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ASYMPTOTIC PROPERTIES FOR CONTINUOUS AND JUMP TYPE'S FEYNMAN-KAC FUNCTIONALS

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1. Introduction

During the past years, theory of measure perturbations of Dirichlet forms and the associated perturbed semigroups - so called *generalized Feynman-Kac semigroups* have been studied by many authors ([1],[2],[3],[7],[24],[25],[31],[32]). Furthermore, Dirichlet forms perturbed by jumps were recently investigated ([3],[24]), and unlike the case of the perturbation by smooth measures, we needed some new concepts and tools to study them. Indeed, when the underlying Markov process X_t is discontinuous, the following important discontinuous additive functionals can be considered ;

$$A_t^F = \sum_{s\leq t} F(X_{s-}, X_s), \tag{1.1}$$

where F is a Borel function on $X \times X$ vanishing on the diagonal. In this case, there are some difficulties in studying the Feynman-Kac semigroup for (1.1) and the corresponding bilinear form, because the basic tools used in the continuous case are not available in this discontinuous case. Song [24] studied the additive functionals of the forms

$$A_t^{\mu,F} = A_t^{\mu} + A_t^F \tag{1.2}$$

for the symmetric α-stable processes, where A^{μ} is the continuous additive functional (or abbreviated as CAF) with μ as its Revuz measure. For $f \in \mathcal{B}(R^d)$, he proved that if μ is a measure in the Kato class and F is a bounded "admissible" function with respect to the base process, then the Feynman-Kac semigroup

$$p_t^{\mu,F} f(x) = E_x\left(\exp\left(-A_t^{\mu,F}\right) f(X_t)\right)$$

is strongly continuous on $L^p(R^d)$. As a generalized approach, Ying [31] introduced the concepts of additive functionals of *extended Kato class* and showed that, if an additive functional A_t of symmetric Markov process X_t belongs to this class, the Feynman-Kac semigroup $p_t^{-A} f(x) = E_x\left((\mathbf{exp}(A))_t\, f(X_t)\right)$ may be extended to a strongly continuous semigroup of bounded operator, where $\mathbf{exp}(A)$ $\left(= e^{A_t^c}\prod_{s\le t}(1+\ \Delta A_s)\right)$ is the Stieltjes exponential of A. He also characterized the perturbed bilinear form associated with (p_t^{-A}).

In 1951, Kac [18] obtained a remarkable relation between the principal eigenvalue of *Schrödinger operators* and an asymptotics of the so called *Feynman-Kac functional.* More precisely, let (B_t, P_x^W) be the d-dimemsional Brownian motion and V a function on R^d such that $\lim_{x\to\infty} V(x) = \infty$. Then, it holds

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\log P_x^W\left(\exp\left(-\int_0^t V(B_s)ds\right)\right)\\ = -\inf_{\substack{u\in H^1(R^d)\\ \|u\|_2=1}}\left(\frac{1}{2}\mathbf{D}(u,u)+\int_{R^d}u^2(x)V(x)dx\right).\end{aligned}\tag{1.3}$$

Here $\mathbf{D}$ is the classical Dirichlet integral and $(1/2)\mathbf{D}$ plays a role as the rate function of the large deviation principle for the Brownian motion. Afterward, this formula became a motivation of Donsker-Varadhan large deviation theory which tells us an asymptotics of occupation time distribution $L_t(\omega,\cdot) = (1/t)\int_0^t \chi_\cdot((B_s(\omega))ds$, and nowadays, the formula (1.3) also can be derived as a corollary of Donsker-Varadhan large deviation theory.

On the other hand, many people recently studied the so called *generalized Schrödinger operator* with a signed smooth measure μ as a potential, $-\frac{1}{2}\Delta+\mu$, and the associated semigroup $E_x^W(\exp(-A_t^\mu) f(B_t))$ ([1],[2],[3],[6]). Moreover, Carmona-Masters-Simon [7] considered the *relativistic Hamiltonian operator* $\sqrt{-\Delta+m^2}-m,\ m>0$ instead of Laplacian. Note that if we want to extend the formula (1.3) to the relativistic Hamiltonian operator and also to the generalized Schrödinger operator, Donsker-Varadhan large deviation principle is no longer available because the additive functional $A_t^{\mu,F}$ is not a function of the occupation time distribution $L_t(\omega,\cdot)$ while $\int_0^t V(X_s)ds$ can be written as $t\int_X V(x)L_t(\omega,dx)$.

An objective of this paper is extend the formula (1.3) with continuous and jump type's Feynman-Kac functionals $\exp\left(-A_t^{\mu,F}\right)$ in the framework of regular Dirichlet space. To do this, we shall concentrate our attention on the modification of "Donsker-Varadhan large deviation principle for symmetric Markov processes with multiplicative functional $\exp\left(-A_t^{\mu,F}\right)$". The two methods may be considered. One can immediately expect on account of [26] that, if μ and F are positive and a symmetric Markov process as a base process explodes so fast that $R_1^{\mu,F}1$ the 1-resolvent of the identity function 1 belongs to $C_\infty(X)$ the space of continuous functions on the state space X vanishing at infinity, then we can use the full large deviation principle[1](Theorem 1.1 and 4.4 in [26]). The other is the choice of special Lévy process satisfying the exponentially

[1]Throughout this paper, "large deviation principle" means a "Donsker-Varadhan's type", otherwise, we shall give a comment.

localized condition (cf. [7]), under which, the *L^p-spectral radius* of the Feynman-Kac semigroup of kernels be ones p-independent. We combine this independent property with the large deviation principle.

More precisely, let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X; m)$ and $\mathbf{M} = (\Omega, X_t, P_x, \zeta, \theta_t)$ the associated Hunt process, where X is a locally compact separable metric space and m is a positive Radon measure on X with full support. Let μ be a signed smooth measure associated with $(\mathcal{E}, \mathcal{F})$ and F a Borel function of bounded below on $X \times X$ vanishing on the diagonal. First we show that if $\bar{F} = \left(1 - e^{-F}\right) \in L^2(X \times X \setminus d\,; Nm)$ and $A^{\mu, \bar{F}}$ is the additive functional of extended Kato class, then the Hunt process $\mathbf{M}$ can be transformed into the $\phi^2 m$-symmetric ergodic process whose Dirichlet form on $L^2(X; \phi^2 m)$ is written as

$$\mathcal{E}^{\phi}(u,u) = \frac{1}{2}\int_X \Gamma^c(u,u)\phi^2 dm + \iint_{X \times X \setminus d} (\tilde{u}(x) - \tilde{u}(y))^2 \phi(x)\phi(y) e^{-F} N(x, dy) m(dx)$$

by means of a supermartingale multiplicative functional

$$N_t^{\phi} = \exp\left(-A_t^{\mu,F}\right) \frac{\phi(X_t)}{\phi(X_0)} \exp\left(-\int_0^t \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_s) ds\right). \tag{1.4}$$

Using this fact, we prove the following formula for the symmetric Lévy process on R^d with its exponent being the so-called α-relativistic Hamiltonian

$$(p^2 + m^2)^{\frac{\alpha}{2}} - m^{\alpha}, \qquad m > 0,\ 0 < \alpha \le 2;$$

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t} \log E_x \left(\exp\left(-A_t^{\mu,F}\right)\right) \\ = &- \inf_{\substack{u \in \mathcal{F}^{\mu,F} \\ \|u\|_2 = 1}} \left(\mathcal{E}(u,u) + \int_{R^d} u(x)^2\, d\mu(x) \right. \\ &\left. + \iint_{R^d \times R^d \setminus d} u(x)u(y)(1 - e^{-F}) J(dx\, dy)\right) \end{aligned} \tag{1.5}$$

holding for all $x \in R^d$, where $\mathcal{F}^{\mu,F} = \{u \in \mathcal{F}\,; u \in L^2(R^d; |\mu| + |\rho|)\}$, $\rho(A) = \int_A \cdot \int_{R^d} (1 - e^{-F(x,y)}) J(dxdy)$, $A \in \mathcal{B}(R^d)$.

We shall also discuss some of its applications. When the base process is a symmetric stable process of index α, $(1 < \alpha \le 2)$, the surface measure of the sphere on R^3 is in the (extended) Kato class ([6]). Since the pseudo-differential operator $|-\Delta|^{\alpha/2} (1 < \alpha \le 2)$ and the α-relativistic Hamiltonian operator $(-\Delta + m^2)^{\alpha/2} - m^{\alpha}$ $(m > 0, 1 < \alpha \le 2)$ define the same Kato class, the formula (1.5) gives us the asymptotic behaviour of the local time for the surface measure of the sphere on R^3 (see Example 5.1 and 5.2). As mentioned above, when μ and F are positive and $R_1^{\mu,F} 1$ belongs to $C_{\infty}(X)$, we can also consider the full large deviation formula. We shall describe the asymptotics of the number of jumps for symmetric Markov chains on a symmetric region of the state space and give some sufficient conditions to ensure $R_1^F 1 \in C_\infty(X)$ in establishing full large deviation principle (see Example 5.3 and Theorem 5.2).

