Imada, M.
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# PERIODIC POINTS ON THE BOUNDARIES OF ROTATION DOMAINS OF SOME RATIONAL FUNCTIONS

MITSUHIKO IMADA

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**Abstract**

We are interested in periodic points on the boundaries of rotation domains of rational functions. R. Pérez-Marco showed that there are no periodic points on the boundaries of Siegel disks having Jordan neighborhoods with certain properties [12]. In this paper, we consider periodic points on the boundaries of rotation domains under more weakly conditions.

## 1. Introduction and the main theorem

In this paper, we deal with one-dimensional complex dynamical systems, especially iterated dynamical systems of rational functions on the Riemann sphere $\hat{\mathbb{C}}$. The dynamics on a periodic Fatou component is well understood, actually there are three possibilities. They are the attracting case, the parabolic case or the irrational rotation case. However, it is difficult to see the dynamics on the boundary of a periodic Fatou component. A positive answer to the question of local connectivity of the boundary sometimes gives a model of the dynamics. Even when the boundary fails to be locally connected, we are interested in the dynamics of the boundary. Especially, we may ask can the boundary have a dense orbit or a periodic orbit?

It is interesting that the periodic points on the boundary $\partial\Omega$ of an immediate attracting or parabolic basin $\Omega$ are dense in $\partial\Omega$ [14, Theorem A]. According to [18, Theorem 1], if $\Omega$ is a bounded Fatou component of a polynomial that is not eventually a Siegel disk, then the boundary $\partial\Omega$ is a Jordan curve. For a geometrically finite rational function with connected Julia set, the Julia set is locally connected [22, Theorem A], and thus every Fatou component is locally connected.

We are interested in the topological structures of the boundaries of rotation domains and the dynamics on the boundaries. There are some results about the Julia sets which contain the boundaries of Siegel disks (see for example [1, 5, 11, 15, 16, 17]).

If the boundary $\partial\Omega$ of a Siegel disk $\Omega$ is locally connected, then it follow from the Carathéodory's theorem in the theory of conformal mappings that $\partial\Omega$ is a Jordan

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curve and the dynamics on $\partial\Omega$ is topologically conjugate to an irrational rotation. In particular, there are no periodic points on the boundary $\partial\Omega$.

R. Pérez-Marco has shown that the injectivity on a simply connected neighborhood of the closure of a Siegel disk implies that no periodic points on the boundary of the Siegel disk. More precisely, we have the following proposition [12, Theorem IV.4.2].

**Proposition 1.1.** *Let $\Omega$ be an invariant Siegel disk of a rational function $R$, and let $U$ be a neighborhood of $\overline{\Omega}$ so that the boundary $\partial U$ consists of a Jordan curve $\gamma$. If $R$ is injective on a neighborhood of $\overline{U}$, and both of $\gamma$ and $R(\gamma)$ are contained in a component of $\hat{\mathbb{C}} - \overline{\Omega}$, then the boundary $\partial\Omega$ contains no periodic points.*

In general, it may be hard to find a Jordan domain where the function is injective. So we shall show the following theorem which is the main result in this paper.

**Theorem 1.1.** *Let $\Omega$ be an invariant rotation domain of a rational function $R$, and let $U$ be a neighborhood of $\overline{\Omega}$. If $R$ is injective on $U$, then the boundary $\partial\Omega$ contains no periodic points except the Cremer points.*

The above theorem means that there are still no periodic points except the Cremer points on the boundary of invariant rotation domains even when the injective neighborhood is not a Jordan domain.

In the last section, we will discuss some related topics.

## 2. Basic definitions

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, and let $R\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of degree at least two. We define the *Fatou set* of $R$ as the union of all open sets $U \subset \hat{\mathbb{C}}$ such that the family of iterates $\{R^n\}$ is equicontinuous on $U$, and the *Julia set* of $R$ as the complement of the Fatou set of $R$. We denote the Julia set of $R$ by $J(R)$ and the Fatou set of $R$ by $F(R)$. The Fatou set $F(R)$ is a completely invariant open set and the Julia set $J(R)$ is a completely invariant compact set. Their fundamental properties can be found in [2, 9].

