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SIMPLEX MOVES ON ELEMENTARY SURFACES

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1. Introduction

In this paper, a surface in $R^4 = \{(x_1, x_2, x_3, t) | x_1, x_2, x_3, t \in R\}$ means a closed (oriented or not and connected or not) PL 2-manifold embedded in R^4 locally flatly. For two surfaces F and F' in R^4, the following conditions are mutually equivalent (cf. [3]).

(1) F is ambient isotopic to F'.
(2) F is related with F' by a sequence of simplex moves on surfaces in R^4.

On the other hand, it is usual to describe a surface in R^4 by use of a motion picture method [1]; taking the t-coordinate as a height function, we consider a surface to be a one-parameter family of subsets in R^3 that are the intersections of the surface and the parallel hyperplanes. A surface in R^4 is said to be elementary if all of its critical points are elementary (that is, minimal points, maximal points, and saddle points).

Let $\varphi_\theta : R^4 \longrightarrow R^4$ be a rotation about the x_1x_2-plane by an angle θ. If p is an elementary (resp. non-elementary) critical point of a surface F, then $\varphi_\theta(p)$ is also an elementary (resp. non-elementary) critical point of $\varphi_\theta(F)$ for a sufficiently small positive angle θ. In particular, if F is elementary, then $\varphi_\theta(F)$ is also elementary.

The purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let F and F' in R^4 be two elementary surfaces. The following conditions are mutually equivalent.*

(1) *F is ambient isotopic to F'.*
(2) *$\varphi_\theta(F)$ is related with $\varphi_\theta(F')$ by a sequence of simplex moves on elementary surfaces in R^4 for a sufficiently small positive angle θ.*

In Section 2, we introduce the notion of a degree of a point of a surface in R^4. We give a sufficient condition to decide which critical points are elementary (Lemma 2.3). Section 3 is devoted to examining how a 3-simplex move changes the degree of a point of a surface (Lemma 3.1). In Section 4, we define a Λ-move, which is a deformation to "pick up" a critical point and change it into some elementary critical points. This deformation was used in [2]. We show that a Λ-move is decomposed into some 3-simplex moves (Lemma 4.2). In Section 5, we prove Theorem 1.1.

Throughout this paper, we work in the piecewise linear category.

2. Critical Points

Let $\pi : R^4 \longrightarrow R^3$ be the projection defined by $\pi(x_1, x_2, x_3, t) = (x_1, x_2, x_3)$. We use the notation $t(p)$ for the t-coordinate of a point p in R^4. We consider the following condition for a compact polyhedron P in R^4:

(2.1) Any two vertices v and v' of P satisfy that $\pi(v) \neq \pi(v')$ and $t(v) \neq t(v')$.

We notice that $\varphi_\theta(P)$ satisfies the condition (2.1) for a sufficiently small positive angle θ. In this section, we assume that a surface F in R^4 satisfies (2.1).

For a subset A of R^3 and a subset B of R, we denote the subset $A \times B \subset R^3 \times R = R^4$ by AB. If B consists of one point t, we use the notation $A[t]$ for $A\{t\}$.

The intersection $F \cap R^3[t]$ is an *ordinary cross-section* if it is the empty set or a closed 1-manifold in $R^3[t]$. The intersection $F \cap R^3[t]$ is an *exceptional cross-section* if it is not an ordinary cross-section.

If $F \cap R^3[t]$ is an exceptional cross-section, then there is a unique point p that has no neighborhood in $F \cap R^3[t]$ homeomorphic to an interval.Such a point p is called a *critical point* of F. We note that a critical point must be a vertex of F, that is, a 0-simplex of any triangulation of F.

In this paper, maximal points, minimal points, and saddle points are called *elementary critical points*, where a saddle point is the singular point illustrated in Figure 2.1. The points of F except critical points are called the *ordinary points*. We say that F is an *elementary surface* if all the critical points of F are elementary.