2. Preliminaries and Assumptions

Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X; m)$ and $\mathbf{M} = (\Omega, X_t, P_x, \zeta)$ a corresponding Hunt process.[2] Here Ω is the space of all right continuous functions from $[0, \infty)$ to X_Δ with the left hand limit on $(0, \infty)$ such that $\omega(t) = \Delta$ for all $t \geq \zeta(\omega) = \inf\{s \geq 0 ; \omega(s) = \Delta\}$ and X_t is the coordinate maps, $X_t(\omega) = \omega(t)$. The stopping time ζ is called the *life time* of $\mathbf{M}$ and Δ is called a *cemetery point* adjoined to X. Let $(p_t)_{t>0}$ be the transition semigroup of $\mathbf{M}$, $p_t f(x) = E_x(f(X_t))$, and $(R_\alpha)_{\alpha>0}$ the resolvent of $\mathbf{M}$, $R_\alpha f(x) = E_x\left(\int_0^\infty e^{-\alpha t} f(X_s) ds\right)$. The *capacity* associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is defined by

$$\text{Cap}(O) = \inf\left\{\mathcal{E}(u,u) + \int_X u^2 dm \ ; \ u \in \mathcal{F}, u \geq 1 \ m\text{-a.e. on } O\right\}$$

for an open set O, and is extended to any set as an outer capacity. We say that an increasing sequence $\{F_n\}$ of closed sets is a *generalized nest* (simply say, a *nest*) if $\lim_{n\to\infty} \text{Cap}(K \setminus F_n) = 0$ for any compact set K. A positive Borel measure μ on X is said to be *smooth* if μ charges no set of zero capacity and there exists a nest $\{F_n\}$ such that $\mu(F_n) < \infty$, for all n (§2.2 in [16]). Let us denote by $\mathcal{S}$ the totality of smooth measures. Given a signed Borel measure $\mu = \mu^+ - \mu^-$, we say μ the *signed* smooth measure and write $\mu \in \mathcal{S} - \mathcal{S}$ if $\mu^+ \in \mathcal{S}$ and $\mu^- \in \mathcal{S}$. For a Borel function f on X, we define the quasi-norm by

$$\|f\|_q = \inf_{\text{Cap}(N)=0} \sup_{x \in X \setminus N} |f(x)|.$$

Note that, if f is quasi-continuous, $\|f\|_q$ is the same as $\|f\|_\infty$ the usual L^∞-norm.

A signed Radon measure μ is said to be in the *Kato class* $\mathcal{S}_K$ if

$$\lim_{t \downarrow 0} \|E_x\left(A_t^\mu\right)\|_q = 0, \tag{2.1}$$

which is the extension of the classical definition of the Kato class for Brownian motion. Now, we introduce more general concepts of the Kato class (2.1)

[2]The Dirichlet form is defined as a Markovian closed "symmetric" form and hence, the associated Hunt process is always assumed to be "m-symmetric" (cf. [16]).

DEFINITION 2.1. A positive additive functional A of $\mathbf{M}$ is said to be in the *extended Kato class* if it belongs to

$$\mathcal{A}_K^{ad} = \left\{ A \, ; \, \inf_{t>0} \|E_x(A_t)\|_q < \frac{1}{2} \right\}.$$

A signed smooth measure $\mu = \mu^+ - \mu^-$ (resp. A Borel function $F = F^+ - F^-$ on $X \times X \setminus d$) is said to belong to the *extended Kato class* if $A^\mu = A^{\mu^+} - A^{\mu^-} \in \mathcal{A}_K^{\mathrm{ad}} - \mathcal{A}_K^{\mathrm{ad}}$ (resp. $A^F = A^{F^+} - A^{F^-} \in \mathcal{A}_K^{\mathrm{ad}} - \mathcal{A}_K^{\mathrm{ad}}$).

From now on, we assume that F is a symmetric (i.e. $F(x,y) = F(y,x)$) Borel function on $X \times X \setminus d$ and for convenience, $F(x, \Delta) = 0, \; x \in X$. Put $\bar{F} = \left(1 - e^{-F}\right)$. For the additive functional $A^{\mu,\bar{F}}$ which belongs to the extended Kato class, we define the symmetric bilinear form $(\mathcal{E}^{\mu,F}, \mathcal{F}^{\mu,F})$ on $L^2(X;m)$ by

$$\begin{cases} \mathcal{E}^{\mu,F}(u,v) = \mathcal{E}(u,v) + \displaystyle\int_X \tilde{u}(x)\tilde{v}(x)\,\mu(dx) + \iint_{X\times X\setminus d} \tilde{u}(x)\tilde{v}(y)\bar{F}J(dx\,dy) \\ \hfill \text{for } \; u,v \in \mathcal{F}^{\mu,F} \\ \mathcal{F}^{\mu,F} = \{u \in \mathcal{F} \, ; \, \tilde{u} \in L^2(X; |\mu| + |\rho|)\}, \end{cases}$$

where $\rho(B) = \int_B \cdot \int_X \bar{F}(x,y)J(dxdy), B \in \mathcal{B}(X)$, i.e., the marginal measure of $\bar{F}J$ and $\tilde{u}$ means a quasi continuous m-version of u with respect to the capacity Cap. Then $(\mathcal{E}^{\mu,F}, \mathcal{F}^{\mu,F})$ is a lower semibounded closed symmetric bilinear form on $L^2(X;m)$ because its perturbed semigroup, the so-called *Feynman-Kac semigroup*,

$$p_t^{\mu,F} f(x) = E_x\left(\exp\left(-A_t^{\mu,F}\right) f(X_t)\right) \quad x \in X. \tag{2.2}$$

is a strongly continuous symmetric semigroup of bounded operators on $L^2(X;m)$ (Theorem 3.2 in [31]).

Now, in order to obtain our main results, we make the following assumptions on $(\mathcal{E}, \mathcal{F})$ and $\mathbf{M}$ which are always available throughout this paper.

Assumptions

(I) The energy measure corresponding to $(\mathcal{E}, \mathcal{F})$ is absolutely continuous with respect to the base measure m, that is, $\mathcal{E}$ can be written as

$$\begin{aligned} \mathcal{E}(u,u) &= \frac{1}{2}\int_X \Gamma^c(u,u)dm + \frac{1}{2}\iint_{X\times X\setminus d} (u(x)-u(y))^2 N(x,dy)m(dx) \\ &\quad + \int_X u(x)^2 k(x)m(dx) \quad \text{ for } u \in \mathcal{F}\cap C_0(X), \end{aligned}$$

where $C_0(X)$ is the set of continuous functions on X with compact support.

(II) (Irreducibility of **M**) If a Borel set $B \in \mathcal{B}(X)$ satisfies

$$\chi_B p_t f = p_t(\chi_B f) \quad \text{for } \forall f \in L^2(X;m) \cap \mathcal{B}(X) \text{ and } t > 0,$$

then $m(B) = 0$ or $m(X \setminus B) = 0$.

(III) The transition probability $p_t(x, dy)$ of **M** is absolutely continuous with respect to m for each $t > 0$ and $x \in X$.

(IV) p_t is a bounded operator from $L^1(X;m)$ to $L^\infty(X;m)$ for any $t > 0$.

The next lemma asserts that under the extended Kato class conditions for $A^{\mu,\bar{F}}$ determined by μ and F, the Feynman-Kac semigroup (2.2) inherits the L^p-boundedness and L^p-smoothing properties of the original semigroup.

Lemma 2.1. *Suppose* $A_t^{\mu,\bar{F}} \in \mathcal{A}_K^{\mathrm{ad}} - \mathcal{A}_K^{\mathrm{ad}}$. *Then*

(i) *There exist positive constants* c *and* $\beta(\mu, F)$ *such that for all* $t > 0$,

$$\left\| p_t^{\mu,F} \right\|_{p,p} \le c e^{\beta(\mu,F)t}, \quad 1 \le \forall p \le \infty,$$

where $\| \ \|_{p,p}$ *means the operator norm from* L^p *to* L^p.
(ii) *For any* $1 \le p, q \le \infty$ *and* $t > 0$, $\left\| p_t^{\mu,F} \right\|_{p,q} < \infty$.

Proof. (i) Since $\mathbf{exp}\left(-A^{\mu,\bar{F}}\right)_t = e^{-A_t^\mu} \prod_{s \le t} (1 + e^{-F(X_{s-},X_s)} - 1) = \exp\left(-A_t^{\mu,F}\right)$, we see by Lemma 2.1 in [31] that there exist positive constants c and $\beta(\mu, F)$ such that for any $t > 0$,

$$\left\| E_x \left(\exp\left(-A_t^{\mu,F}\right)\right)\right\|_q \le c e^{\beta(\mu,F)t} \qquad x \in X.$$

Now, we have the result by using interpolations.

(ii) It is enough to show that $p_t^{\mu,F}$ is bounded operator from $L^1(X;m)$ to $L^p(X;m)$, $(1 \le p \le \infty)$ for any $t > 0$. The assumption (IV) say that there exists a positive constant C_t such that for any $t > 0$,

$$\|p_t\|_{1,\infty} = C_t < \infty.$$

Now, for any $f \in L^q(X;m), (1 \le q \le \infty)$, we have

$$\left| p_t^{\mu,F} f \right| \le \left(E_\cdot \left(e^{-pA_t^{\mu,F}} \right) \right)^{\frac{1}{p}} (p_t |f|^q)^{\frac{1}{q}} \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Using (i), we get $\left\|p_t^{\mu,F}\right\|_{q,\infty} \le \exists C'_t,\ t>0$ and its dual property give the conclusion.
□

3. Transformation to Ergodic Processes by Multiplicative Functional

In this section, we transform the Hunt process $\mathbf{M}$ into the ergodic subprocess by certain multiplicative functional, which will play a crucial role in the coming section. First, we define the resolvent $\{R_\alpha^{\mu,F}\}_{\alpha>\beta(\mu,F)}$ by

$$R_\alpha^{\mu,F}f(x) = E_x\left(\int_0^\infty \exp(-\alpha t - A_t^{\mu,F})f(X_t)dt\right) \quad \text{for } f \in \mathcal{B}_b(X)$$

and the generator $\mathcal{L}^{\mu,F}$ by

$$\mathcal{L}^{\mu,F}u = \alpha u - f \quad \text{for } u = R_\alpha^{\mu,F}f,\ f \in C_b(X)$$

where $\mathcal{B}_b(X)$ (resp. $C_b(X)$) is the set of all bounded Borel (resp. continuous) functions and $\beta(\mu,F)$ is the constant in Lemma 2.1. Set the domain of $\mathcal{L}^{\mu,F}$

$$\mathcal{D}_+^2(\mathcal{L}^{\mu,F}) = \left\{R_\alpha^{\mu,F}f\ ;\ \alpha > \beta(\mu,F),\ f \in L^2(X;m)\cap C_b(X),\ f \ge 0 \text{ and } f \not\equiv 0\right\}.$$

Note that any function in $\mathcal{D}_+^2(\mathcal{L}^{\mu,F})$ is strictly positive on X by assumption (II) and (III) (cf. Theorem 4.6.6 in [16]).