For each periodic point $z_0$ with period $k$, the *multiplier* is defined as $(R^k)'(z_0)$ and we denote it by $\lambda$. A connected component of the Fatou set $F(R)$ is called a *Fatou component*.

A periodic point $z_0$ with period $k$ is called *attracting* if $|\lambda| < 1$. Then the point $z_0$ is contained in the Fatou set $F(R)$. The Fatou component $\Omega$ containing the point $z_0$ is called the *immediate attracting basin* of $z_0$. Then $\{(R^k)^n\}$ converges locally uniformly to $z_0$ on $\Omega$.

A periodic point $z_0$ with period $k$ is called *parabolic* if $\lambda$ is a root of unity, or equivalently there exists an rational number $p/q$ such that $\lambda = e^{2\pi i p/q}$. Then the point $z_0$ is contained in the Julia set $J(R)$. A Fatou component $\Omega$ whose boundary contains the point $z_0$ is called an *immediate parabolic basin* of $z_0$ if $\{(R^{kq})^n\}$ converges locally uniformly to $z_0$ on $\Omega$.

A periodic point $z_0$ with period $k$ is called *irrationally indifferent* if $|\lambda| = 1$ but $\lambda$ is not a root of unity, or equivalently there exists an irrational number $\theta$ such that $\lambda = e^{2\pi i\theta}$. Then we distinguish between two possibilities. If the point $z_0$ lies in the Fatou set $F(R)$, then $z_0$ is called a *Siegel point*. The Fatou component $\Omega$ containing the Siegel point $z_0$ is called the *Siegel disk* with *center* $z_0$. Then $\Omega$ is conformally isomorphic to the unit disk $\mathbb{D}$, and the dynamics of $R^k$ on $\Omega$ corresponds to the dynamics of the irrational rotation $\lambda z$ on $\mathbb{D}$. Otherwise, if the point $z_0$ belongs to the Julia set $J(R)$, then $z_0$ is called a *Cremer point*.

A periodic point $z_0$ is called *weakly repelling* if $\lambda = 1$ or $|\lambda| > 1$, in particular, is called *repelling* if $|\lambda| > 1$. It is well known that the repelling periodic points are dense in the Julia set $J(R)$ and the non-repelling periodic points are finite.

A periodic Fatou component $\Omega$ with period $k$ is called a *Herman ring* if $\Omega$ is conformally isomorphic to some annulus $\mathbb{A}_r = \{z\colon 1/r < |z| < r\}$. Then the dynamics of $R^k$ on $\Omega$ corresponds to the dynamics of an irrational rotation on $\mathbb{A}_r$. We say that a Siegel disk or a Herman ring is a *rotation domain*. It is well known that every Fatou component is eventually periodic, and a periodic Fatou component is either an immediate attracting basin or an immediate parabolic basin or a Siegel disk or a Herman ring.

## 3. Local surjectivity

In this section, we shall see local surjectivity of a rational function $R$ of degree at least two. The notion of local surjectivity is referred from [19].

DEFINITION 3.1. Let $\Omega$ be a Fatou component, and let $z_0 \in \partial\Omega$. We say $R$ is *locally surjective* for $(z_0, \Omega)$, if there exists $\epsilon > 0$ such that $R(N \cap \Omega) = R(N) \cap R(\Omega)$ for any neighborhood $N \subset B_\epsilon(z_0) = \{z\colon d(z, z_0) < \epsilon\}$ of $z_0$.

