. Thus,
\[
\int_{r \in \gamma^{-1}(D_k)} \frac{1}{r} dr = \int_{r \in \gamma^{-1}(A|z_{j+l}|)} \frac{1}{r} dr = \int_{A^{-1}|z_j|}^{A|z_{j+l}|} \frac{1}{\gamma^{-1}(r) \gamma'(\gamma^{-1}(r))} dr
\]
\[
\leq \int_{A^{-1}|z_j|}^{A|z_{j+l}|} \frac{1}{r} dr \leq 2d \ln A,
\]
where $\gamma^{-1}(t)$ is the inverse function of $\gamma(t)$.

According to the calculation above, the $L^p$-boundedness for $\mathcal{M}_{D_k}$ is established,
\[
\|\mathcal{M}_{D_k} f\|_{L^p} \leq C \|f\|_{L^p}, \quad \text{for } 1 < p < \infty, \ k \in \mathbb{K}.
\]

3.2. The $L^p$-boundedness for $\mathcal{M}_{G_j}$. Next, we verify that $\mathcal{M}_{G_j}$ is $L^p$-bounded for $j \in \mathcal{J}$. The maximal operators $\mathcal{M}_{G_j}$ can be expressed as
\[
\mathcal{M}_{G_j} f(u) = \sup_{k \in \mathbb{Z}} \int_{|y| < \rho^k y^{-1}(G_j) \cap (1, \rho)} |f(x - \rho^k y, s - P(\gamma(|\rho^k y|)))| \Omega(y) |dy|
\]

Set $I_{k,j} = (1, \rho] \cap \rho^{-k}\gamma^{-1}(G_j)$, and define the measure $\mu_{k,j}$ by
\[
\langle \mu_{k,j}, \psi \rangle = \int_{|y| < \rho^k y^{-1}(G_j) \cap (1, \rho)} \psi(\rho^k y, P(\gamma(|\rho^k y|))) |\Omega(y)| dy
\]
for $\psi \in \mathcal{S}(\mathbb{R}^{n+1})$. Then, for $j \in \mathcal{J}$, $\mathcal{M}_{G_j} f$ also can be rewritten as
\[
\mathcal{M}_{G_j} f(u) = \sup_{k \in \mathbb{Z}} \mu_{k,j} * |f|(u).
\]

We also need to define the measure $\sigma_{k,j}$ by
\[
\langle \sigma_{k,j}, \psi \rangle = \frac{\widehat{\mu_{k,j}}(0)}{|A_{k+1,j} B|} \int_{A_{k+1,j} B} \psi(u) du,
\]
where $B = \{u \in \mathbb{R}^{n+1} : |u| \leq n + 1\}$. 
3.2.1. Fourier transform estimates for related measures.

Proposition 3.1. For $j \in J$ and $k \in \mathbb{Z}$, then there exists $C > 0$ and $\beta > 0$ independent of $j$ and $k$ such that

\begin{equation}
|\hat{\mu}_{k,j}(\xi)|, |\hat{\sigma}_{k,j}(\xi)| \leq C \max\{|A_{k,j}\xi|^{-1}, |A_{k,j}\xi|^{-\beta}\}
\end{equation}

and

\begin{equation}
|\hat{\mu}_{k,j}(\xi) - \hat{\sigma}_{k,j}(\xi)| \leq C|A_{k+1,j}\xi|.
\end{equation}

Proof. The main idea of the following proof comes from the work of Bez (see [1]). For completeness, we show more details.

Let $\xi = (\xi, \eta)$, where $\xi \in \mathbb{R}^d$ and $\eta \in \mathbb{R}$. For $k \in \mathbb{Z}$ and $j \in J$, we have

\[
|\hat{\mu}_{k,j}(\xi)| = \left| \int_{[0,1]^d} e^{-i(\rho^d y \cdot \xi + \eta (\rho^d y \cdot v))} |\Omega(y)| \, dy \right| 
\]

\[
\leq \int_{[0,1]^d} \left| \int_{S^{d-1}} e^{-i\rho^d z \cdot (y' - \xi)} |\Omega(y')| \, d\sigma(y') \right| dt.
\]

Set $I_k(t) = \int_{S^{d-1}} e^{-i\rho^d y \cdot (y' - \xi)} |\Omega(y')| \, d\sigma(y')$, by Hölder’s inequality,

\[
|\hat{\mu}_{k,j}(\xi)|^2 \leq C \int_{[0,1]^d} |I_k(t)|^2 \, dt 
\]

\[
\leq C \int_{S^{d-1}} |\Omega(y')| \, d\sigma(y') \int_{[0,1]^d} e^{i\rho^d \xi \cdot (y' - \xi)} \, d\sigma(y') \, d\sigma(y').
\]

By van der Corput’s lemma, for any $\alpha \in (0, 1)$, we have

\[
\left| \int_{[0,1]^d} e^{i\rho^d \xi \cdot (y' - \xi)} \, dt \right| \leq C \min\{1, |\rho^d \xi \cdot (y' - \xi)|^{-1}\}
\]

\[
\leq C (\rho^d |\xi|)^{-\alpha} |\xi' \cdot (y' - \xi)|^{-\alpha}.
\]