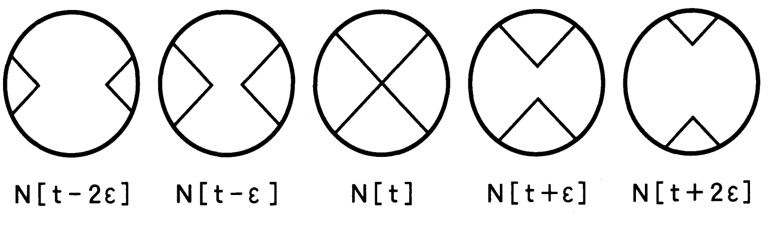

Figure 2.1

For any point p of F, the number of the edges in the 1-dimensional polyhedron $F \cap R^3[t(p)]$ around p is even.

DEFINITION 2.2. The *degree* of p of F is the half number of such edges and denoted by $d(p; F)$.

The degree $d(p; F)$ is 0 (resp. 1) if and only if p is a maximal point or a minimal point (resp. an ordinary point) of F. If $d(p; F) \geq 3$, then p is a non-elementary critical point of F. In the case of $d(p; F) = 2$, p is not necessarily a saddle point of F.

Lemma 2.3. *Let K be a triangulation of F which contains a vertex p. If the number of the edges in K around p is less than or equal to five, then p is an elementary critical point or an ordinary point.*

Proof. Let $|pv_1|, |pv_2|, \cdots, |pv_n|$ be the 1-simplices in K such that the link Lk(p; F) $= |\mathrm{Lk}(p; K)|$ is $|v_1v_2| \cup |v_2v_3| \cup \cdots \cup |v_nv_{n+1}|$ $(v_{n+1} = v_1)$. Since $2d(p; F)$ is equal to the number

$$\sharp\{i | t(v_i) < t(p) < t(v_{i+1}) \text{ or } t(v_i) > t(p) > t(v_{i+1})\},$$

we have $d(p; F) \leq 2$. It suffices to consider the case of $d(p; F) = 2$.

We take a small cylindrical neighborhood $N[a, b]$ of p in R^4, where N is a convex linear 3-ball in R^3 and $a < t(p) < b$. Taking $b - a$ to be a sufficiently small positive number, we may assume that the side $(\partial N)[a, b]$ is disjoint from $|pv_i|$ $(i = 1, \cdots, n)$. Let $T_a(p; F)$ and $T_b(p; F)$ be two tangles $(N[a], F \cap N[a])$ and $(N[b], F \cap N[b])$ respectively. Because of $d(p; F) = 2$, $T_k(p; F)$ is a 2-string tangle $(k = a, b)$. Each string of $T_k(p; F)$ has one or two vertices corresponding to $|pv_i| \cap N[k]$, and in total two strings of $T_k(p; F)$ have two or three vertices in int$N[k]$ $(k = a, b)$. Therefore we see that both $T_a(p; F)$ and $T_b(p; F)$ are trivial tangles.

We identify $\partial T_a(p; F)$ with $\partial T_b(p; F)^*$, where $T_b(p; F)^*$ is the mirror image of $T_b(p; F)$. Since $T_a(p; F)$ and $T_b(p; F)$ are trivial 2-string tangles and the union $T_a(p; F) \cup_\partial T_b(p; F)^*$ is a trivial knot, there exists an isotopy $\{h_s\}$ $(0 \leq s \leq 1)$ of $N[a] = N[b]$ such that $h_1(T_a(p; F))$ and $h_1(T_b(p; F))$ have the forms $N[t - \varepsilon]$ and $N[t + \varepsilon]$ in Figure 2.1, respectively. This isotopy is extended to a level-preserving isotopy of R^4, and hence p is a saddle point of F. This completes the proof.

REMARK 2.4. We have the following equation:

$$\sum_{p \in F} \{d(p; F) - 1\} = -\chi(F),$$

where $\chi(F)$ is the Euler number of F. Since $d(p; F) - 1 = 0$ for any ordinary point p, the sum is finite.