**Lemma 3.1.** *Let $\Omega$ be a Fatou component, and let $z_0 \in \partial\Omega$. Assume that $R$ is locally surjective for $(z_0, \Omega), (R(z_0), R(\Omega)), \dots, (R^{n-1}(z_0), R^{n-1}(\Omega))$. Then $R^n$ is locally surjective for $(z_0, \Omega)$.*

Proof. It follows from the assumption that there exists $\epsilon > 0$ such that

$$R(N \cap \Omega) = R(N) \cap R(\Omega),$$
$$R(R(N) \cap R(\Omega)) = R(R(N)) \cap R(R(\Omega)),$$
$$\dots$$
$$R(R^{n-1}(N) \cap R^{n-1}(\Omega)) = R(R^{n-1}(N)) \cap R(R^{n-1}(\Omega)),$$

for any neighborhood $N \subset B_\epsilon(z_0)$ of $z_0$. So $R^n(N \cap \Omega) = R^n(N) \cap R^n(\Omega)$. □

The following two propositions are described in [19]. Since the proofs are not given in [19], we will give proofs for the sake of completeness.

**Proposition 3.1.** *Let $\Omega$ be a Fatou component, and let $z_0 \in \partial\Omega$. Assume that $z_0$ is not a critical point, and there exists a Fatou component $\Omega' \neq \Omega$ such that $z_0 \in \partial\Omega'$ and $R(\Omega') = R(\Omega)$. Then $R$ is not locally surjective for $(z_0, \Omega)$.*

Proof. Since $z_0$ is not a critical point, for any $\epsilon > 0$ there is a sufficiently small neighborhood $N \subset B_\epsilon(z_0)$ of $z_0$ such that $R|_N \colon N \to R(N)$ is a homeomorphism. Then $R(N \cap \Omega) \cap R(N \cap \Omega') = \emptyset$ and $R(N \cap \Omega') \subset R(N) \cap R(\Omega') = R(N) \cap R(\Omega)$. Therefore, $R(N \cap \Omega) \subset R(N) \cap R(\Omega) - R(N \cap \Omega') \subsetneq R(N) \cap R(\Omega)$. □

**Proposition 3.2.** *Let $\Omega$ be a Fatou component, and let $z_0 \in \partial\Omega$. Assume that $R$ is not locally surjective for $(z_0, \Omega)$. Then there exists a Fatou component $\Omega' \neq \Omega$ such that $z_0 \in \partial\Omega'$ and $R(\Omega') = R(\Omega)$.*

Proof. From the assumption, for each $n \in \mathbb{N}$ there exists a neighborhood $N_n \subset B_{1/n}(z_0)$ of $z_0$ such that $R(N_n \cap \Omega) \subsetneq R(N_n) \cap R(\Omega)$. Hence, there is a point $z_n \in N_n - \Omega$ so that $R(z_n) \in R(N_n) \cap R(\Omega) - R(N_k \cap \Omega)$. Let $\Omega_n$ be the Fatou component contains $z_n$. Then, $\Omega_n \neq \Omega$ and $R(\Omega_n) = R(\Omega)$. Thus, we can set $\Omega' = \Omega_{n_i}$ for a subsequence $\{n_i\}$. Then $z_{n_i} \in \Omega'$ and $\lim_{i\to+\infty} z_{n_i} = z_0$, therefore, $z_0 \in \partial\Omega'$. □

As it has been pointed out in [19], the above proposition implies that if $\Omega$ is a completely invariant Fatou component and $z_0 \in \partial\Omega$, then $R$ is locally surjective for $(z_0, \Omega)$.

**Lemma 3.2.** *Let $\Omega$ be a Fatou component, and let $z_0 \in \partial\Omega$. If $R$ is injective on a neighborhood $V$ of the boundary $\partial\Omega$, then $R$ is locally surjective for $(z_0, \Omega)$.*

Proof. Since $R$ is injective on $V$, there are no Fatou components of $R^{-1}(R(\Omega))$ which contain $z_0$ on their boundaries, except the component $\Omega$. By the contraposition of Proposition 3.2, the proof is finished. □

For a Fatou component whose boundary contains no critical point, the injectivity on the closure implies local surjectivity.