If $q = \infty$, it is trivial, we set $\beta = 1/2$. For $q \in (1, \infty)$, specially, we choose a positive constant $\alpha$ so that $\alpha q' < 1$. By Hölder’s inequality, we get

\[
|\hat{\mu}_{k,j}(\xi)|^2 \leq C (\rho^d |\xi|)^{-\alpha} \int_{S^{d-1}} |\Omega(y')| \, d\sigma(y') \int_{S^{d-1}} |\Omega(z')| \, d\sigma(z') \frac{d\sigma(y') \, d\sigma(z')}{|\xi' \cdot (y' - z')|^{\alpha q'}}
\]

\[
\leq C (\rho^d |\xi|)^{-\alpha} \left( \int_{S^{d-1}} |\Omega(y')|^q \, d\sigma(y') \right)^{1/q'} \left( \int_{S^{d-1}} |\Omega(z')|^q \, d\sigma(z') \right)^{1/q'}
\]

\[
\times \left( \int_{S^{d-1}} \frac{d\sigma(y') \, d\sigma(z')}{|\xi' \cdot (y' - z')|^{\alpha q'}} \right)^{1/q'}
\]

\[
\leq C \|\Omega\|_{L^q(S^{d-1})}^2 (\rho^d |\xi|)^{-\alpha}.
\]
Finally, there exists a constant $\beta \in (0, 1/(2q'))$ such that

\[(3.3) \quad |\hat{u}_{k,j}(\xi)| \leq C(\rho^k|\xi|)^{-\beta}.
\]

**Case 1.** $j \in J \setminus \{1\}$. If $\xi$ satisfies $4\rho^k|\xi| \geq |p_j|\gamma^j(\rho^k)|\eta|$, then $|A^*_{k,j}\xi| \leq \sqrt{77}\rho^k|\xi|$. Therefore, (3.3) implies $|\hat{u}_{k,j}(\xi)| \leq C|A^*_{k,j}\xi|^{-\beta}$.

If $\xi$ satisfies $4\rho^k|\xi| < |p_j|\gamma^j(\rho^k)|\eta|$, in order to estimate $|\hat{u}_{k,j}(\xi)|$, we need the following lemma which is Lemma 2.2 in [1].

**Lemma 3.2.** For all $j \in J \setminus \{1\}$, the function

$$t \mapsto P''(\gamma(\rho^k t))\gamma'(\rho^k t)^2 + P'(\gamma(\rho^k t))\gamma''(\rho^k t)$$

is single-signed on $I_{k,j}$.

On the other hand,

$$|\hat{u}_{k,j}(\xi)| \leq \int_{S^{n-1}} \int_{I_{k,j}} e^{-|\rho^k y' \cdot \xi + \eta P(\gamma(\rho^k t))|} dt \Omega(y')|d\sigma(y').$$

For fixed $y' \in S^{n-1}$, let $h_k(t) = \rho^k y' \cdot \xi + \eta P(\gamma(\rho^k t))$. For $t \in I_{k,j}$, by (2.1) and the convexity of $\gamma$, we have

\[(3.4) \quad |h'_{k}(t)| \geq |\rho^k P(\gamma(\rho^k t))\gamma'(\rho^k t)\eta| - |\rho^k \xi|
\]

$$\geq \frac{1}{2} j |p_j|\rho^k \gamma^j(\rho^k t)\gamma'(\rho^k t)|\eta| - \rho^k |\xi| \geq \frac{1}{2} j |p_j|\gamma^j(\rho^k)|\eta| - \rho^k |\xi|.$$  

Note that $4\rho^k|\xi| < |p_j|\gamma^j(\rho^k)|\eta|$, and $|A^*_{k,j}\xi| \leq (\sqrt{77}/|p_j|)\gamma^j(\rho^k)|\eta|$. Hence,

\[(3.5) \quad |h'_{k}(t)| \geq \frac{1}{4} j |p_j|\gamma^j(\rho^k)|\eta| \geq \frac{1}{\sqrt{77}} |A^*_{k,j}\xi|.$$  

For $j \in J \setminus \{1\}$, $h_k(t)$ is monotone on $I_{k,j}$ by Lemma 3.2. By van der Corput’s lemma and (3.5), we get

$$|\hat{u}_{k,j}(\xi)| \leq C \lVert \Omega \rVert_{L^1(S^{n-1})} (|p_j|\gamma^j(\rho^k)|\eta|)^{-1} \leq C|A^*_{k,j}\xi|^{-1}.$$  

**Case 2.** $j = 1$. If $\xi$ satisfies $|\xi| \geq (1/4)|p_1|\gamma'(\rho^k)|\eta|$, by the convexity of $\gamma$, then $\rho^k|\xi| \geq (1/4)|p_1|\gamma(\rho^k)|\eta|$ and $|A^*_{k,1}\xi| \leq \sqrt{77}\rho^k|\xi|$. According to (3.3), we obtain

$$|\hat{u}_{k,1}(\xi)| \leq C|A^*_{k,1}\xi|^{-\beta}.$$
If $\zeta$ satisfies $|\xi| < (1/4)|p_1|\gamma'(\rho^k)|\eta|$, (3.4) implies

\[
(3.6) \quad |h_k'(t)| \geq \frac{1}{2}|p_1|\rho^k\gamma'(\rho^k t)|\eta| - \rho^k|\xi| \geq \frac{1}{4}|p_1|\rho^k\gamma'(\rho^k t)|\eta| \geq \frac{1}{4}|p_1|\rho^k\gamma'(\rho^k)|\eta|.
\]