3. Simplex Move

Let P be a p-manifold in a q-manifold with $p < q$ and σ^{p+1} be a $(p+1)$-simplex such that $P \cap \sigma^{p+1} = P \cap \partial\sigma^{p+1}$ is the union of some p-faces of σ^{p+1}. Let P' be the

p-manifold $\mathrm{cl}(P \cup \partial\sigma^{p+1} - P \cap \partial\sigma^{p+1})$. Then we say that P' is obtained from P by the $(p+1)$-*simplex move* associated with σ^{p+1}.

Suppose that F and F' are two surfaces in R^4 which satisfy (2.1)and that F' is obtained from F by a 3-simplex move associated with σ^3.

Lemma 3.1. *For any point p of $F \cap F'$, we have*

$$|d(p; F') - d(p; F)| \leq 1.$$

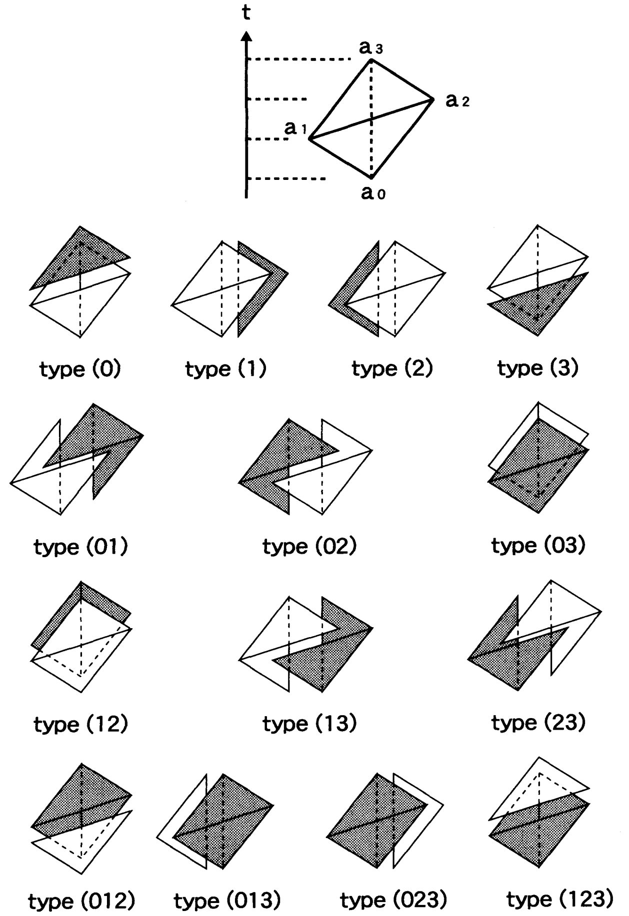

Figure 3.1

Proof. Let a_0, a_1, a_2 and a_3 be the vertices of σ^3 with

$$t(a_0) < t(a_1) < t(a_2) < t(a_3)$$

and τ_i^2 the 2-face of σ^3 such that $a_i * \tau_i^2 = \sigma^3$ $(i = 0, 1, 2, 3)$. We say that the type of the 3-simplex move is (i), (ij), or (ijk) if $F \cap \sigma^3 = \tau_i, \tau_i \cup \tau_j$, or $\tau_i \cup \tau_j \cup \tau_k$ for distinct $i, j, k \in \{0, 1, 2, 3\}$ respectively; see Figure 3.1. In the figure, the black faces (resp. the white faces) indicate $F \cap \sigma^3$ (resp. $F' \cap \sigma^3$).

Suppose that the type of the 3-simplex move is (0); namely, $F \cap \sigma^3$ consists of $\tau_0^2 = |a_1 a_2 a_3|$. If p is any point of $F \cap F'$ except a_1, a_2 and a_3, then it is obvious that $d(p; F') - d(p; F) = 0$. Consider the case $p = a_1$. Since $\mathrm{Lk}(a_1; F')$ is obtained from $\mathrm{Lk}(a_1; F)$ by replacing $|a_2 a_3|$ with $|a_2 a_0| \cup |a_0 a_3|$, the difference $d(a_1; F') - d(a_1; F)$ is $+1$. Similarly, if $p = a_2$ or a_3, we have $d(p; F') - d(p; F) = 0$. Note that a_0 is not in F but is in F' as a minimal point of F'.