For $\phi = R_\alpha^{\mu,F}g \in \mathcal{D}_+^2(\mathcal{L}^{\mu,F})$, put

$$M_t^{\mu,F,\phi} = e^{-A_t^{\mu,F}}\phi(X_t) - \phi(X_0) - \int_0^t e^{-A_s^{\mu,F}}\mathcal{L}^{\mu,F}\phi(X_s)\,ds.$$

Then, $M_t^{\mu,F,\phi}$ is a martingale with respect to P_x, $x \in X$ because

$$E_x(M_t^{\mu,F,\phi}) = 0 \quad \text{and} \quad M_{s+t}^{\mu,F,\phi} = M_s^{\mu,F,\phi} + e^{-A_s^{\mu,F}}M_t^{\mu,F,\phi}(\theta_s).$$

Lemma 3.1. *$M_t^{\mu,F,\phi}$ can be also written as*

$$M_t^{\mu,F,\phi} = \int_0^t e^{-A_{s-}^{\mu,F}}dM_s^{[\phi]} - \int_0^t e^{-A_{s-}^{\mu,F}}dL_s^{\phi,\bar{F}}, \quad P_x\text{-a.e. } x \in X,$$

where

$$L_t^{\phi,\bar{F}} = \sum_{s\le t}\phi(X_s)\bar{F}(X_{s-},X_s) - \int_0^t\int_X N(X_s,dy)\phi(y)\bar{F}(X_s,y)\,ds$$

and $M^{[\phi]}$ is the martingale part for Fukushima decomposition of $\phi(X_t)-\phi(X_0)$.

Proof. We apply Itô formula to $G(x,y)=xy$. Since $de^{-A_t^{\mu,F}} = -(e^{-A_t^{\mu,F}}dA_t^{\mu} + e^{-A_{t-}^{\mu,F}}dA_t^{\bar{F}})$, we get

$$
\begin{aligned}
e^{-A_t^{\mu,F}}\phi(X_t)-\phi(X_0) &= G(e^{-A_t^{\mu,F}},\phi(X_t)) - G(e^{-A_0^{\mu,F}},\phi(X_0)) \\
&= \int_0^t e^{-A_{s-}^{\mu,F}} dM_s^{[\phi]} + \int_0^t e^{-A_{s-}^{\mu,F}} dN_s^{[\phi]} \\
&\quad - \int_0^t \phi(X_s)e^{-A_s^{\mu,F}} dA_s^{\mu} - \int_0^t \phi(X_s)e^{-A_{s-}^{\mu,F}} dA_s^{\bar{F}},
\end{aligned} \tag{3.1}
$$

where $A_t^{\bar{F}} = \sum_{s\le t}\bar{F}(X_{s-},X_s)$. Put $A_t^{\phi,\bar{F}} = \sum_{s\le t}\phi(X_s)\bar{F}(X_{s-},X_s)$ and consider its dual predictable projection $\left(A_t^{\phi,\bar{F}}\right)^p = \int_0^t\int_X N(X_s,dy)\phi(y)\bar{F}(X_s,y)ds$. Then

$$
L_t^{\phi,\bar{F}} = \sum_{s\le t}\phi(X_s)\bar{F}(X_{s-},X_s) - \int_0^t\int_X N(X_s,dy)\phi(y)\bar{F}(X_s,y)ds
$$

is a martingale with respect to $P_x, x\in X$, and hence, the last term of the right hand side of (3.1) (discontinuous part) equals

$$
\begin{aligned}
&\int_0^t e^{-A_{s-}^{\mu,F}} dA_s^{\phi,\bar{F}} \\
&= \int_0^t e^{-A_{s-}^{\mu,F}} dL_s^{\phi,\bar{F}} + \int_0^t e^{-A_{s-}^{\mu,F}} \int_X N(X_s,dy)\phi(y)\bar{F}(X_s,y)ds.
\end{aligned}
$$

We have the lemma. □

Let us define the multiplicative functional N^ϕ by

$$
N_t^\phi = \exp\left(-A_t^{\mu,F}\right)\frac{\phi(X_t)}{\phi(X_0)}\exp\left(-\int_0^t \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_s)ds\right). \tag{3.2}
$$

Then N^ϕ is a supermartingale multiplicative functional. Indeed, put $K_n = \{x\in X\,;\,\phi(x)\ge \frac{1}{n}\}$ and denote by K_n^o the fine interior of K_n. By noting

$$
\begin{aligned}
&d\left(e^{-A_t^{\mu,F}}\phi(X_s)\exp\left(-\int_0^t \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right)\right) \\
&= \exp\left(-\int_0^t \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right)\left(d(e^{-A_t^{\mu,F}}\phi(X_s)) - e^{-A_t^{\mu,F}}\mathcal{L}^{\mu}\phi(X_s)ds\right),
\end{aligned}
$$

we can immediately check that for each n

$$N^{\phi}_{t\wedge\tau_n} - 1 \tag{3.3}$$
$$= \int_0^{t\wedge\tau_n} \frac{1}{\phi(X_0)} \exp\left(-\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right) dM_s^{\mu,F,\phi} \quad P_x\text{-a.e. } x \in X,$$

where τ_n is the first leaving time from K_n^o, $\tau_n = \inf\{t > 0 : X_t \notin K_n^o\}$. Therefore, we see from (3.3) that

$$E_x(N_t^{\phi}) = E_x(N^{\phi}_{t\wedge\zeta}) \leq \liminf_{n\to\infty} E_x(N^{\phi}_{t\wedge\tau_n}) = 1, \quad x \in X.$$

Let us denote by $\mathbf{M}^{\phi} = (\Omega, X_t, P_x^{\phi}, \zeta)$ the transformed process of $\mathbf{M}$ by N^{ϕ}.

Lemma 3.2. *$\mathbf{M}^{\phi}$ is a $\phi^2 m$-symmetric right process on X.*

Proof. For a path $\omega \in \Omega$ with $t < \zeta(\omega)$, let r_t be a reversal operator on Ω which is defined by

$$r_t(\omega)(s) = \omega((t-s)-) \quad \text{if} \quad (0 \leq s \leq t), \qquad r_t(\omega)(s) = \omega(0) \quad \text{if} \quad (s > t).$$

Note that $\mathbf{M}^{\phi}$ is a right process (see [21]) and is reversible under P_m^{ϕ}-a.e., because for any $\mathcal{F}_t$-measurable function f,

$$E_m(f(r_t\cdot)\ ;\ t < \zeta) = E_m(f(\cdot)\ ;\ t < \zeta).$$

Since F is symmetric on $X \times X \setminus d$, $A_t^{\mu,F}(r_t\omega) = A_t^{\mu,F}(\omega)$, P_m-a.e. on account of Theorem 5.4.1 in [16]. Now for $f, g \in \mathcal{B}(X)$,

$$\begin{aligned}
&(p_t^{\phi} f, g)_{\phi^2 m} \\
&= \left(E_\cdot\left(e^{-A_t^{\mu,F}} \frac{\phi(X_t)}{\phi(X_0)} \exp\left(-\int_0^t \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_s)\,ds\right) f(X_t)\right), g\right)_{\phi^2 m} \\
&= E_m\left(e^{-A_t^{\mu,F}} \phi(X_t)\phi(X_0) f(X_0) g(X_t) \exp\left(-\int_0^t \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_s)\,ds\right)\right) \\
&= (p_t^{\phi} g, f)_{\phi^2 m},
\end{aligned}$$

which completes the proof. □

Denote by $(\mathcal{E}^{\phi}, \mathcal{F}^{\phi})$ the Dirichlet form on $L^2(X; \phi^2 m)$ associated with $\mathbf{M}^{\phi}$. Note that K_n^o is also the fine interior of K_n with respect to $\mathbf{M}^{\phi}$ because $P_x = P_x^{\phi}$ on $\mathcal{F}^*_{0+} (= \cap_{t>0}\mathcal{F}_t^*$, $\mathcal{F}_t^*$ is the universal completion of $\mathcal{F}_t^0 = \sigma\{X_s\, ;\, 0 \leq s \leq t\})$. Let $\mathbf{M}^n$ (resp. $\mathbf{M}^{\phi,n}$) be the part process of $\mathbf{M}$ (resp. $\mathbf{M}^{\phi}$) on K_n^o and $(\mathcal{E}^n, \mathcal{F}^n)$ (resp. $(\mathcal{E}^{\phi,n}, \mathcal{F}^{\phi,n})$) the Dirichlet form generated by $\mathbf{M}^n$ (resp. $\mathbf{M}^{\phi,n}$).

Lemma 3.3 ([19]). *Under the identification of* $L^2(K_n^o;\phi^2 m)$ *with* $L^2_{K_n^o}(X;\phi^2 m)=\{u\in L^2(X;\phi^2 m)\ ;\ u=0\ m\text{-a.e. on } X\setminus K_n^o\}$,

$$\mathcal{F}^{\phi,n}=\{u\in\mathcal{F}^{\phi}\ ;\ u=0\ \phi^2 m\text{-a.e. on } X\setminus K_n\}$$

and $\mathcal{E}^{\phi,n}=\mathcal{E}^{\phi}$. *In particular,* $\mathcal{F}^{\phi,n}$ *is included in* $\mathcal{F}^{\phi}$.

In order to reach our final goal of this section, we make full use of the following expression of transformed Dirichlet form. We would like to emphasize that $\mathcal{F}^{\phi}$ the domain of the transformed Dirichlet form in Proposition 3.1 includes $\mathcal{F}$, which makes it possible to show that the identity function 1 belongs to $\mathcal{F}^{\phi}$ even if the expression (3.4) itself follows from Itô formula.