Integration by parts and (3.6) show that

\[
\left| \int_{I_{k,1}} e^{-i[\rho^k t + \xi + \eta P(\gamma t^k)]} dt \right| = \left| \int_{I_{k,1}} e^{-ih_k(t)}h_k'(t) dt \right| \leq 8(|p_1|\rho^k\gamma'(\rho^k)|\eta|)^{-1} + \int_{I_{k,1}} \frac{|h_k''(t)|}{|h_k'(t)|^2} dt.
\]

Essentially, we just need to consider the second term, which can be dominated by

\[
\int_{I_{k,1}} \frac{\rho^k|\eta||P'(\gamma t^k)|\gamma''(\rho^k t)}{h_k'(t)^2} dt + \int_{I_{k,1}} \frac{\rho^k|\eta||P''(\gamma(\rho^k t))|\gamma'(\rho^k t)^2}{h_k'(t)^2} dt := \alpha_1 + \alpha_2.
\]

In order to estimate the term $\alpha_1$, we define $\varphi_k(t) = \rho^k t|\xi| + |p_1|\gamma(\rho^k t)|\eta|$, then, $\varphi_k'(t) = \rho^k|\xi| + |p_1|\rho^k\gamma'(\rho^k t)|\eta|$. By (3.6), for $t \in I_{k,1}$, it is obvious that

\[
(3.7) \quad |\varphi_k'(t)| \leq \frac{5}{4}|p_1|\gamma'(\rho^k t)\rho^k|\eta| \leq 5h_k'(t).
\]

On the other hand, for $t \in I_{k,1}$,

\[
(3.8) \quad |\varphi_k(t)| \geq |p_1|\rho^k\gamma'(\rho^k t)|\eta| - \rho^k|\xi| \geq \frac{3}{4}|p_1|\rho^k\gamma'(\rho^k t)|\eta|.
\]

Also, by (2.1), for $t \in I_{k,1}$,

\[
(3.9) \quad \varphi_k''(t) = |p_1|\rho^{2k}\gamma''(\rho^k t)|\eta| \geq \frac{1}{2}\rho^{2k}|\eta||P'(\gamma(\rho^k t))|\gamma''(\rho^k t).
\]

Thus, in view of (3.7), (3.9) and (3.8), we have

\[
(3.10) \quad \alpha_1 \leq C \int_{I_{k,1}} \frac{\varphi_k''(t)}{\varphi_k'(t)^2} dt \leq C(|p_1|\rho^k\gamma'(\rho^k)|\eta|)^{-1}.
\]

For $\alpha_2$, by (3.6) and (2.1),

\[
\alpha_2 \leq C \int_{I_{k,1}} \frac{\rho^{2k}|\eta||P''(\gamma(\rho^k t))|\gamma'(\rho^k t)^2}{|p_1|\rho^k\gamma'(\rho^k t)|\eta|^2} dt \\
\leq C \int_{I_{k,1}} |p_1|^{-1}|P''(\gamma(\rho^k t))|\rho^k\gamma'(\rho^k t) \frac{1}{|p_1|\rho^k\gamma'(\rho^k t)|\eta|} dt \\
\leq C(|p_1|\rho^k\gamma'(\rho^k)|\eta|)^{-1} \int_{G_1} \frac{1}{g_1} P''(t) dt \\
\leq C(|p_1|\rho^k\gamma'(\rho^k)|\eta|)^{-1}.
\]
Note that $|A_{k,l}^s \xi| \leq (\sqrt{17}/4)|p_1|\rho^k \gamma' (\rho^k) |\eta|$. Then, (3.10) and (3.11) imply

$$|\hat{\mu}_{k,1}(\xi)| \leq C|A_{k,1}^s \xi|^{-1}. $$

For $\hat{\sigma}_{k,j}$, we have

$$|\hat{\sigma}_{k,j}(\xi)| = \frac{|\hat{\mu}_{k,j}(0)|}{|B|} \left| \int_B e^{-iu A_{k+1,j}^s} du \right| \leq C|A_{k,j}^s \xi|^{-1}. $$

According to the estimates for $\hat{\mu}_{k,j}$ and $\hat{\sigma}_{k,j}$ above, we obtain (3.1). (3.2) can be proved as follows,

$$|\hat{\mu}_{k,j}(\xi) - \hat{\sigma}_{k,j}(\xi)| \leq |\hat{\mu}_{k,j}(\xi) - \hat{\mu}_{k,j}(0)| + |\hat{\mu}_{k,j}(0)| |\hat{\sigma}_{k,j}(\xi) - 1|$$

$$\leq \int_{|y| \leq r} |e^{-i(\rho^k \gamma (\eta') |\eta| + \rho^k |\eta|)} - 1| |\Omega(y)| dy$$

$$+ \frac{\|\Omega\|_{L^1(S^{n-1})}}{|B|} \int_B |e^{-iu A_{k+1,j}^s} - 1| du$$

$$\leq C|A_{k+1,j}^s \xi|. \quad \Box$$

3.2.2. The $L^p$-norm of $\mathcal{M}_{G,f}$. For the maximal operators $\mathcal{M}_{G,1}$, it can be dominated by

$$\mathcal{M}_{G,f}(u) \leq \sup_{k \in \mathbb{Z}} \sigma_{k,j} * f(u) + \sup_{k \in \mathbb{Z}} (\mu_{k,j} - \sigma_{k,j}) * f |(u)$$

$$\leq \mathcal{M}_s f(u) + \sup_{k \in \mathbb{Z}} (\mu_{k,j} - \sigma_{k,j}) * f |(u),$$

where $\mathcal{M}_s$ denotes the strong maximal function.