The other types are similarly examined as shown in Table 3.1. In the table, the notation $\times$ means that the difference $d(a_i; F') - d(a_i; F)$ has no sense because a_i is not in both of F and F'. This completes the proof.

type	(0)	(1)	(2)	(3)
$d(a_0; F') - d(a_0; F)$	$\times$	0	0	0
$d(a_1; F') - d(a_1; F)$	$+1$	$\times$	0	0
$d(a_2; F') - d(a_2; F)$	0	0	$\times$	$+1$
$d(a_3; F') - d(a_3; F)$	0	0	0	$\times$

type	(01)	(02)	(03)	(12)	(13)	(23)
$d(a_0; F') - d(a_0; F)$	0	0	0	0	0	0
$d(a_1; F') - d(a_1; F)$	$+1$	0	0	0	0	-1
$d(a_2; F') - d(a_2; F)$	-1	0	0	0	0	$+1$
$d(a_3; F') - d(a_3; F)$	0	0	0	0	0	0

type	(012)	(013)	(023)	(123)
$d(a_0; F') - d(a_0; F)$	0	0	0	$\times$
$d(a_1; F') - d(a_1; F)$	0	0	$\times$	-1
$d(a_2; F') - d(a_2; F)$	-1	$\times$	0	0
$d(a_3; F') - d(a_3; F)$	$\times$	0	0	0

Table 3.1

In the case of $d(p; F') - d(p; F) = 0$ in Lemma 3.1, we have the following.

Lemma 3.2. *Let p be a point of $F \cap F'$. If p is an elementary critical point (resp. an ordinary point) of Fand $d(p; F) - d(p; F') = 0$, then p is also an elementary critical point (resp. an ordinary point) of F'.*

Proof. If p is a maximal point or a minimal point, then $d(p;F) = d(p;F') = 0$ and hence p is a maximal point or a minimal point of F'. If p is an ordinary point of F, then $d(p;F) = d(p;F') = 1$ and hence p is an ordinary point of F'.

Suppose that p is a saddle point of F. We use the notations in the proof of Lemma 2.3. Let D_k be $\sigma^3 \cap N[k]$ $(k = a, b)$. If $D_k = \phi$, then $T_k(p;F) = T_k(p;F')$. If $D_k \neq \phi$, then D_k is a 2-disk. In this case, we see that $T_k(p;F)$ and $T_k(p;F')$ are ambient isotopic and that $T_k(p;F')$ is a trivial tangle; see Figure 3.2. Hence p is a saddle point of F'. This completes the proof.

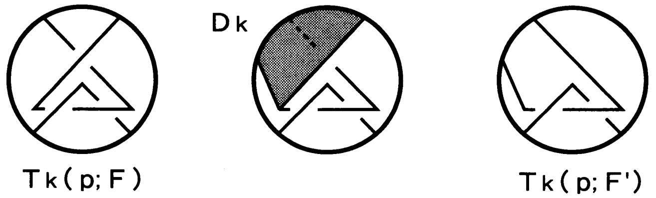

Figure 3.2

Two p-manifolds P and P' in a q-manifold Q with $p < q$ are *related by a sequence of simplex moves on p-manifolds in Q* if there exists a sequence of p-manifolds in Q

$$P = P_1 \longrightarrow P_2 \longrightarrow \cdots \longrightarrow P_n = P'$$

such that P_{i+1} is obtained from P_i by a $(p+1)$-simplex move $(i = 1, 2, \cdots, n-1)$. Two elementary surfaces F and F' in R^4 are *related by a sequence of simplex moves on elementary surfaces in* R^4 if there exists a sequence of elementary surfaces in R^4

$$F = F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_n = F'$$

such that F_{i+1} is obtained from F_i by a 3-simplex move $(i = 1, 2, \cdots, n-1)$. Kamada, Kawauchi and Matumoto proved the following theorem in [3].