Proposition 3.1. *Fix a Borel function* F *of bounded below on* $X\times X\setminus d$. *Suppose that* $\bar{F}\in L^2(X\times X\setminus d\ ;\ Nm)$, *where* $Nm(dxdy)=N(x,dy)m(dx)$ *and* $A_t^{\mu,\bar{F}}\in\mathcal{A}_K^{\mathrm{ad}}-\mathcal{A}_K^{\mathrm{ad}}$. *Then the Dirichlet space* $\mathcal{F}^{\phi}$ *includes* $\mathcal{F}$ *and for* $u\in\mathcal{F}$,

$$\begin{aligned}\mathcal{E}^{\phi}(u,u) &= \frac{1}{2}\int_X \Gamma^c(u,u)\phi^2 dm \\ &\quad + \iint_{X\times X\setminus d}(\tilde{u}(x)-\tilde{u}(y))^2\phi(x)\phi(y)e^{-F}N(x,dy)m(dx).\end{aligned}\tag{3.4}$$

Proof. First, for $\phi=R_\alpha^{\mu,F}g\in\mathcal{D}_+^2(\mathcal{L}^{\mu,F})$ and $u\in\mathcal{F}^n\cap L^\infty(X;m)$, we put

$$\begin{aligned}&(u-E_\cdot(N_t^{\phi}u(X_t)\ ;\ t<\tau_n),u)_{\phi^2 m}\\ =\ &(u-E_\cdot(u(X_t)\ ;\ t<\tau_n),u)_{\phi^2 m}-(E_\cdot((N_t^{\phi}-1)u(X_t)\ ;\ t<\tau_n),u)_{\phi^2 m}\\ =\ &(I)_t-(II)_t.\end{aligned}$$

From (3.3), $(II)_t$ equals

$$E_{u\phi m}\left(u(X_t)\int_0^t \exp\left(-\int_0^s\frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right)dM_s^{\mu,F,\phi}\ ;\ t<\tau_n\right).$$

Put

$$(III)_t=E_{u\phi m}\left(u(X_t)\int_0^{t\wedge\tau_n}\exp\left(-\int_0^s\frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right)dM_s^{\mu,F,\phi}\ ;\ t\geq\tau_n\right).$$

Then, by Lemma 3.1,

$$
\begin{aligned}
\frac{1}{t^2}(III)_t^2 \;\le\; & \frac{1}{t}E_{|u|\phi m}\Big(\int_0^{t\wedge\tau_n}\exp\Big(-2A_s^{\mu,F} \\
& -2\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\Big)d\,\langle M^{[\phi]}-L^{\phi,\bar F}\rangle_s\Big) \\
& \times\ \frac{1}{t}E_{|u|\phi m}(u(X_t)^2\ ;\ t\ge\tau_n)
\end{aligned} \tag{3.5}
$$

For $\phi, \varphi \in \mathcal{F}$, we let

$$\Gamma(\phi,\varphi)(x) = \Gamma^c(\phi,\varphi)(x) + \int_X (\tilde\phi(x)-\tilde\phi(y))(\tilde\varphi(x)-\tilde\varphi(y))N(x,dy) + \tilde\phi(x)\tilde\varphi(x)k(x).$$

Then the joint quadratic variation $\langle M^{[\phi]}, M^{[\varphi]}\rangle_t$ is equal to

$$\langle M^{[\phi]}, M^{[\varphi]}\rangle_t = \int_0^t \Gamma(\phi,\varphi)(X_s)ds, \qquad \phi,\ \varphi\in\mathcal{F}$$

(Theorem 5.3.1 in [16]). Now we claim that the first factor of the right hand side of (3.5) is bounded. By Lemma 2.1 (i) and the fact that $\left|\frac{\mathcal{L}^{\mu,F}\phi}{\phi}\right| \le \exists c < \infty$ on K_n^o,

$$
\begin{aligned}
& \frac{1}{t}E_{|u|\phi m}\left(\int_0^{t\wedge\tau_n}\exp\left(-2A_s^{\mu,F}-2\int_0^s\frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right)d\,\langle M^{[\phi]}\rangle_s\right) \\
\le\; & \frac{1}{t}\exp(2ct)E_{|u|\phi m}\left(\int_0^t \exp(-2A_s^{\mu,F})\Gamma(\phi,\phi)(X_s)ds\right) \\
\le\; & \frac{1}{t}\exp(2ct)\|u\phi\|_\infty\int_0^t \|p_s^{2\mu,2F}(\Gamma(\phi,\phi))\|_1 ds \\
\le\; & \frac{1}{t}\exp(2ct)\|u\phi\|_\infty\int_0^t \|p_s^{2\mu,2F}\|_{1,1}\|\Gamma(\phi,\phi)\|_1 ds \\
\le\; & c'\frac{1}{t}\exp(2ct)\|u\phi\|_\infty\|\Gamma(\phi,\phi)\|_1\int_0^t e^{\beta(2\mu,2F)s}ds < \infty.
\end{aligned}
$$

Also, since $\langle L^{\phi,\bar F}\rangle_t = \int_0^t\int_X \phi^2(y)\bar F^2(X_s,y)N(X_s,dy)ds$,

$$
\begin{aligned}
& \frac{1}{t}E_{|u|\phi m}\left(\int_0^{t\wedge\tau_n}\exp\left(-2A_s^{\mu,F}-2\int_0^s\frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right)d\,\langle L^{\phi,\bar F}\rangle_s\right) \\
\le\; & \frac{1}{t}\exp(2ct)E_{|u|\phi m}\left(\int_0^t \exp\left(-2A_s^{\mu,F}\right)\int_X \phi^2(y)\bar F^2(X_s,y)N(X_s,dy)ds\right) \\
\le\; & \frac{1}{t}\exp(2ct)\|u\phi^3\|_\infty\int_0^t \|p_t^{2\mu,2F}(\int_X \bar F^2(\cdot,y)N(\cdot,dy))\|_1 ds
\end{aligned}
$$

$$\leq \frac{1}{t}\exp(2ct)\|u\phi^3\|_\infty \int_0^t \|p_t^{2\mu,2F}\|_{1,1}\| \int_X \bar{F}^2(\cdot,y)N(\cdot,dy)\|_1 ds$$
$$\leq c'\frac{1}{t}\exp(2ct)\|u\phi^3\|_\infty \int_0^t e^{\beta(2\mu,2F)s}ds \iint_{X\times X\backslash d} \bar{F}^2(x,y)N(x,dy)m(dx) < \infty.$$

Now, Kunita-Watanabe inequality gives the boundedness of the first factor of (3.5). The second factor of (3.5) is equal to $\frac{1}{t}(|u|\phi, p_t u^2 - u^2)_m - \frac{1}{t}(|u|\phi, p_t^n u^2 - u^2)_m$, which tends to $\mathcal{E}(|u|\phi, u^2) - \mathcal{E}^n(|u|\phi, u^2) = 0$ as $t \to 0$ because $|u|\phi$ and u^2 are elements of $\mathcal{F}^n$ and we can appeal to Theorem 4.4.2 in [16]. Hence we get

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}(II)_t \\
&= \lim_{t\to 0}\frac{1}{t}E_{u\phi m}\left(u(X_t)\int_0^{t\wedge\tau_n} \exp\left(-\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right) dM_s^{\mu,F,\phi}\right) \\
(3.6)\quad &= \lim_{t\to 0}\frac{1}{t}E_{u\phi m}\Bigg((u(X_t) - u(X_0)) \\
&\qquad \times \int_0^{t\wedge\tau_n} \exp\left(-\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right) dM_s^{\mu,F,\phi}\Bigg).
\end{aligned}$$

Now, let $u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}$ be the Fukushima decomposition (cf. Theorem 5.2.2 in [16]). Then,

$$\lim_{t\to 0}\frac{1}{t}E_{u\phi m}\left(N_t^{[u]}\int_0^{t\wedge\tau_n} \exp\left(-\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right) dM_s^{\mu,F,\phi}\right) = 0$$

because $N^{[u]}$ is of zero-energy (p.201 in [16]), and thus the right hand side of (3.6) is equal to

$$(3.7)\quad \begin{aligned}
&\lim_{t\to 0}\frac{1}{t}E_{u\phi m}\Bigg(\int_0^{t\wedge\tau_n} \exp\Bigg(-A_s^{\mu,F} \\
&\qquad -\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\Bigg) d\left\langle M^{[u]}, M^{[\phi]} - L^{\phi,\bar{F}}\right\rangle_s\Bigg).
\end{aligned}$$

Moreover, since

$$\left|\frac{\exp\left(-\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right) - 1}{s}\right| \leq \exists M < \infty, \quad s \leq t\wedge\tau_n,$$

$$\begin{aligned}
&\frac{1}{t^2}E_{u\phi m}\left(\int_0^{t\wedge\tau_n} e^{-A_s^{\mu,F}}\left(e^{-\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du}-1\right)d\left\langle M^{[u]},L^{\phi,\bar{F}}\right\rangle_s\right)^2\\
\le\ &\frac{1}{t^2}\|u\phi\|_\infty^2 E_m\left(\int_0^t e^{-A_s^{\mu,F}}d\left\langle L^{\phi,\bar{F}}\right\rangle_s\right)E_m\left(\int_0^{t\wedge\tau_n}\left(e^{-\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du}-1\right)d\left\langle M^{[u]}\right\rangle_s\right)\\
\le\ & M^2\|u\phi\|_\infty^2 E_m\left(\int_0^t e^{-A_s^{\mu,F}}d\left\langle L^{\phi,\bar{F}}\right\rangle_s\right)E_m\left(\int_0^t \Gamma(u,u)(X_s)ds\right)\\
\le\ & M^2\|u\phi^3\|_\infty^2\int_0^t \left\|p_s^{\mu,F}\left(\int_X \bar{F}^2(\cdot,y)N(\cdot,dy)\right)\right\|_1 ds\int_0^t \|p_s\left(\Gamma(u,u)\right)\|_1 ds\\
\le\ & M'\|u\phi^3\|_\infty^2\int_0^t e^{\beta s}ds\iint_{X\times X\setminus d}\bar{F}^2(x,y)N(x,dy)m(dx)\int_0^t \|p_s\left(\Gamma(u,u)\right)\|_1 ds\\
&\longrightarrow 0 \ \text{ as } \ t\to 0.
\end{aligned}$$