We first consider the $L^2$-estimates for $\mathcal{M}_{G,1}$. It is known that $\mathcal{M}_s$ is $L^p$ bounded for $1 < p \leq \infty$, thus, it suffices to consider the $L^2$-norm of $\sup_{k \in \mathbb{Z}} |(\mu_{k,j} - \sigma_{k,j}) * f |$. In view of Lemma 2.3, we have

$$|(\mu_{k,j} - \sigma_{k,j}) * f |$$

$$\leq \left| \sum_{l \leq 0} \mu_{k,j} * S_{l+k,j} f \right| + \left| \sum_{l \leq 0} \sigma_{k,j} * S_{l+k,j} f \right| + \left| \sum_{l=1}^\infty (\mu_{k,j} - \sigma_{k,j}) * S_{l+k,j} f \right|$$

$$:= A_{k,j} + B_{k,j} + C_{k,j}.$$

The $L^2$-norm of the supremums of $A_{k,j}$, $B_{k,j}$ and $C_{k,j}$ are considered separately. Now, the supremum of $A_{k,j}$ is controlled by

$$\sup_{k \in \mathbb{Z}} A_{k,j} \leq \sum_{l \leq 0} \sup_{k \in \mathbb{Z}} |\mu_{k,j} * S_{l+k,j} f | \leq \sum_{l \leq 0} \left( \sum_{k \in \mathbb{Z}} |\mu_{k,j} * S_{l+k,j} f |^2 \right)^{1/2} := \sum_{l=\infty}^0 \mathcal{E}_{l,j} f.$$. 
For each integer \( l \leq 0 \), by Plancherel’s theorem, (3.1) and (2.2),

\[
\| \mathcal{E}_{l,j} f \|_{L^2} = \left( \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} |\hat{\mu}_{k,j}(\xi)|^2 |m_{l+k,j}(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \leq C \rho^{|l|} \| f \|_{L^2}.
\]

Then, by the triangle inequality in \( L^2 \), we have

\[
\left\| \sup_{k \in \mathbb{Z}} A_{k,j} \right\|_{L^2} \leq C \| f \|_{L^2}.
\]

The \( L^2 \)-norm of \( \sup_{k \in \mathbb{Z}} B_{k,j} \) can be considered in the same way, therefore,

\[
\left\| \sup_{k \in \mathbb{Z}} B_{k,j} \right\|_{L^2} \leq C \| f \|_{L^2}.
\]

Similarly, for \( \sup_{k \in \mathbb{Z}} C_{k,j} \), we have

\[
\sup_{k \in \mathbb{Z}} C_{k,j} \leq \sum_{l=1}^{\infty} \left( \sum_{k \in \mathbb{Z}} \left| \left( \mu_{k,j} - \sigma_{k,j} \right) \ast S_{l+k,j} f \right|^2 \right)^{1/2} = \sum_{l=1}^{\infty} \mathcal{F}_{l,j} f.
\]

For each integer \( l \geq 1 \), by Plancherel’s theorem, (3.2) and (2.3), \( \| \mathcal{F}_{l,j} f \|_{L^2} \leq C \rho^{|l|} \| f \|_{L^2} \). Furthermore,

\[
\left\| \sup_{k \in \mathbb{Z}} C_{k,j} \right\|_{L^2} \leq C \| f \|_{L^2}.
\]

Then, combining (3.12), (3.14), (3.15) with (3.16), we have

\[
\| \mathcal{M}_{G_j} f \|_{L^2} \leq C \| f \|_{L^2}.
\]

For the \( L^p \)-boundedness of \( \mathcal{M}_{G_j} \), with \( p \neq 2 \), we need the following lemma, which is Lemma 4 in [8].

**Lemma 3.3.** Suppose that \( U_k f = u_k \ast f \) is a sequence of positive operators uniformly bounded on \( L^{\infty} \) and \( U^* f = \sup_{k \in \mathbb{Z}} \left| u_k \ast f \right| \) is bounded on \( L^r \), then, for \( p > 2r/(1 + r) \), there exists a positive constant \( C_p \) such that

\[
\left\| \left( \sum_{k \in \mathbb{Z}} |u_k f_k|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \left\| \left( \sum_{k \in \mathbb{Z}} |f_k|^2 \right)^{1/2} \right\|_{L^p}, \quad \{f_k\} \in L^p(l^2).
\]
By (3.17), Lemma 3.3 and Lemma 2.3, for $p > 4/3$, we get

$$
\| \mathcal{E}_{l,j} \|_{L^p} = \left( \sum_{k \in \mathbb{Z}} |\mu_{k,j} \ast S_{l+k,j}f|^2 \right)^{1/2}
$$

(3.18)

Interpolation between (3.13) and (3.18), and the triangle inequality in $L^p$ imply that

$$
\left( \sum_{k \in \mathbb{Z}} |S_{l+k,j}f|^2 \right)^{1/2} \leq C \| f \|_{L^p}.
$$

(3.19)

For $p > 3/4$, we obtain

$$
\left\| \sup_{k \in \mathbb{Z}} A_{k,j} \right\|_{L^p} \leq C \| f \|_{L^p}, \quad p > \frac{3}{4}.
$$

(3.20)

So, according to the $L^p$-boundedness of $\mathcal{M}$, (3.19) and (3.20), we have $\| \mathcal{M}_G, f \|_{L^p} \leq C \| f \|_{L^p}$ for $p > 4/3$.