**Theorem 3.1.** *Let $\Omega$ be a Fatou component. Assume that $R$ is injective on $\Omega$ and the boundary $\partial\Omega$ contains no critical points. Then, either $R$ is injective on the boundary $\partial\Omega$ or there exists $z_0 \in \partial\Omega$ such that $R$ is not locally surjective for $(z_0, \Omega)$.*

Proof. Suppose that $R$ is injective on $\partial\Omega$ and let $z_0 \in \partial\Omega$. Then, $R$ is injective on a neighborhood $V$ of the boundary $\partial\Omega$ (see also [6, Lemma 3.1]). Therefore, $R$ is locally surjective for $(z_0, \Omega)$ by Lemma 3.2.

Now suppose that $R$ is not injective on $\partial\Omega$. Then, there are two distinct points $z_0 \in \partial\Omega$ and $w_0 \in \partial\Omega$ such that $R(z_0) = R(w_0)$. Since the boundary $\partial\Omega$ contains no critical points, there exists $\epsilon > 0$ such that $B_\epsilon(z_0) \cap B_\epsilon(w_0) = \emptyset$ and $R|_{B_\epsilon(z_0)}\colon B_\epsilon(z_0) \to R(B_\epsilon(z_0))$ is a homeomorphism. Let $w_n \in \Omega$ be a sequence so that $\lim_{n\to+\infty} w_n = w_0$. For any neighborhood $N \subset B_\epsilon(z_0)$ of $z_0$, the image $R(N)$ is a neighborhood of $R(z_0)$. Since $\lim_{n\to+\infty} R(w_n) = R(w_0) = R(z_0)$, there is some point $R(w_n)$ in $R(N)$. From the injectivity of $R|_\Omega$, there is no point in $N \cap \Omega$ whose image is equal to the point $R(w_n)$. Then, $R(w_n) \in R(N) \cap R(\Omega) - R(N \cap \Omega)$, and thus $R(N \cap \Omega) \subsetneq R(N) \cap R(\Omega)$. Therefore, $R$ is not locally surjective for $(z_0, \Omega)$. □

Since $R$ is injective on a rotation domain, the following corollary argues that the injectivity on the boundary implies local surjectivity.

**Corollary 3.1.** *Let $\Omega$ be an invariant rotation domain. Assume that the boundary $\partial\Omega$ contains no critical points. Then, either $R$ is injective on the boundary $\partial\Omega$ or there exists $z_0 \in \partial\Omega$ such that $R$ is not locally surjective for $(z_0, \Omega)$.*

## 4. The proof of the main theorem

DEFINITION 4.1. Let $\Omega \subset \hat{\mathbb{C}}$ be a Fatou component. A point $z \in \partial\Omega$ is called *accessible* from $\Omega$ if there exists a continuous curve $\gamma\colon [0,1) \to \Omega$ such that $\lim_{s\nearrow 1} \gamma(s) = z$. We say that such a curve $\gamma$ is a *periodic curve* if $R^k(\gamma) \subset \gamma$ or $R^k(\gamma) \supset \gamma$ for some $k$.

We show Theorem 1.1 by using the following key proposition [19, Theorem 1].

**Proposition 4.1.** *Let $\Omega$ be an invariant Fatou component, and let $z_0 \in \partial\Omega$ be a weakly repelling fixed point. If $R$ is locally surjective for $(z_0, \Omega)$, then $z_0$ is accessible from $\Omega$ by a periodic curve.*

So we have the following lemma.