Theorem 3.3. *Let P and P' be two p-manifolds in a q-manifold Q with $p < q$. The following conditions are mutually equivalent.*

(1) *P is ambient isotopic to P'.*
(2) *P is related with P' by a sequence of simplex moves on p-manifolds in Q.*

If two elementary surfaces F and F' in R^4 are ambient isotopic, then there exists a sequence of 3-simplex moves on surfaces in R^4

$$F = F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_n = F'$$

by Theorem 3.3. However F_i does not necessarily satisfy (2.1) ($i = 2, \cdots, n-1$). Taking a sufficiently small positive angle θ, we obtain a sequence of 3-simplex moves

$$\varphi_\theta(F) = \varphi_\theta(F_1) \longrightarrow \varphi_\theta(F_2) \longrightarrow \cdots \longrightarrow \varphi_\theta(F_n) = \varphi_\theta(F').$$

such that $\varphi_\theta(F_i)$ satisfies (2.1); nevertheless $\varphi_\theta(F_i)$ is not necessarily an elementary surface. Our theorem (Theorem 1.1) asserts that we can replace the intermediate surfaces of the above sequence with another ones which are all elementary.

4. Λ -move

For a point p of a surface F which satisfies (2.1), we take a sufficiently small cylindrical neighborhood $N[a,b]$ of p in R^4 such that the bottom $N[a]$ and the top $N[b]$ are disjoint from F, where N is a convex linear 3-ball in R^3 (this is different from the one defined in the proof of Lemma 2.3). We remove the 2-ball $F \cap N[a,b]$ and replace it by a cone $\widehat{p} * \{F \cap (\partial N)[a,b]\}$ so that we obtain a new surface F', where $\widehat{p}$ is in int$N[b]$. We say that F' is obtained from F by a *Λ-move at p*, and denote F' by F_p.

In comparison between the vertices of F and F_p, p is not in F_p and $v_1, \cdots, v_n$ and $\widehat{p}$ are in F_p, where v_i ($i = 1, \cdots, n$) are the vertices of the polygonal curve $F \cap (\partial N)[a,b]$. Taking an appropriate 3-ball N, we make F_p satisfy (2.1). Throughout this paper we may assume that, if F satisfies (2.1), then F_p also satisfies (2.1).

We see that $\widehat{p}$ is a maximal point of F_p and that v_i is an elementary critical point or an ordinary point of F_p by Lemma 2.3. Hence we have the following (cf. [2]).

Lemma 4.1. *If all the critical points of F except p are elementary, then F_p is an elementary surface. In particular, if F is elementary, then F_p is also elementary.*

Lemma 4.2. *If F is elementary, then F and F_p are related by a sequence of simplex moves on elementary surfaces.*

Proof. Let $\ell(p; F)$ be a polygonal curve $F \cap (\partial N)[a,b]$ in $(\partial N)[a,b]$. By Theorem 3.3, if p is a maximal point, an ordinary point, or a saddle point, then there exists a sequence of 2-simplex moves on polygonal curves in int$(\partial N)[a,b]$

$$\ell(p; F) = \ell_1 \longrightarrow \ell_2 \longrightarrow \cdots \longrightarrow \ell_n = \partial\tau^2$$

such that

(1) τ^2 is a 2-simplex in int$(\partial N)[a, t(p)]$,
(2) ℓ_{i+1} is obtained from ℓ_i by a 2-simplex move associated with τ_i^2 ($i = 1, 2, \cdots, n-1$),

(3) $\{p\} \cup \ell_i$ satisfies (2.1) ($i = 1, 2, \cdots, n$), and
(4) $\sharp\{\ell_1 \cap (\partial N)[t(p)]\} \geq \sharp\{\ell_2 \cap (\partial N)[t(p)]\} \geq \cdots \geq \sharp\{\ell_n \cap (\partial N)[t(p)]\} = 0$.