Finally, by a bootstrap argument, we can apply Lemma 3.3 inductively to show that

$$
\| \mathcal{M}_G, f \|_{L^p} \leq C \| f \|_{L^p}, \quad 1 < p < \infty.
$$

4. The $L^p$-boundedness for $T$

Similar to the maximal functions $\mathcal{M}$, the singular integrals $T$ can be decomposed as

$$
Tf(u) = \sum_{k \in \mathcal{K}} T_{D_k}f(u) + \sum_{j \in \mathcal{J}} T_{G_j}f(u).
$$

Then, the $L^p$-boundedness for $T_{D_k}$ and $T_{G_j}$ will be considered separately for each $k \in \mathcal{K}$ and $j \in \mathcal{J}$.

4.1. The $L^p$-boundedness for $T_{D_k}$. For $k \in \mathcal{K}$, by Minkowski’s inequality, we have

$$
\| T_{D_k}f \|_{L^p} \leq \int_{\mathbb{R}^{n+1}} |K(y)| \left( \int_{\mathbb{R}^{n+1}} |f(x - y, s - P(y(y)))|^p \, du \right)^{1/p} \, dy
$$

(4.1)

$$
\leq \| f \|_{L^p} \int_{\mathbb{R}^{n+1}} |\Omega(y')| \, d\sigma(y') \int_{S^{n+1}} \frac{1}{r} \, dr.
$$

As the $L^p$-estimates for $\mathcal{M}_{D_k}$ in Subsection 3.1, we get the $L^p$-boundedness of $T_{D_k}$,

$$
\| T_{D_k}f \|_{L^p} \leq C \| f \|_{L^p}, \quad 1 < p < \infty.
$$
4.2. The $L^p$-boundedness for $\mathcal{T}_{G_j}$. For $j \in J$, $\mathcal{T}_{G_j} f$ can be rewritten as

$$\mathcal{T}_{G_j} f(u) = \sum_{k \in \mathbb{Z}} v_{k,j} \ast f(u),$$

where the measure $v_{k,j}$ is given by

$$\langle v_{k,j}, \psi \rangle = \int_{|y| \leq I_{k,j}} \psi(\rho^k y, P(y(|\rho^k y|))) K(y) \, dy$$

for $\psi \in \mathcal{S}(\mathbb{R}^{n+1})$.

For the estimates of $\hat{v}_{k,j}$, we have the following proposition.

**Proposition 4.1.** For $j \in J$ and $k \in \mathbb{Z}$, then there exists $C > 0$ and $\beta > 0$ independent of $j$ and $k$ such that

$$|\hat{v}_{k,j}(\xi)| \leq C \max\{|A^{\ast}_{k,j} \xi|^{-1}, |A^{\ast}_{k,j} \xi|^{-\beta}\}$$

and

$$|\hat{v}_{k,j}(\xi)| \leq C |A^{\ast}_{k+1,j} \xi|.$$

Proof. (4.2) can be proved by using the same method as (3.1). It is trivial to verify (4.3). In fact, by (1.1),

$$|\hat{v}_{k,j}(\xi)| = \left| \int_{|y| \leq I_{k,j}} \left[ e^{-i(\rho^k y \cdot \xi + \eta \rho^k y |y|}) - e^{-i \eta P(y |y|)} \right] K(y) \, dy \right|$$

$$\leq \int_{|y| \leq I_{k,j}} |e^{i(\rho^k y \cdot \xi)} - 1| |K(y)| \, dy \leq C \| \Omega \|_{L^1(\mathbb{R}^{n+1})} \rho^{k+1} |\xi|$$

$$\leq C |A^{\ast}_{k+1,j} \xi|.$$

By Lemma 2.3, we can decompose $\mathcal{T}_{G_j}$ as

$$\mathcal{T}_{G_j} f = \sum_{k \in \mathbb{Z}} \sum_{l \geq 1} v_{k,j} \ast S_{l+k,j} f + \sum_{k \in \mathbb{Z}} \sum_{l \leq 0} v_{k,j} \ast S_{l+k,j} f := \mathcal{D}_j + \mathcal{G}_j.$$