We get

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}E_{u\phi m}\left(\int_0^{t\wedge\tau_n}\exp\left(-A_s^{\mu,F}-\int_0^s\frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right)d\left\langle M^{[u]},L^{\phi,\bar{F}}\right\rangle_s\right)\\
=\ &\lim_{t\to 0}\frac{1}{t}E_{u\phi m}\left(\int_0^{t\wedge\tau_n}\exp\left(-A_s^{\mu,F}\right)d\left\langle M^{[u]},L^{\phi,\bar{F}}\right\rangle_s\right),
\end{aligned}$$

and also similarly,

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}E_{u\phi m}\left(\int_0^{t\wedge\tau_n}\exp\left(-A_s^{\mu,F}-\int_0^s\frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right)d\left\langle M^{[u]},M^{[\phi]}\right\rangle_s\right)\\
=\ &\lim_{t\to 0}\frac{1}{t}E_{u\phi m}\left(\int_0^{t\wedge\tau_n}\exp\left(-A_s^{\mu,F}\right)d\left\langle M^{[u]},M^{[\phi]}\right\rangle_s\right).
\end{aligned}$$

Therefore, (3.7) is equal to

$$\lim_{t\to 0}\frac{1}{t}E_{u\phi m}\left(\int_0^{t\wedge\tau_n}\exp\left(-A_s^{\mu,F}\right)\left(d\left\langle M^{[u]},M^{[\phi]}\right\rangle_s-d\left\langle M^{[u]},L^{\phi,\bar{F}}\right\rangle_s\right)\right).$$

Now, since $\lim_{t\to 0}P_{|u|\phi m}(t\ge\tau_n)=0$,

$$\begin{aligned}
&\frac{1}{t^2}E_{|u|\phi m}\left(\int_0^t \exp(-A_s^{\mu,F})d\left\langle M^{[u]},L^{\phi,\bar{F}}\right\rangle_s; t\ge\tau_n\right)^2\\
\le\ &\frac{1}{t^2}P_{|u|\phi m}(t\ge\tau_n)E_{|u|\phi m}\left(\left(\int_0^t \exp(-A_s^{\mu,F})d\left\langle M^{[u]},L^{\phi,\bar{F}}\right\rangle_s\right)^2\right)\\
\le\ &\frac{1}{t^2}P_{|u|\phi m}(t\ge\tau_n)\|u\phi\|_\infty^2 E_m\left(\int_0^t\Gamma(u,u)(X_s)ds\right)E_m\left(\int_0^t\exp(-2A_s^{\mu,F})d\left\langle L^{\phi,\bar{F}}\right\rangle_s\right)
\end{aligned}$$

$$\leq \frac{1}{t^2} P_{|u|\phi m}(t \geq \tau_n) \|u\phi^3\|_\infty^2 \int_0^t \|p_s^{2\mu,2F}(\int_X \bar{F}^2(\cdot,y)N(\cdot,dy))\|_1 ds$$
$$\times \int_0^t \|p_s(\Gamma(u,u))\|_1 ds$$
$$\longrightarrow 0 \quad \text{as } t \to 0,$$

and also

$$\lim_{t\to 0} \frac{1}{t} E_{u\phi m}\left(\int_0^{t\wedge\tau_n} \exp\left(-A_s^{\mu,F}\right) d\langle M^{[u]}, M^{[\phi]}\rangle_s; t \geq \tau_n\right) = 0.$$

Therefore, we conclude that

$$\begin{aligned}
&\lim_{t\to 0} \frac{1}{t}(II)_t \\
&= \lim_{t\to 0} \frac{1}{t} E_{u\phi m}\left(\int_0^t \exp(-A_s^{\mu,F})\Gamma(u,\phi)(X_s)ds\right) \\
&\quad - \lim_{t\to 0} \frac{1}{t} E_{u\phi m}\left(\int_0^t \exp(-A_s^{\mu,F}) \int_X \phi(y)\bar{u}\bar{F}(X_s,y)N(X_s,dy)ds\right) \\
&= \lim_{t\to 0} \frac{1}{t}(u\phi, \int_0^t p_s^{\mu,F}(\Gamma(u,\phi))ds)_m \\
&\quad - \lim_{t\to 0} \frac{1}{t}(u\phi, \int_0^t p_s^{\mu,F}(\int_X \phi(y)\bar{u}\bar{F}(\cdot,y)N(\cdot,dy))ds)_m \\
&= (u\phi, \Gamma(u,\phi))_m - (u\phi, \int_X \phi(y)\bar{u}\bar{F}(\cdot,y)N(\cdot,dy))_m \\
&= \int_X u\phi\Gamma(u,\phi)dm - \iint_{X\times X\setminus d} u(x)\phi(x)\phi(y)\bar{u}\bar{F}(x,y)N(x,dy)m(dx),
\end{aligned}$$

where $\bar{u}(x,y) = u(y) - u(x)$. On the other hand, $\lim_{t\to 0} \frac{1}{t}(I)_t = \mathcal{E}(u, u\phi^2)$. Hence,

$$\begin{aligned}
&\lim_{t\to 0} \frac{1}{t}(u - E_\cdot(N_t^\phi u(X_t); t < \tau_n), u)_{\phi^2 m} \\
&= \frac{1}{2}\int_X \Gamma(u,u\phi^2)dm + \frac{1}{2}\int_X u^2\phi^2 k dm - \int_X u\phi\Gamma(u,\phi)dm \\
&\quad + \iint_{X\times X\setminus d} u(x)\phi(x)\phi(y)\bar{u}\bar{F}(x,y)N(x,dy)m(dx) \\
&= \frac{1}{2}\int_X \Gamma^c(u,u)\phi^2 dm \\
&\quad + \iint_{X\times X\setminus d} (\tilde{u}(x) - \tilde{u}(y))^2\phi(x)\phi(y)e^{-F(x,y)}N(x,dy)m(dx),
\end{aligned}$$

and consequently, $u \in \mathcal{F}^{\phi,n}$, which implies that $\mathcal{F}^n \cap L^\infty(X;m) \subset \mathcal{F}^\phi$ by virtue of Lemma 3.3. Noting that

$$\mathcal{E}^\phi(u,u) \le \|\phi\|_\infty^2 \mathcal{E}(u,u) \qquad u \in \mathcal{F}^n \cap L^\infty(X;m).$$

Since $\cup_n(\mathcal{F}^n \cap L^\infty(X;m))$ is dense in $\mathcal{F}$, we arrive at the lemma. □

Proposition 3.2. *For $\phi \in \mathcal{D}_+^2(\mathcal{L}^{\mu,F})$, the identity function* 1 *belongs to $\mathcal{F}^\phi$ and* $\mathcal{E}^\phi(1,1) = 0$.

Proof. This proposition follows from Proposition 3.1 with the same argument as in Lemma 6.3.3 in [16]. □

Lemma 3.4 ([14],[19],[22]). *Let $(\mathcal{E},\mathcal{F})$ be the Dirichlet form associated with an m-symmetric right process (Ω, X_t, P_x) on X. Suppose that*

$$1 \in \mathcal{F} \quad \text{and} \quad \mathcal{E}(1,1) = 0.$$

Then, the following statements are equivalent each other;
(i) $(\mathcal{E},\mathcal{F})$ *is irreducible.*
(ii) *If $\mathcal{E}(u,u) = 0$, then u is constant m-a.e.*
(iii) *If $T_t u = u$ m-a.e. for all $t > 0$, then u is constant m-a.e.*
(iv) $(\Omega, P_m, \mathcal{F}^0, \theta_t)$ *is ergodic, (i.e., if $\Lambda \in \mathcal{F}^0$ is θ_t-invariant, $(\theta_t)^{-1}(\Lambda) = \Lambda$, then $P_x(\Lambda) = 0$ for all $x \in X$ or $P_x(\Lambda) = 1$ for all $x \in X$. Here $\mathcal{F}^0 = \sigma\{X_t : 0 \le t < \infty\}$ and θ_t is the shift operater on Ω).*

Theorem 3.1. *The transformed process $\mathbf{M}^\phi$ is ergodic in the sense of Lemma 3.4.*

Proof. On account of the positivity of N^ϕ up to the life time ζ, $(\mathcal{E}^\phi, \mathcal{F}^\phi)$ is irreducible. Hence, it follows from Lemma 3.4 and Proposition 3.2 that $P^\phi_{\phi^2 m}(\Lambda) = 0$ or $P^\phi_{\phi^2 m}(\Omega \setminus \Lambda) = 0$. Moreover, by assumption (III), $\mathbf{M}^\phi$ also admits a transition density. Hence $P^\phi_x(\Lambda) = 0$ or 1. □

Assume that μ and F are positive. Then by combining Theorem 3.2 in [31] for the multiplicative functional $\exp\left(-A_t^{\mu,F}\right)$ with Theorem 6.3.1 in [16], we can immediately check that Theorem 3.1 is also derived without the assumption (I).

4. Lower and Upper Estimations

In this section, we now describe the asymptotic behaviour of $E_x\left(\exp\left(-A_t^{\mu,F}\right)\right)$ by applying the results of previous sections. We first give the lower estimate of its large deviation principle governed by the perturbed Dirichlet form $\mathcal{E}^{\mu,F}$, and next consider the upper estimate for the symmetric Lévy process with its exponent (4.1) below.