**Lemma 4.1.** *Let $\Omega$ be an invariant Fatou component, and let $z_0 \in \partial\Omega$ be a parabolic fixed point. If $R$ is locally surjective for $(z_0, \Omega)$, then $z_0$ is accessible from $\Omega$ by a periodic curve.*

*Proof.* Let $\lambda = e^{2\pi i p/q}$ be the multiplier at $z_0$. It is clear that $\Omega$ is an invariant Fatou component for $R^q$. So $(R^q)'(z_0) = \lambda^q = 1$ and thus $z_0$ is a weakly repelling fixed point of $R^q$. Since $R^n(z_0) = z_0$ and $R^n(\Omega) = \Omega$ for $0 \le n \le q$, Lemma 3.1 implies that $R^q$ is locally surjective for $(z_0, \Omega)$. From Proposition 4.1, $z_0$ is accessible from $\Omega$ by a periodic curve for $R^q$. This curve is periodic for $R$. □

Proof of Theorem 1.1. We give the proof by contradiction. Suppose that the boundary $\partial\Omega$ contains a periodic point $z_0$ with period $k$ which is not a Cremer point. So the point $z_0$ is a parabolic or repelling fixed point of $R^k$. It is clear that $R^n(\Omega) = \Omega$ and $R^n(z_0) \in \partial\Omega$ for $0 \leq n \leq k$, and thus $\Omega$ is an invariant Fatou component for $R^k$. Since $R$ is injective on $U$, it follows from Lemma 3.2 that $R$ is locally surjective for $(z_0, \Omega), (R(z_0), \Omega), \dots, (R^{k-1}(z_0), \Omega)$. Lemma 3.1 implies that $R^k$ is locally surjective for $(z_0, \Omega)$. By Proposition 4.1 and Lemma 4.1, the point $z_0$ is accessible from $\Omega$ by a periodic curve for $R^k$. This contradicts that $\Omega$ is a rotation domain. □

## 5. Some related topics

In this section, we shall give some results on related topics. First, similarly to Proposition 1.1, we formulate the following proposition related to Herman rings.

**Proposition 5.1.** *Let $\Omega$ be an invariant Herman ring of a rational function $R$, and let $U$ be a neighborhood of $\overline{\Omega}$ so that the boundary $\partial U$ consists of two Jordan curves $\gamma$ and $\gamma'$ which are separated by invariant curves in the Herman ring $\Omega$. If $R$ is injective on a neighborhood of $\overline{U}$, and both of $\gamma$ and $R(\gamma)$ are contained in a component $V$ of $\hat{\mathbb{C}} - \overline{\Omega}$, and both of $\gamma'$ and $R(\gamma')$ are contained in a component $V'$ of $\hat{\mathbb{C}} - \overline{\Omega}$, then the boundary $\partial\Omega$ contains no periodic points.*

Proof. This proof is referred from the proof of [12, Theorem IV.4.2]. We give the proof by contradiction. Suppose that the boundary $\partial\Omega$ contains a periodic point with period $k$. Then, the periodic orbit $O = \{z_1, z_2, \dots, z_k\}$ is contained in a component $L$ of the boundary $\partial\Omega$. Let $\{K_n\}$ be a sequence of invariant closed annuli in the Herman ring $\Omega$ such that $K_n \subset \operatorname{Int} K_{n+1}$ and $\bigcup_{n=1}^{+\infty} K_n = \Omega$. Then $\{K_n\}$ converges to $\overline{\Omega}$ in the sense of Hausdorff convergence. Let $\tilde{\Omega}$ be the filled set of $\overline{\Omega}$ such that $\tilde{\Omega} = \hat{\mathbb{C}} - (V \cup V')$. By the assumption, we note that $R|_{\tilde{\Omega}} \colon \tilde{\Omega} \to \tilde{\Omega}$ is a homeomorphism.