By the triangle inequality in $L^p$ and Lemma 2.3, we have

$$\|\mathcal{D}_j\|_{L^p} \leq \sum_{l \geq 1} \left\| \sum_{k \in \mathbb{Z}} v_{k,j} \ast S_{l+k,j} f \right\|_{L^p} \leq C \sum_{l \geq 1} \|\mathcal{H}_{l,j}\|_{L^p},$$
where \( \mathcal{H}_{l,j} = (\sum_{k \in \mathbb{Z}} |v_{k,j} \ast S_{l+k,j} f|^2)^{1/2} \). Plancherel’s theorem, (4.3) and (2.3) give

\[
\|\mathcal{H}_{l,j}\|_{L^2} = \left( \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} |m_{l+k,j}(\xi)|^2 |\hat{v}_{k,j}(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \leq C \rho^{-l} \|f\|_{L^2}.
\]

On the other hand, note that \( |v_{k,j} \ast g| \leq C \mu_{k,j} \ast |g| \). For \( 1 < p < \infty \), by the \( L^p \)-boundedness of \( \mathcal{M}_{G_j} \), Lemma 3.3 and Lemma 2.3, we obtain

\[
\|\mathcal{H}_{l,j}\|_{L^p} \leq C \left( \sum_{k \in \mathbb{Z}} |S_{l+k,j} f|^2 \right)^{1/2} \leq C \|f\|_{L^p}.
\]

Interpolation between (4.6) and (4.7), and (4.5) imply that

\[
\|\mathcal{D}_j\|_{L^p} \leq C \|f\|_{L^p}, \quad 1 < p < \infty.
\]

The \( L^p \)-norm of \( G_j \) can be obtained in the same way. For \( l \leq 0 \), using Plancherel’s theorem, (4.2) and (2.2), we have \( \|\mathcal{H}_{l,j}\|_{L^2} \leq C \rho^{l} \|f\|_{L^2} \). Further, (4.7) still holds. Interpolation and the triangle inequality in \( L^p \) show that

\[
\|G_j\|_{L^p} \leq C \|f\|_{L^p}, \quad 1 < p < \infty.
\]

Combining (4.8) and (4.9), we prove the \( L^p \)-boundedness for \( \mathcal{T}_{G_j} \).

5. The \( L^p \)-boundedness for \( \mathcal{T}^* \)

Let \( \mathcal{K} \) and \( \mathcal{J} \) be given as in the second section. Then, we have the following majorization

\[
\mathcal{T}^* f(u) \leq \sum_{k \in \mathcal{K}} \sup_{0 < e < 1} \left| \int_{|y| \in y^{-1}(D_k) \cap \{t \geq e\}} f(x - y, s - P(y(|y|))) K(y) \, dy \right|
\]
\[
\quad + \sum_{j \in \mathcal{J}} \sup_{0 < e < 1} \left| \int_{|y| \in y^{-1}(G_j) \cap \{t \geq e\}} f(x - y, s - P(y(|y|))) K(y) \, dy \right|
\]
\[
= \sum_{k \in \mathcal{K}} \mathcal{T}_{D_k}^* f(u) + \sum_{j \in \mathcal{J}} \mathcal{T}_{G_j}^* f(u).
\]

In the same way, we just need to show that \( \mathcal{T}_{D_k}^* \) and \( \mathcal{T}_{G_j}^* \) are \( L^p \) bounded for \( k \in \mathcal{K} \) and \( j \in \mathcal{J} \).

For \( k \in \mathcal{K} \), let \( e(u) \) be some measurable function from \( \mathbb{R}^{n+1} \) to \( \mathbb{R}^+ \) such that

\[
\mathcal{T}_{D_k}^* f(u) \leq 2 \left| \int_{|y| \in y^{-1}(D_k) \cap \{t \geq e(u)\}} f(x - y, s - P(y(|y|))) K(y) \, dy \right|.
\]
Then, the \( L^p \)-boundedness for \( \mathcal{T}^n_{b} \) can be proved in the same way as (4.1).

For \( j \in J \), it is trivial that

\[
\mathcal{T}^n_{b} f(u) \leq \mathcal{M}_{G_{j}} f(u) + \sup_{i \in \mathbb{Z}} \left| \sum_{k \geq i} v_{k,j} \ast f(u) \right|
\]

By the \( L^p \)-boundedness for \( \mathcal{M}_{G_{j}} \), it suffices to consider the latter term. Let \( \Phi \in \mathcal{S}(\mathbb{R}^n) \) be such that \( \hat{\Phi}(\xi) = 1 \) for \( |\xi| \leq 1 \) and \( \hat{\Phi}(\xi) = 0 \) for \( |\xi| \geq 2 \). Write \( \Phi_i(\xi) = \hat{\Phi}(\rho^i\xi) \), and denote by \( \ast \) convolution in the first \( n \) variables. For \( i \in \mathbb{Z} \), the truncated singular integrals can be split as

\[
\sum_{k \geq i} v_{k,j} \ast f = \Phi_i \ast \left( \mathcal{T}_{G} f - \sum_{k < i} v_{k,j} \ast f \right) + (\delta - \Phi_i) \ast \sum_{k \geq i} v_{k,j} \ast f =: \mathcal{A}_{i,j} + \mathcal{B}_{i,j},
\]

where \( \delta \) is the Dirac measure in \( \mathbb{R}^n \). Then, we just need to estimate \( \sup_{i \in \mathbb{Z}} |\mathcal{A}_{i,j}| \) and \( \sup_{i \in \mathbb{Z}} |\mathcal{B}_{i,j}| \) for \( j \in J \).