The lower estimate

Let $\mathcal{M}_1(X)$ be the set of probability measures on X equipped with the weak topology. Define the function $I_{\mathcal{E}^{\mu,F}}$ on $\mathcal{M}_1(X)$ by

$$I_{\mathcal{E}^{\mu,F}}(\nu) = \begin{cases} \mathcal{E}^{\mu,F}(\sqrt{f},\sqrt{f}) & \text{if } \nu = f\cdot m,\ \sqrt{f}\in\mathcal{F}^{\mu,F} \\ \infty & \text{otherwise.}\end{cases}$$

For $\omega\in\Omega$ and $0<t<\zeta(\omega)$, also define the occupation distribution

$$L_t(\omega)(A) = \frac{1}{t}\int_0^t \chi_A(X_s(\omega))ds, \qquad A\in\mathcal{B}(X).$$

Theorem 4.1. *Fix a Borel function F of bounded below on $X\times X\setminus d$. Suppose that $\bar{F}\in L^2(X\times X;Nm)$ and $A^{\mu,\bar{F}}\in\mathcal{A}_K^{\text{ad}}-\mathcal{A}_K^{\text{ad}}$. Then for any open set $G\subset\mathcal{M}_1(X)$,*

$$\liminf_{t\to\infty}\frac{1}{t}\log E_x\left(\exp\left(-A_t^{\mu,F}\right)\ ;\ L_t\in G, t<\zeta\right) \geq -\inf_{\nu\in G} I_{\mathcal{E}^{\mu,F}}(\nu) \quad \textit{for all } x\in X.$$

Proof. Using Theorem 3.1, we have the theorem with exactly the same argument as in [11],[27]. Take $\phi = R_\alpha^{\mu,F}f\in\mathcal{D}_+^2(\mathcal{L}^{\mu,F})$ and $\phi^2m\in G$. For the multiplicative functional N^ϕ defined by (3.2) and $x\in X$, we have

$$\begin{aligned}
& E_x\left(\exp\left(-A_t^{\mu,F}\right)\ ;\ L_t\in G, t<\zeta\right) \\
= \ & E_x^\phi\left(\left(N_t^\phi\right)^{-1}\exp\left(-A_t^{\mu,F}\right)\ ;\ L_t\in G\right) \\
\geq \ & \exp\left(t\left(\int_X \phi\mathcal{L}^{\mu,F}\phi dm - \varepsilon\right)\right) E_x^\phi\left(\frac{\phi(X_0)}{\phi(X_t)}\ ;\ S(t,\varepsilon)\right) \\
\geq \ & \exp\left(t\left(\int_X \phi\mathcal{L}^{\mu,F}\phi dm - \varepsilon\right)\right) \frac{\phi(x)}{\|\phi\|_\infty}\left(1 - P_x^\phi(\Omega\setminus S(t,\varepsilon))\right),
\end{aligned}$$

where

$$S(t,\varepsilon) = \left\{\omega\in\Omega\ ;\ \left|\int_X \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(x)L_t(\omega,dx) - \int_X \phi\mathcal{L}^{\mu,F}\phi dm\right| < \varepsilon\right\}\cap\{\omega\in\Omega\ ;\ L_t(\omega)\in G\}.$$

Put

$$\begin{aligned}\Omega_1 &= \left\{\omega \in \Omega\,;\ \lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_s(\omega))ds = \int_X \phi\mathcal{L}^{\mu,F}\phi\, dm\right\},\\ \Omega_2 &= \{\omega \in \Omega\,;\ L_t(\omega) \text{ converges to } \phi^2 m \text{ as } t\to\infty\}.\end{aligned}$$

Now, on account of the shift invariance of Ω_i $(i = 1, 2)$ due to Theorem 3.1, we know $P_x^\phi(\Omega_i) = 1$, $\phi^2 m$-a.e. Hence,

$$P_x^\phi(\Omega \setminus S(t,\varepsilon)) \longrightarrow 0 \quad \text{as} \quad t \to \infty \quad \text{for } \forall x \in X.$$

Consequently, we have

$$\liminf_{t\to\infty}\frac{1}{t}\log E_x\left(\exp\left(-A_t^{\mu,F}\right)\ ;\ L_t \in G, t < \zeta\right) \geq \int_X \phi\mathcal{L}^{\mu,F}\phi dm - \varepsilon.$$

Note that the set $\{\phi \in \mathcal{D}_+^2(\mathcal{L}^{\mu,F})\,;\ \|\phi\|_2 = 1\}$ is dense in $\{\phi \in \mathcal{F}^{\mu,F}\,;\ \phi \geq 0, \|\phi\|_2 = 1\}$ with respect to $\mathcal{E}_{\alpha_0}^{\mu,F}$ $(\alpha_0 > \beta(\mu, F))$ which completes the proof. □

We can not treat the upper estimate of $E_x\left(\exp\left(-A^{\mu,F}\right)\right)$ with large deviation theory without additional conditions. By the same argument of Proposition 4.2 in [27], we have the followings.

REMARK 4.1. Let us assume the hypotheses in Theorem 4.1 are satisfied. In addition, let assume that the symmetric Markov process $\mathbf{M}$ as a base process is conservative and the transition probability p_t of $\mathbf{M}$ satisfies the strong Feller property. Then for any compact subset K of $\mathcal{M}_1(X)$,

$$\limsup_{t\to\infty}\frac{1}{t}\log E_x\left(\exp\left(-A_t^{\mu,F}\right)\ ;\ L_t \in K\right) \leq -\inf_{\nu\in K} I_{\mathcal{E}^{\mu,F}}(\nu) \quad \textit{for all} \quad x \in X.$$

Moreover, if the base process explodes so fast in the sense that $R_1^{\mu,F}1$ belongs to $C_\infty(X)$, it can be extended for any closed subset of $\mathcal{M}_1(X)$.

We shall consider the upper estimate of large deviation principle without above two conditions. To do this, we apply the fact that the L^p-spectral radius of the Feynman-Kac semigroup of kernels is p-independent under the symmetric Lévy process whose Lévy exponent is the so-called α-relativistic Hamiltonian (4.1) below.

The upper estimate

Symmetric convolution semigroups $\{\nu_t, t \geq 0\}$ of infinitely divisible probability measures on R^d define a Markovian semigroup p_t by

$$p_t f(x) = \int_{R^d} f(x+y)\nu_t(dy) \qquad f \in \mathcal{B}_b(R^d),$$

which is symmetric with respect to the Lebesgue measure, and the Lévy-Khinchin formula under the above conditions for ν_t leads as follows ;

$$\int_{R^d} e^{i(x,y)}\nu_t(dy) = \exp(-t\psi(x))$$
$$\psi(x) = \frac{1}{2}(Sx,x) + \int_{R^d}(1-\cos(x,y))J(dy),$$

where S is a non-negative definite $d \times d$ symmetric matrix and J is a symmetric measure on $R^d \setminus \{0\}$ such that $\int_{R^d \setminus \{0\}} \min(1, |x|^2)J(dx) < \infty$. Here, the function ψ is called the Lévy exponent and the measure J is called the Lévy measure. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ associated with the semigroup p_t is written as

$$\mathcal{E}(u,u) = \frac{1}{2}\int_{R^d}(S\nabla u, \nabla u)(x)\,dx + \frac{1}{2}\iint_{R^d \times R^d \setminus d}(u(x)-u(y))^2 J(dy-x)\,dx,$$
$$\mathcal{D}(\mathcal{E}) = \left\{u \in L^2(R^d)\ ;\ \mathcal{E}(u,u) < \infty\right\}.$$

The Hunt process associated with $(\mathcal{E}, \mathcal{F})$ is called the Lévy process and denote it by $\mathbf{M} = (\Omega, X_t, P_x)$. In this situation, we add the following assumption ;

$$\textbf{(V)} \qquad \int_{R^d} e^{-t\psi(x)}dx < \infty, \qquad \forall t > 0.$$

Then, under this assumption, the function

$$p_t(x) = \frac{1}{(2\pi)^d}\int_{R^d} e^{iy\cdot x}e^{-t\psi(y)}dy (\in C_\infty(R^d)), \qquad x \in R^d$$

is the density of the measure ν_t for each $t > 0$ and is an analytic function of t on $(0, \infty)$ ([7]). Furthermore, the transition density of $\mathbf{M}$, $p_t(x-y)$ satisfies all assumptions in section 2. Indeed, the irreducibility of $\mathbf{M}$ is derived by noting the periodicity of $\psi(x)$ on R^d. Therefore, we can apply the lower estimate of large deviation principle for the Lévy process $\mathbf{M}$.

DEFINITION 4.1. A Lévy measure J is said to be exponentially localized if there exists a constant $\delta > 0$ such that

$$\int_{|x|>1} e^{\delta|x|} J(dx) < \infty.$$

Lemma 4.1 ([7]). *Let the Lévy measure J of $\mathbf{M}$ be exponentially localized. Then there exist positive constants c_1, c_2 such that*

$$E_0\left(e^{\delta \sup_{0\le s\le t}|X_s|}\right) \le c_1 e^{c_2 t}.$$

We now concentrate on the symmetric Lévy process $\mathbf{M} = (\Omega, X_t, P_x)$ on R^d with the so-called *α-relativistic Hamiltonian* as Lévy exponent

$$(p^2 + m^2)^{\frac{\alpha}{2}} - m^\alpha, \qquad m > 0,\ 0 < \alpha \le 2 \tag{4.1}$$

which was recently investigated by Carmona, Masters, and Simon ([7]). Note that the Lévy measure corresponding to the Lévy exponent (4.1) is exponentially localized because (4.1) has an analytic continuation to some strip (see Proposition II.1 in [7]). With the Lévy process $\mathbf{M}$ as the base process, we then have the following upper estimate.