Note that $\sharp\{\ell_i \cap (\partial N)[t(p)]\}$ is equal to $2d(p; F_i)$ and hence $\sharp\{\ell_1 \cap (\partial N)[t(p)]\}$ is equal to 0, 2, or 4 . If p is a minimal point, we replace "int$(\partial N)[a, t(p)]$" in (1) by "int$(\partial N)[t(p), b]$".Then we have a sequence of surfaces in R^4

$$\begin{aligned} F = F_1 \longrightarrow F_2 \longrightarrow \quad \cdots \quad &\longrightarrow F_n \\ \longrightarrow (F_n)_p \longrightarrow \quad \cdots \quad &\longrightarrow (F_2)_p \longrightarrow (F_1)_p = F_p \end{aligned}$$

such that

(5) F_{i+1} is obtained from F_i by a 3-simplex move associated with $p * \rho_i^2$, where ρ_i^2 is a 2-simplex in R^4 ($i = 1, 2, \cdots, n-1$),
(6) F_i satisfies (2.1) ($i = 2, \cdots, n$), and
(7) $(p * \rho_i^2) \cap (\partial N)[a, b] = \tau_i^2$ ($i = 1, 2, \cdots, n-1$).

Using this sequence, we prove that F and F_n, F_n and $(F_n)_p$, $(F_n)_p$ and F_p are related by a sequence of simplex moves on elementary surfaces, respectively.

First, p is an elementary critical point or an ordinary point of F_i by (4) and Lemma 3.2. Moreover, the new vertices of F_i generated by the 3-simplex move associated with $p * \rho_{i-1}^2$ are elementary critical points or ordinary points of F_i by Lemma 2.3. Hence F_i is an elementary surface. It follows that F and F_n are related by a sequence of simplex moves on elementary surfaces.

Second, let F_n' be a surface obtained from F_n by the 3-simplex move associated with $p * \tau^2$. Then $(F_n)_p$ is obtained from F_n' by the 3-simplex move associated with $\widehat{p} * \tau^2$. We see that F_n' and $(F_n)_p$ are elementary surfaces by Lemma 2.3, and hence F_n and $(F_n)_p$ are related by a sequence of simplex moves on elementary surfaces.

Finally, we notice that $(F_i)_p$ is an elementary surface by Lemma 4.1. We remove the 3-simplex $p * \tau_i^2$ from $p * \rho_i^2$ and replace it by the 3-simplex $\widehat{p} * \tau_i^2$ so that we obtain the 3-ball B_i^3 ($i = 1, 2, \cdots, n-1$). Then two elementary surfaces $(F_{i+1})_p$ and $(F_i)_p$ differ by B_i^3.

By assuming Lemma 4.3 which is stated below, we see that $(F_{i+1})_p$ and $(F_i)_p$ are related by a sequence of simplex moves on elementary surfaces. It follows that $(F_n)_p$ and F_p are related by a sequence of simplex moves on elementary surfaces, and we have the conclusion.

Let $a_0 * \rho^2 = |a_0 a_1 a_2 a_3|$ be a 3-simplex in R^4 which satisfies (2.1). We take a 2-simplex $\tau^2 = |b_1 b_2 b_3|$ in $a_o * \rho^2$ which satisfies (2.1), where b_i is an interior point of $|a_0 a_i|$ and close to a_0 ($i = 1, 2, 3$). Let b_0 be a point in R^4 such that b_0 is joinable with τ^2, $\mathrm{cl}(a_0 * \rho^2 - a_0 * \tau^2) \cap (b_0 * \tau^2) = \tau^2$, and $t(b_0) > t(b_i)$ ($i = 1, 2, 3$). Let F and F_B be two elementary surfaces such that F_B is obtained from F by a 3-cellular move associated with a 3-ball $B^3 = (a_0 * \rho - a_0 * \tau^2) \cup (b_0 * \tau^2)$. Suppose that $F \cap B^3$ is a 2-ball which is $T_1, T_2, T_3, T_{12}, T_{13}$ or T_{23}, where