### 5.1. The \( L^p \)-estimates of \( \sup_{i \in \mathbb{Z}} |\mathcal{A}_{i,j}| \)

By a linear transformation and (1.1), we observe that

\[
\Phi_i \ast \sum_{k < i} v_{k,j} \ast f(u)
\]

\[
= \int_{\mathbb{R}^n} \Phi_i(x - y) \sum_{k < i} \int_{|z| \in \rho^k I_{i,j}} f(y - z, s - P(\gamma(|z|))) K(z) \, dz \, dy
\]

\[
= \sum_{k < i} \int_{|z| \in \rho^k I_{i,j}} K(z) \int_{\mathbb{R}^n} \Phi_i(x - y - z) f(y, s - P(\gamma(|z|))) \, dy \, dz
\]

\[
= \sum_{k < i} \int_{|z| \in \rho^k I_{i,j}} K(z) \int_{\mathbb{R}^n} [\Phi_i(x - y - z) - \Phi_j(x - y)] f(y, s - P(\gamma(|z|))) \, dy \, dz.
\]

Note that \( \Phi \in \mathcal{S}(\mathbb{R}^n) \), then, for any \( N > 0 \),

\[
\left| \Phi_i \ast \sum_{k < i} v_{k,j} \ast f(u) \right|
\]

\[
\leq \int_{|z| \in (0, \rho') \cap \gamma^{-1}(G_j)} |K(z)| \int_{\mathbb{R}^n} \frac{|z| \rho^{-n}}{\rho^i(1 + \rho^{-i}|x - y|)^N} |f(y, s - P(\gamma(|z|)))| \, dy \, dz
\]

\[
\leq \int_{\mathbb{R}^n} \frac{\rho^{-in}}{(1 + |\rho^{-i}| |x - y|)^N} \frac{1}{\rho^i} \int_{|z| \in (0, \rho') \cap \gamma^{-1}(G_j)} |f(y, s - P(\gamma(|z|)))| \frac{|\Omega(z)|}{|z|^{n-1}} \, dz \, dy.
\]

For the inner integral in \( z \), by a rotation,

\[
\frac{1}{\rho^i} \int_{|z| \in (0, \rho') \cap \gamma^{-1}(G_j)} |f(y, s - P(\gamma(|z|)))| \frac{|\Omega(z)|}{|z|^{n-1}} \, dz \leq \|\Omega\|_{L^1(\mathbb{R}^n)} \mathcal{N}_j f(y, s),
\]
where $\mathcal{N}_j$ is defined by

$$\mathcal{N}_j g(s) = \sup_{i \in \mathbb{Z}} \frac{1}{\rho^i} \int_{t \in (0, \rho^i) \cap \gamma^{-1}(G_j)} |g(s - P(y(t)))| \, dt.$$ 

Thus, we obtain

$$\sup_{i \in \mathbb{Z}} |\phi_{i,j}| \leq C[\|\Omega\|_{L^{p'}(R^n)}(\mathcal{N}_j f)^*(u) + (T_{G_j} f)^*(u)],$$

where $f^*(x, s)$ is the Hardy–Littlewood maximal function of $f(y, s)$ in the first $n$ variables.

**Proposition 5.1.** For $j \in \mathcal{J}$, $\mathcal{N}_j$ is a bounded operator on $L^p(R^n)$, $1 < p < \infty$.

Proof. We denote $P(y(t))$ by $\gamma(t)$ for short, then, $\gamma(t)' = P'(y(t))\gamma'(t)$. Note that $P(s)$ has no null point on $G_j$, then, it is singed-signed. For $t \in \gamma^{-1}(G_j)$, $\gamma(t) \in G_j$, by (2) of Lemma 2.1, $P'(y(t))$ is also singed-signed on $\gamma^{-1}(G_j)$. By $\gamma'(0) \geq 0$ and the convexity of $\gamma$, $\gamma'(t) > 0$ for $t > 0$. Then, $\gamma(t)$ is monotous on $\gamma^{-1}(G_j)$. Suppose that $\gamma(t)$ is increasing on $\gamma^{-1}(G_j)$, then

$$\frac{1}{\rho^i} \int_{t \in (0, \rho^i) \cap \gamma^{-1}(G_j)} |g(s - \gamma(t))| \, dt = \frac{1}{\rho^i} \int_{t \in (0, \gamma(t')) \cap P(G_j)} |g(s - t)| \frac{dt}{\gamma'(\gamma^{-1}(t))}$$

$$= \int_0^\infty |g(s - t)| \phi_{i,j}(t) \, dt.$$ 

For $j \in \mathcal{J} \setminus \{1\}$, by Lemma 3.2, $\gamma(t)'$ is monotous on $\gamma^{-1}(G_j)$. If $\gamma'(t)$ is increasing on $\gamma^{-1}(G_j)$, then, for $i \in \mathbb{Z}$, $\phi_{i,j}(t)$ is nonnegative and decreasing on $P(G_j)$. Furthermore, one should note that

$$\int_0^\infty \phi_{i,j}(t) \, dt \leq \frac{1}{\rho^i} \int_{t \in (0, \gamma(t'))} \frac{dt}{\gamma'(\gamma^{-1}(t))} = 1.$$ 