The component $L$ contains either $\partial V$ or $\partial V'$. For the sake of convenience, we may assume that $L$ contains $\partial V$, and furthermore, $V$ contains infinity $\infty$. Let $V_n$ be the component of $\hat{\mathbb{C}} - K_n$ which contains $\infty$. Since $\{K_n\}$ converges to $\overline{\Omega}$ in the sense of Hausdorff convergence, $\{V_n\}$ converges to $V$ with respect to $\infty$ in the sense of Carathéodory kernel convergence. We consider the following conformal isomorphisms

$$\Phi_n \colon \hat{\mathbb{C}} - \overline{\mathbb{D}} \to V_n, \quad \Phi \colon \hat{\mathbb{C}} - \overline{\mathbb{D}} \to V$$

so that $\Phi_n(\infty) = \Phi(\infty) = \infty$, $\lim_{z\to\infty} \Phi_n(z)/z > 0$ and $\lim_{z\to\infty} \Phi(z)/z > 0$. Then, $\{\Phi_n\}$ converges locally uniformly to $\Phi$ by the Carathéodory kernel theorem (see for example [13, Theorem 1.8]). There exists $r > 1$ such that $\Phi(r\mathbb{S}^1) \subset U$ and $\Phi_n(r\mathbb{S}^1) \subset U$ for all large enough $n$. It follows from the assumption that $R(\Phi_n(r\mathbb{S}^1)) \subset V_n$ and $R(\Phi(r\mathbb{S}^1)) \subset V$. Hence, $g_n = \Phi_n^{-1} \circ R \circ \Phi_n$ and $g = \Phi^{-1} \circ R \circ \Phi$ are defined and injective on $\{z\colon 1 < |z| < r\}$. By the reflection principle, $g_n$ and $g$ are extended and injective on

$\mathbb{A}_r$. We fix $r'$ such that $1 < r' < r$. Since $\{\Phi_n\}$ converges locally uniformly to $\Phi$, $\{g_n\}$ converges uniformly to $g$ on $r'\mathbb{S}^1$. Thus, $\{g_n\}$ converges uniformly to $g$ on $(1/r')\mathbb{S}^1$. By the maximum principle, $\{g_n\}$ converges uniformly to $g$ on $\overline{\mathbb{A}}_{r'}$, particularly on the unit circle $\mathbb{S}^1$.

Let $L_n$ be the component of $\partial K_n$ which is close to $L$. We notice that the dynamics of $g_n$ on $\mathbb{S}^1$ corresponds to the dynamics of $R$ on $L_n$. Since $L_n$ is an invariant curve in the Herman ring $\Omega$, the dynamics of $R$ on $L_n$ corresponds to the dynamics of an irrational rotation $z \mapsto e^{2\pi i\theta}z$. Therefore, the rotation number $\mathrm{Rot}(g|_{\mathbb{S}^1})$ is calculated as follows:

$$\mathrm{Rot}(g|_{\mathbb{S}^1}) = \lim_{n\to+\infty} \mathrm{Rot}(g_n|_{\mathbb{S}^1}) = \lim_{n\to+\infty} \theta = \theta.$$

Now let $O'_n = \Phi_n^{-1}(O)$, so $O'_n$ is a periodic orbit of $g_n$ with period $k$. Since $\{K_n\}$ converges to $\overline{\Omega}$ in the sense of Hausdorff convergence, we see that $O'_n$ get close to $\mathbb{S}^1$ as $n \to +\infty$. More precisely, there are subsequence $\{O'_{n_i}\}$ and a set $O' \subset \mathbb{S}^1$ so that $\{O'_{n_i}\}$ converges to $O'$ in the sense of Hausdorff convergence. Since $O'_{n_i} = \Phi_{n_i}^{-1}(O)$ are finite sets, so the limit set $O'$ is a finite set. Moreover, $g_{n_i}(O'_{n_i}) = O'_{n_i}$ implies that $g(O') = O'$ (see also [12, Lemma III.1.2]), and thus $g$ has a periodic point on $\mathbb{S}^1$. This contradicts that the rotation number $\mathrm{Rot}(g|_{\mathbb{S}^1}) = \theta$ is irrational. □

We consider the topology of the boundary of a Siegel disk.

DEFINITION 5.1. Let $K \subset \hat{\mathbb{C}}$ be a non-degenerate continuum. We say $z_0 \in K$ is a *cut point* of $K$ if $K - \{z_0\}$ is disconnected.