Theorem 4.2. *For any $A^{\mu,\bar{F}} \in \mathcal{A}_K^{\mathrm{ad}} - \mathcal{A}_K^{\mathrm{ad}}$ and $x \in R^d$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log E_x\left(\exp\left(-A_t^{\mu,F}\right)\right) \le - \inf_{\substack{u\in\mathcal{F}^{\mu,F} \\ \|u\|_2=1}} \mathcal{E}^{\mu,F}(u,u).$$

Proof. Let

$$\lambda = -\log \left\|p_1^{\mu,F}\right\|_{2,2} = \inf_{\substack{u\in\mathcal{F}^{\mu,F} \\ \|u\|_2=1}} \mathcal{E}^{\mu,F}(u,u).$$

We also put

$$\begin{aligned} & E_x\left(\exp\left(-A_t^{\mu,F}\right)\right) \\ = \;& E_x\left(\exp\left(-A_t^{\mu,F}\right)\chi_{B_r(x)}(X_t)\right) + E_x\left(\exp\left(-A_t^{\mu,F}\right)\left(1-\chi_{B_r(x)}\right)(X_t)\right) \\ = \;& (I)_t + (II)_t, \end{aligned}$$

where $B_r(x)$ is the ball centered at x with radius r. Then, from Lemma 4.1,

$$(II)_t \;\le\; E_x\left(\exp\left(-2A_t^{\mu,F}\right)\left(1-\chi_{B_r(x)}\right)(X_t)\right)^{1/2} E_x\left(\left(1-\chi_{B_r(x)}\right)(X_t)\right)^{1/2}$$

$$
\begin{aligned}
&\leq \left\| p_t^{2\mu,2F} \right\|_{\infty,\infty}^{1/2} P_0 \left(\sup_{0\leq s\leq t} |X_s| \geq r \right)^{1/2} \\
&\leq C \exp\left(\beta(2\mu, 2F)t - \frac{\delta}{2}r + \frac{c_2}{2}t \right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(I)_t &= p_t^{\mu,F} \chi_{B_r(x)}(x) \\
&= p_1^{\mu,F} \cdot p_{t-1}^{\mu,F} \chi_{B_r(x)}(x) \\
&\leq \left\| p_1^{\mu,F} \right\|_{2,\infty} \left\| p_{t-1}^{\mu,F} \chi_{B_r(x)} \right\|_2 \\
&\leq \left\| p_1^{\mu,F} \right\|_{2,\infty} e^{-\lambda(t-1)} \left\| \chi_{B_r(x)} \right\|_2 \\
&\leq C' r^{d/2} e^{-\lambda t},
\end{aligned}
$$

where $C' = e^{\lambda} \left\| p_1^{\mu,F} \right\|_{2,\infty}$. Now, take $k > 0$ so that $-\beta(2\mu, 2F) + \frac{\delta}{2}k - \frac{c_2}{2} > \lambda$ and put $r = kt$. We then have

$$
E_x \left(\exp\left(-A_t^{\mu,F} \right) \right) \leq K(1 + t^{d/2}) e^{-\lambda t},
$$

for some positive constant K. The proof is complete. □

Consequently, when the underlying process is the symmetric Lévy process with its Lévy exponent (4.1), Theorem 4.1 and Theorem 4.2 lead us to the following theorem.

Theorem 4.3. *Fix a Borel function F of bounded below on $R^d \times R^d \setminus d$. Suppose that $\bar{F} \in L^2(R^d \times R^d; Nm)$ and $A^{\mu,\bar{F}} \in \mathcal{A}_K^{\mathrm{ad}} - \mathcal{A}_K^{\mathrm{ad}}$. Then*

$$
\lim_{t\to\infty} \frac{1}{t} \log E_x \left(\exp\left(-A_t^{\mu,F} \right) \right) = - \inf_{\substack{u \in \mathcal{F}^{\mu,F} \\ \|u\|_2 = 1}} \mathcal{E}^{\mu,F}(u,u) \qquad \text{for } \forall x \in R^d. \tag{4.2}
$$

5. Applications

Let $\mathbf{M}' = (\Omega, X_t, P_x)$ be a symmetric stable process of index $\alpha, (1 < \alpha \leq 2)$ on R^d with its exponent $|z|^{\alpha}$. It is known in [24] that the definition of Kato class measure μ, (2.1) is equivalent to the classical definition of the Kato class

$$
\lim_{\delta\to 0} \sup_{x\in R^d} \int_{|x-y|<\delta} \frac{1}{|x-y|^{d-\alpha}} |\mu|(dy) = 0, \qquad |\mu| = \mu^+ + \mu^-
$$

under $\mathbf{M}'$. So, we see from [6] that the surface measure σ of the sphere on R^3 is in the (extended) Kato class. We shall first apply the formula (4.2) to the surface measure σ and give the asymptotics of local time for σ. The problem is arised for the fact that at least, we cannot say anything about the formula (4.2) under $\mathbf{M}'$ because its Lévy measure is not exponentially localized. To show that the surface measure σ belongs to the Kato class corresponding to the α-relativistic Hamiltonian operator, we need to apply the following fact essentially due to [7],[32] ; the Kato class measure is same whether we deal with the pseudo-differential operator $|-\Delta|^{\alpha/2}(1<\alpha\leq 2)$ or the α-relativistic Hamiltonian operator $(-\Delta+m^2)^{\alpha/2}-m^{\alpha}(m>0, 1<\alpha\leq 2)$.

Let us denote g_α by the Green's density of $\mathbf{M}'$. A measurable function f on R^d is said to be in $l^1(L^\infty)$ if $\|f\|_{l^1(L^\infty)}=\sum_{k\in Z^d}\sup_{x\in C(k)}|f(x)|<\infty$, where $C(k)$ denotes the cube centered at $k\in Z^d$ with sides of length 1. Note that the transition density $p_t(\cdot)$ of $\mathbf{M}'$ belongs to $l^1(L^\infty)$ and for each fixed $\delta>0$,

$$\sup_t \left\|\chi_{\{|y|>\delta\}}p_t u\right\|_{l^1(L^\infty)}<\infty \tag{5.1}$$

([29]). Note that most of the transition densities of the Lévy processes we are interested indeed, satisfy the above properties ([7]).

Theorem 5.1. *Let μ be a smooth measure such that $\sup_{x\in R^d}\mu(x+C(0))<\infty$. Suppose that the assumption (V) and the condition (5.1) are satisfied by the Lévy process. If*

$$\lim_{\delta\to 0}\sup_{x\in R^d}\int_{|x-y|<\delta} g_\alpha(x-y)\mu(dy)=0,$$

then μ belongs to the (extended) Kato class.

Proof. For fixed $\delta>0$, arbitrary $\varepsilon>0$ and sufficiently large enough $\beta>0$, we have from (5.1)

$$\begin{aligned}
&\sup_{x\in R^d}\int_{|x-y|\geq\delta} g_\beta(x-y)\mu(dy)\\
=\ &\sup_{x\in R^d}\int_{R^d}\chi_{\{|u|\geq\delta\}}p_s u\,\mu(x-du)\int_0^\infty e^{-\beta s}ds\\
=\ &\sup_{x\in R^d}\sum_{k\in Z^d}\int_{C(k)}\chi_{\{|u|\geq\delta\}}p_s u\,\mu(x-du)\int_0^\infty e^{-\beta s}ds\\
\leq\ &\sup_{x\in R^d}\mu(x+C(0))\sup_{s>0}\|\chi_{\{|u|\geq\delta\}}p_s u\|_{l^1(L^\infty)}\int_0^\infty e^{-\beta s}ds\\
<\ &\frac{\varepsilon}{4}.
\end{aligned}$$

On the other hand, for such $\beta > 0$,

$$\sup_{x\in R^d}\int_{|x-y|<\delta} g_\beta(x-y)\mu(dy) \le \sup_{x\in R^d}\int_{|x-y|<\delta} g_\alpha(x-y)\mu(dy)\ (< \frac{\varepsilon}{4}).$$

Hence, we can choose $\beta > 0$ large enough so that

$$\begin{aligned}\sup_{x\in R^d} E_x\left(A_t^\mu\right) &\le \sup_{x\in R^d} E_x\left(e^{\beta t}\int_0^\infty e^{-\beta s}dA_s^\mu\right)\\ &= \sup_{x\in R^d} e^{\beta t}R_\beta\mu(x)\\ &= \sup_{x\in R^d} e^{\beta t}\int_{R^d} g_\beta(x-y)\mu(dy) < e^{\beta t}\frac{\varepsilon}{2}.\end{aligned}$$

Now, to reach the conclusion, we only choose sufficiently small t so that $e^{\beta t} < 2$. The proof is complete. □

In fact, the above theorem is held equivalently because for sufficiently large enough $\beta > 0$ and some constant $c > 0$,

$$\int_{R^d} g_\beta(x-y)\mu(dy) \le \|p_s\|_{l^1(L^\infty)} \sup_{x\in R^d}\mu(x+C(0))\left\{\int_0^\infty e^{-\beta s}ds\right\} < \frac{\varepsilon}{c}$$

and there exists constant $\delta > 0$ such that if $|x-y| \le \delta$ then $g_\alpha(x-y) \le cg_\beta(x-y)$ (Lemma III.3 in [7]).

EXAMPLE 5.1. Note that the Theorem 5.1 implies that the definition of the Kato class measure depends only on the behaviour of the exponent function $\psi(u)$ when $u \to \infty$ because the Green's density $g_\alpha(x)$ is equal to

$$\frac{1}{(2\pi)^d}\int_{R^d}\frac{1}{\alpha+\psi(u)}e^{-iux}du = K_R + \frac{1}{(2\pi)^d}\int_{|u|\ge R}\frac{1}{\alpha+\psi(u)}e^{-iux}du.$$

So the Kato class is the same whether we deal with $\psi(u) = |u|^\alpha, (1 < \alpha \le 2)$ or $\psi(u) = (u^2+m^2)^{\alpha/2} - m^\alpha, (m > 0,\ 1 < \alpha \le 2)$. This means that the surface measure $\mu = \sigma(|x| = R)$ of the sphere on R^3 also belongs to the (extended) Kato class when the base process is the relativistic Hamiltonian of index α, $(1 < \alpha \le 2)$. Let us denote by $\ell_R(t)$ the local time corresponding to μ. We have

$$\lim_{t\to\infty}\frac{1}{t}\log E_x\left(\exp(-\ell_R(t))\right) = -\inf_{\substack{u\in\mathcal{F}^\mu\\ \|u\|_2=1}}\left(\int_{R^3}\psi(x)|\hat{u}(x)|^2dx + \int_{|x|=R}\tilde{u}(x)^2d\mu(x)\right)$$

where $\mathcal{F}^\mu = \{u \in \mathcal{F} : \tilde{u} \in L^2(R^3; |\mu|)\}$, $\psi(x) = (x^2 + m^2)^{\alpha/2} - m^\alpha (m > 0,\ 1 < \alpha \leq 2)$, and $\hat{}$ means its Fourier transformation. □