Therefore, for $i \in \mathbb{Z}$, we have

$$\frac{1}{\rho^i} \int_{t \in (0, \rho^i) \cap \gamma^{-1}(G_j)} |g(s - \gamma(t))| \, dt \leq CM g(s).$$

If $\gamma'(t)$ is decreasing on $\gamma^{-1}(G_j)$, write

$$\int_0^\infty |g(s - t)| \phi_{i,j}(t) \, dt = \int_0^\infty |\tilde{g}(s + t)| \tilde{\phi}_{i,j}(-t) \, dt = \int_{-\infty}^0 |\tilde{g}(s - t)| \tilde{\phi}_{i,j}(t) \, dt,$$
where \( \tilde{g} \) denotes the reflection of \( g \). Notice that \( \tilde{\phi}_{i,j}(t) \) is nonnegative and decreasing on \( -P(G_j) \). Also, \( \|\tilde{\phi}_{i,j}\|_{L^1} \leq 1 \). Similarly,

\[
\frac{1}{\rho'} \int_{t \in (0, \rho'[\gamma^{-1}(G_i)])} |g(s - \gamma(t))| \, dt \leq CM\tilde{g}(-s).
\]

For \( j = 1 \), note that \( \gamma(t) \) and \( \gamma(t) \) are increasing on \( \gamma^{-1}(G_1) \) and \( \mathbb{R}^+ \), respectively. Then, \( P(s) \) is increasing on \( G_1 \), that is, \( P'(s) > 0 \). According to (2.1), \( (1/2)|p_1| \leq P(t) \leq 2|p_1| \), furthermore, \( (1/2)|p_1| |t| \leq P(t) \leq 2|p_1||t| \) for \( t \in G_1 \). Therefore, combining the convexity of \( \gamma \), we get

\[
\frac{1}{\rho'} \int_{t \in (0, \gamma([\gamma^{-1}(G_i)] \cap P(G_i)))} |g(s - t)| \frac{dt}{\gamma'([\gamma^{-1}(G_1]])} \leq \frac{1}{\rho'} \int_{t \in (0, \gamma([\gamma^{-1}(G_i)] \cap P(G_i)))} |g(s - t)| \left( \frac{dt}{\gamma'([\gamma^{-1}(G_1]])} \right) \leq CMg(p_{1/2}(2 |p_1|)^{s}),
\]

where \( g[p_{1/2}(s)] = g(|p_1|t/2) \).

Thus, for \( j \in J \), \( \mathcal{N}_j \) is bounded on \( L^p(\mathbb{R}) \), \( 1 < p < \infty \).

Finally, by Lemma 5.1 and the \( L^p \)-boundedness for \( T_{G_j} \), we obtain

\[
\left\| \sup_{i \in \mathbb{Z}} |\mathcal{A}_{i,j}| \right\|_{L^p} \leq C \|f\|_{L^p}.
\]

5.2. The \( L^p \)-estimates of \( \sup_{i \in \mathbb{Z}} |\mathcal{B}_{i,j}| \). \( \sup_{i \in \mathbb{Z}} |\mathcal{B}_{i,j}| \) is dominated by

\[
\sup_{i \in \mathbb{Z}} |\mathcal{B}_{i,j}| \leq \sum_{l \geq 0} \sup_{i \in \mathbb{Z}} |(\delta - \Phi_i) \ast \psi_{l+i,j} \ast f| := \sum_{l \geq 0} \mathcal{P}_{l,j}.
\]

The maximal operator \( \mathcal{P}_{l,j} \) is uniformly bounded on \( L^p \), \( 1 < p < \infty \), since

\[
\mathcal{P}_{l,j} \leq C(\mathcal{M}_{G_j} f)^\ast.
\]

On the other hand, for \( p = 2 \), we have

\[
\|\mathcal{P}_{l,j}\|_{L^2} \leq \left( \sum_{l \geq 0, i \in \mathbb{Z}} |(\delta - \Phi_i) \ast \psi_{l+i,j} \ast f|^2 \right)^{1/2} \leq \left( \sum_{l \geq 0} \int_{\mathbb{R}^{d+1}} |1 - \tilde{\Phi}(\rho^j \xi)|^2 |\tilde{\psi}_{l+i,j}(\xi)|^2 |\tilde{f}(\xi)|^2 d\xi \right)^{1/2}.
\]
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\[
\begin{align*}
&\leq C \left( \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^{n+1}} \chi_{|\rho^j \xi| \geq 1} (\xi) \rho^{j+1} |\xi|^{-2\beta} |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} \\
&\leq C \rho^{-d\beta} \left( \int_{\mathbb{R}^{n+1}} \sum_{i: |\rho^i \xi| \leq |\xi|} |\rho^i \xi|^{-2\beta} |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} \\
&\leq C \rho^{-d\beta} \| f \|_{L^2},
\end{align*}
\]

where the fact \( |\hat{u}_{i,j}(\xi)| \leq C (\rho^j |\xi|)^{-\beta} \) can be proved in the same way as (3.3).

Interpolation and the triangle inequality in \( L^p \) imply that

\[
\sup_{i \in \mathbb{Z}} \| R_{i,j} \|_{L^p} \leq \sum_{l \geq 0} \| R_{l,j} \|_{L^p} \leq C \| f \|_{L^p}, \quad 1 < p < \infty.
\]

References

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