Theorem 1.1 implies the following corollary, which asserts that the finiteness of cut points on the boundary of a Siegel disk follows from the injectivity of a neighborhood of the boundary.

**Corollary 5.1.** *Let $\Omega$ be an invariant Siegel disk of a rational function $R$, and let $U$ be a neighborhood of $\overline{\Omega}$. If $R$ is injective on $U$, then there are at most finitely many cut points of the boundary $\partial\Omega$.*

Proof. Assume that $z_0 \in \partial\Omega$ is a cut point of the boundary $\partial\Omega$. Then, $z_0$ is bi-accessible from $\Omega$, and thus $z_0$ is a periodic point (see [6, Definition 1.1 and Proposition 1.1]). It follows from Theorem 1.1 that $z_0$ must be a Cremer point. Since there are at most finitely many Cremer points, the proof is finished. □

Now we consider the following two functions. Let $P(z) = e^{2\pi i\theta}z + z^2$ be a quadratic polynomial with $\theta \in \mathbb{R} - \mathbb{Q}$. Let $B(z) = e^{2\pi i\tau(\theta)}z^2(z-a)/(1-\bar{a}z)$ be a cubic Blaschke product so that $|a| > 3$ and the rotation number $\mathrm{Rot}(B|_{\mathbb{S}^1}) = \theta \in \mathbb{R} - \mathbb{Q}$. We compare the dynamics of $P$ and the Julia set $J(P)$ with the dynamics of $B$ and the Julia set $J(B)$.

DEFINITION 5.2. If there exists a local holomorphic change of coordinate $z = \Phi(w)$, with $\Phi(0) = 0$, such that $\Phi^{-1} \circ P \circ \Phi$ is the irrational rotation $w \mapsto e^{2\pi i\theta} w$ near the origin, then we say that $P$ is *linearizable* at the origin.

The origin is either a Siegel point or a Cremer point, according to whether $P$ is linearizable at the origin or not.

DEFINITION 5.3. If there exists an analytic circle diffeomorphism $\Phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ such that $\Phi^{-1} \circ B \circ \Phi$ is the irrational rotation $w \mapsto e^{2\pi i\theta} w$, then we say that $B$ is *linearizable* on the unit circle.

The unit circle is contained in either the Fatou set $F(B)$ or the Julia set $J(B)$, according to whether $B$ is linearizable on the unit circle or not.

Suppose that $P$ is not linearizable at the origin and $B$ is not linearizable on the unit circle. It follows from [12, Theorem 1 and Theorem V.1.1] that there are Siegel compacta in $J(P)$ and Herman compacta in $J(B)$. There is a recurrent critical point $c_P \in J(P)$ whose forward orbit $\{P^n(c_P)\}_{n\geq 0}$ accumulates the origin, and there is a recurrent critical point $c_B \in J(B)$ whose forward orbit $\{B^n(c_B)\}_{n\geq 0}$ accumulates the unit circle (see [7, Theorem I]).

Let $\Omega_P$ be the immediate attracting basin of infinity with respect to the dynamics of $P$, and let $\Omega_B$ be the immediate attracting basin of infinity with respect to the dynamics of $B$. A. Douady and D. Sullivan [20, Theorem 8] has shown that $\partial\Omega_P = J(P)$ is not locally connected (see also [9, Corollary 18.6]). It follows from [16, Lemma 1.7 and Proposition 1.6] that the unit circle is contained in the boundary $\partial\Omega_B$, and the boundary $\partial\Omega_B$ is not locally connected. In particularly, the Julia set $J(B)$ is not locally connected. Therefore, we conclude that both of the Julia sets $J(P)$ and $J(B)$ are connected but not locally connected.

It is well known that every repelling periodic point on the boundary $\partial\Omega_P = J(P)$ is accessible from $\Omega_P$ by a periodic curve. Furthermore, we have the following proposition.