EXAMPLE 5.2. Let $B_R (= \{x; |x| < R\})$ be the open ball in R^3 and σ the surface measure of the sphere $\partial B_r, (r < R)$. Let X_t be an absorbing symmetric stable process on B_R with index $\alpha, (1 < \alpha \leq 2)$. The surface measure σ is in the (extended) Kato class under the process X_t, and is the killing measure on B_R. Let us denote by $\ell_r(t)$ the positive continuous additive functional corresponding to σ. Since the ball B_R is regular, $\lim_{x \to \partial B_R} E_x (e^{-\tau_R}) = 1$. That is, $R_1^R 1 \in C_\infty(B_R)$ and which implies $R_1^{R,r} 1 \in C_\infty(B_R)$. In this case, we can derive the formula (4.2) by using the full large deviation principle. Therefore,

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t} \log E_x^R (\exp(-\ell_r(t)) \ ; \ t < \zeta) \\ = & - \inf_{\substack{u \in \mathcal{F}^\sigma \\ \|u\|_2 = 1}} \left(\int_{R^3} \psi(x) |\hat{u}(x)|^2 dx + \int_{|x|=r} \tilde{u}(x)^2 d\sigma(x) \right), \end{aligned}$$

where $\mathcal{F}^\sigma = \{u \in \mathcal{F} \ ; \ \tilde{u} \in L^2(R^3; |\sigma|)\}$, $\psi(x) = |x|^\alpha, (1 < \alpha \leq 2)$. □

Now, in the rest of this section, we shall consider the formula (4.2) with symmetric Markov chains. Let I be a countable set equipped with the discrete topology. Let $Q = (q_{ij})$ be an $I \times I$ matrix such that

$$q_{ij} \geq 0 \quad (i \neq j), \qquad \sum_{k \neq i} q_{ik} \leq -q_{ii} < \infty, \qquad \forall i \in I$$

and $m_i q_{ij} = m_j q_{ji}$ for some strictly positive function m_i on I. Let $\mathcal{E}$ be the Dirichlet form on $L^2(I; m)$ defined by

$$\begin{aligned} \mathcal{E}(u,v) &= \frac{1}{2} \sum_{i \neq j} q_{ij} m_i (u(j) - u(i))(v(j) - v(i)) \\ &\quad + \sum_i \left(-q_{ii} - \sum_{j \neq i} q_{ij} \right) m_i u(i) v(i). \end{aligned} \tag{5.2}$$

Denote by $\mathcal{F}^r$ the collection of functions u on I such that $\mathcal{E}(u,u) < \infty$. Let $\mathcal{F}_e$ be the set of functions u in $\mathcal{F}^r$ for which there exists $u_n (n = 1, 2, \cdots)$ with finite support such that

$$u_n \to u \quad \text{and} \quad \sup_n \mathcal{E}(u_n, u_n) < \infty.$$

Then $(\mathcal{E},\mathcal{F})$ becomes a symmetric regular Dirichlet form on $L^2(I;m)$ and the Dirichlet space $\mathcal{F}$ is identified with the space $\mathcal{F}_e \cap L^2(I;m)$ (cf. Theorem 17.2 in [22]). The space $\mathcal{F}_e$ is said to be the *extended Dirichlet space of* $\mathcal{F}$ (p.36 in [16]). Denote by $\mathbf{M} = (P_i, X_t)$ the Hunt process associated with $(\mathcal{E},\mathcal{F})$. Then $\mathbf{M}$ is nothing but the minimal $\mathcal{Q}$-process constructed by W. Feller.

EXAMPLE 5.3. Let us consider a symmetric Markov chain $\mathbf{M} = (P_i, X_t)$ on the finite state space $I = \{1, 2, \cdots, n\}$. In this case, since the state space I is compact and the Dirichlet form on $L^2(I;m)$ associated with $\mathbf{M}$ is

$$\mathcal{E}(u,v) = \frac{1}{2}\sum_{i\neq j} q_{ij} m_i (u(j) - u(i))(v(j) - v(i)),$$

we can establish the following full large deviation principle by Remark 4.1;

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t} \log E_i \left(\exp\left(-\sum_{s\leq t} F(X_{s-}, X_s) \right)\right) \\
&= -\inf_{\substack{u\in\mathcal{F}^F \\ \|u\|_2=1}} \left(\frac{1}{2}\sum_{i\neq j} q_{ij} m_i (u(j) - u(i))^2 \right. \\
&\qquad \left. + \sum_{i\neq j} q_{ij} m_i u(i) u(j) (1 - e^{-F(i,j)}) \right) \qquad (5.3) \\
&= -\inf_{\substack{u\in\mathcal{F}^F \\ \|u\|_2=1}} \left(\frac{1}{2}\sum_{i\neq j} q_{ij} m_i e^{-F(i,j)} (u(j) - u(i))^2 \right. \\
&\qquad \left. + \sum_i u^2(i) \sum_j q_{ij} m_i \left(1 - e^{-F(i,j)}\right) \right),
\end{aligned}$$

where $\mathcal{F}^F = \{u \in \mathcal{F}; u \in L^2(I;|\rho|)\}, \rho(A) = \sum_{i\in A} \cdot \sum_{i\in I} q_{ij} m_i \left(1 - e^{-F(i,j)}\right)$ for all $A \in \mathcal{B}(I)$. Now, let us take $F(i,j) = \chi_B(i,j)$, where $B = \{(1,2), (2,3), \cdots, (k, k+1), (k+1, k), (k, k-1), \cdots, (2,1)\}$ is the symmetric region on $I \times I \setminus d$. Then, the formula (5.3) gives to us the asymptotics of the number of jumps for the symmetric Markov chain $\mathbf{M}$ on B. □

To establish the full large deviation principle in the case of infinite state space, we may need the following conditions on the Lévy kernel N.

Theorem 5.2. *For a positive Borel function F on $X \times X \setminus d$, we put*

$$V(x) = \int_X \left(1 - e^{-2F(x,y)}\right) N(x, dy). \tag{5.4}$$

Suppose that $V(x) \to \infty$ as $|x| \to \infty$ and

$$\lim_{x \to \infty} P_x(\sigma_K \le t) = 0 \tag{5.5}$$

for any compact subset $K \subset X$, σ_K is the first hitting time of K. Then the full large deviation principle is established.

Proof. Let $\tilde{F} = \left(e^{-2F} - 1\right)$. For an increasing sequence of compact sets $\{K_n\}$ on $X \times X \setminus d$ such that $\cup_{n=1}^{\infty} K_n = X \times X \setminus d$, we put $F_n = F\chi_{K_n}$, $\tilde{F}_n = \left(e^{-2F_n} - 1\right)$ and $B_t^{(n)} = \sum_{s \le t} \tilde{F}_n(X_{s-}, X_s)$. Then there exists the dual predictable projection of $B_t^{(n)}$,

$$\left(B_t^{(n)}\right)^p = \int_0^t \int_X N(X_s, dy) \tilde{F}_n(X_s, y)\, ds.$$

Since $B_t^{(n)} - \left(B_t^{(n)}\right)^p$ is a P_x-martingale for every $x \in X$, *Doléans-Dade exponential formula* ([10]) implies that

$$\begin{aligned} Z_t &= e^{B_t^{(n)} - (B_t^{(n)})^p} \prod_{s \le t} (1 + \tilde{F}_n(X_{s-}, X_s)) e^{-\tilde{F}_n(X_{s-}, X_s)} \\ &= e^{-(B_t^{(n)})^p} \prod_{s \le t} (1 + \tilde{F}_n(X_{s-}, X_s)) \\ &= e^{-2\sum_{s \le t} F_n(X_{s-}, X_s) - (B_t^{(n)})^p} \end{aligned}$$

is at least supermartingale multiplicative functional. From Schwarz inequality, we get

$$\begin{aligned} &E_x\left(e^{-\sum_{s \le t} F_n(X_{s-}, X_s)}\right) \\ = \; &E_x\left(e^{-\sum_{s \le t} F_n(X_{s-}, X_s) - (1/2)(B_t^{(n)})^p} e^{(1/2)(B_t^{(n)})^p}\right) \\ \le \; &\left(E_x\left(e^{-2\sum_{s \le t} F_n(X_{s-}, X_s) - (B_t^{(n)})^p}\right)\right)^{1/2} \left(E_x\left(e^{(B_t^{(n)})^p}\right)\right)^{1/2} \\ \le \; &E_x\left(\exp\left(-\int_0^t \int_X N(X_s, dy)\left(1 - e^{-2F_n(X_s, y)}\right) ds\right)\right)^{1/2}. \end{aligned}$$

On account of positivity of F, if n tends to ∞, we have

$$E_x\left(e^{-\sum_{s\le t}F(X_{s-},X_s)}\right)\le E_x\left(e^{-\int_0^t V(X_s)\,ds}\right).$$

By assumption for $V(x)$ and (5.5),

$$R_1^F1(x)=\int_0^\infty e^{-\alpha t}E_x\left(e^{-\sum_{s\le t}F(X_{s-},X_s)}\right)dt\longrightarrow 0$$

which implies that the full large deviation principle is derived by Remark 4.1. □

EXAMPLE 5.4. The typical model of Markov chain which satisfies the hypotheses in Theorem 5.2 is a population model with birth rate $\lambda_n = n\lambda$, $(\lambda_0 = \lambda)$ and death rate $\mu_n = n\mu$, where λ, μ are certain rates which satisfy $\lambda > \mu$. Indeed, since $\liminf_{n\to\infty} m_n > 0, m_n = n^{-1}(\lambda/\mu)^n$ and $E_n(\exp(-\sigma_K)) \in L^2(I; m)$, we get

$$\sum_{n=1}^{\infty} m_n E_n(\exp(-\sigma_K))^2 < \infty$$

and which implies (5.5). Moreover, since $N(i,j)$ of (5.4) is equals to $n\lambda$ in this case, the same problem in Example 5.3 can be also considered on an infinte state space with the finite life time. □

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