**Proposition 5.2.** *Let $B(z) = e^{2\pi i\tau(\theta)} z^2(z-a)/(1-\bar{a}z)$ be a cubic Blaschke product so that $|a| > 3$ and the rotation number $\mathrm{Rot}(B|_{\mathbb{S}^1}) = \theta$, let $\Omega_B$ be the immediate attracting basin of infinity. Assume that $\theta$ is irrational and $B$ is not linearizable on the unit circle. Then, every repelling periodic point on the boundary $\partial\Omega_B$ is accessible from $\Omega_B$ by a periodic curve.*

Proof. Let $z_0$ be a repelling periodic point on the boundary $\partial\Omega_B$ with period $k$. It is clear that $B^n(\Omega_B) = \Omega_B$ and $B^n(z_0) \in \partial\Omega_B$ for $0 \leq n \leq k$, and thus $\Omega_B$ is an invariant Fatou component for $B^k$. Let $\Omega'$ be the Fatou component containing the pole $1/\bar{a}$. Then, $B^{-1}(\Omega_B) = \Omega' \cup \Omega_B$. Since the unit circle $\mathbb{S}^1$ is contained in the Julia set $J(B)$, the Fatou component $\Omega'$ is contained in the unit disk $\mathbb{D}$ and $\Omega_B$ is contained in $\hat{\mathbb{C}} - \overline{\mathbb{D}}$. Therefore, injectivity of $B|_{\mathbb{S}^1}$ implies $\partial\Omega' \cap \partial\Omega_B = \emptyset$.

It follows from the contraposition of Proposition 3.2 that $B$ is locally surjective for $(z_0, \Omega_B), (B(z_0), \Omega_B), \ldots, (B^{k-1}(z_0), \Omega_B)$. Lemma 3.1 implies that $B^k$ is locally surjective for $(z_0, \Omega_B)$. By Proposition 4.1, the point $z_0$ is accessible from $\Omega$ by a periodic curve for $R^k$. □

From the results [21, Theorem 3] and [6, Theorem 1.3] of biaccessibility, we note that each of the repelling periodic points on $\partial\Omega_P = J(P)$ or $\partial\Omega_B$ has only one external ray landing at the point.

Finally, we consider buried points in the Julia sets. It follows from $\partial\Omega_P = J(P)$ that the Julia set $J(P)$ has no buried points, however, we see that the Julia set $J(B)$ has buried points.

DEFINITION 5.4. Let $R\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of degree at least two. A point $z$ in the Julia set $J(R)$ is called *buried* if $z$ is not lying in the boundary of any Fatou component.

Interestingly, we have the following (see [4, Proposition 1.4] and [3, Lemma 1]).

**Proposition 5.3.** *Let $R\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of degree at least two. Then there exists a buried point iff there is no periodic Fatou component $U$ such that $\partial U = J(R)$.*

So we have the following proposition.

**Proposition 5.4.** *Let $B(z) = e^{2\pi i \tau(\theta)} z^2 (z-a)/(1-\bar{a}z)$ be a cubic Blaschke product so that $|a| > 3$ and the rotation number $\mathrm{Rot}(B|_{\mathbb{S}^1}) = \theta$. Assume that $\theta$ is irrational and $B$ is not linearizable on the unit circle. Then there exists a buried point.*

*Proof.* Since $B$ is not linearizable on the unit circle, the circle $\mathbb{S}^1$ is contained in the Julia set $J(B)$. There exist two points in $J(B)$ which are separated by $\mathbb{S}^1$ (for example, the recurrent critical points $c_B$ and $1/\bar{c}_B$). Consequently, there is no periodic Fatou component $U$ such that $\partial U = J(B)$, and there exists a buried point by Proposition 5.3. □

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Department of Mathematics
Tokyo Institute of Technology
2-12-1 Oh-okayama, Meguro-ku
Tokyo 152-8551
Japan
e-mail: imada.m.aa@m.titech.ac.jp