$$\begin{aligned}
T_1 &= (|a_0a_2a_3| - |a_0b_2b_3|) \cup |b_0b_2b_3| \cup |a_1a_2a_3|,\\
T_2 &= (|a_0a_1a_3| - |a_0b_1b_3|) \cup |b_0b_1b_3| \cup |a_1a_2a_3|,\\
T_3 &= (|a_0a_1a_2| - |a_0b_1b_2|) \cup |b_0b_1b_2| \cup |a_1a_2a_3|,\\
T_{12} &= (|a_0a_2a_3| - |a_0b_2b_3|) \cup |b_0b_2b_3|\\
&\cup (|a_0a_1a_3| - |a_0b_1b_3|) \cup |b_0b_1b_3| \cup |a_1a_2a_3|,\\
T_{13} &= (|a_0a_2a_3| - |a_0b_2b_3|) \cup |b_0b_2b_3|\\
&\cup (|a_0a_1a_2| - |a_0b_1b_2|) \cup |b_0b_1b_2| \cup |a_1a_2a_3|, \text{ and}\\
T_{23} &= (|a_0a_1a_3| - |a_0b_1b_3|) \cup |b_0b_1b_3|\\
&\cup (|a_0a_1a_2| - |a_0b_1b_2|) \cup |b_0b_1b_2| \cup |a_1a_2a_3|.
\end{aligned}$$

Lemma 4.3. *In the above situation, F and F_B are related by a sequence of simplex moves on elementary surfaces.*

Proof. We may assume that $t(b_1) < t(b_2) < t(b_3)$. According to the levels of a_1 and b_1, a_2 and b_2, a_3 and b_3, we have four cases;

(i-1) $t(a_1) > t(b_1), t(a_2) > t(b_2), t(a_3) > t(b_3)$,
(i-2) $t(a_1) < t(b_1), t(a_2) > t(b_2), t(a_3) > t(b_3)$,
(i-3) $t(a_1) < t(b_1), t(a_2) < t(b_2), t(a_3) > t(b_3)$, and
(i-4) $t(a_1) < t(b_1), t(a_2) < t(b_2), t(a_3) < t(b_3)$.

According to the levels of a_1, a_2 and a_3, we have six cases;

(ii-1) $t(a_1) < t(a_2) < t(a_3)$,
(ii-2) $t(a_1) < t(a_3) < t(a_2)$,
(ii-3) $t(a_2) < t(a_1) < t(a_3)$,
(ii-4) $t(a_2) < t(a_3) < t(a_1)$,
(ii-5) $t(a_3) < t(a_1) < t(a_2)$, and
(ii-6) $t(a_3) < t(a_2) < t(a_1)$.

If the levels of the vertices of B^3 are of type (i-α) and (ii-β), then say that B^3 is of type (α, β), where $\alpha \in \{1,2,3,4\}$ and $\beta \in \{1,2,3,4,5,6\}$. We notice that there exist no 3-balls B^3 of types (2,3), (2,4), (2,5), (2,6), (3,2), (3,4), (3,5), and (3,6). For each type (α, β), there are six cases according to $F \cap B^3 = T_1$, T_2, T_3, T_{12}, T_{13} and T_{23}.

Case 1. Suppose that B^3 is of type (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1) or (2,2).

First, we consider the case that B^3 is of type $(1,1)$ and $F \cap B^3$ is T_1. As the division of B^3, we take four 3-simplices $\Delta_1^3, \Delta_2^3, \Delta_3^3, \Delta_4^3$, where

$$\Delta_1^3 = |a_1a_2a_3b_1|, \Delta_2^3 = |a_2a_3b_1b_2|, \Delta_3^3 = |a_3b_1b_2b_3|, \text{ and } \Delta_4^3 = |b_0b_1b_2b_3|.$$

Then F and F_B are related by a sequence of simplex moves on surfaces which satisfy the condition (2.1);

$$F = F_1 \xrightarrow{\Delta_1} F_2 \xrightarrow{\Delta_2} F_3 \xrightarrow{\Delta_3} F_4 \xrightarrow{\Delta_4} F_5 = F_B,$$

see Figure 4.1. Then the difference of the degrees of the vertices of B^3 is given in Table 4.1.

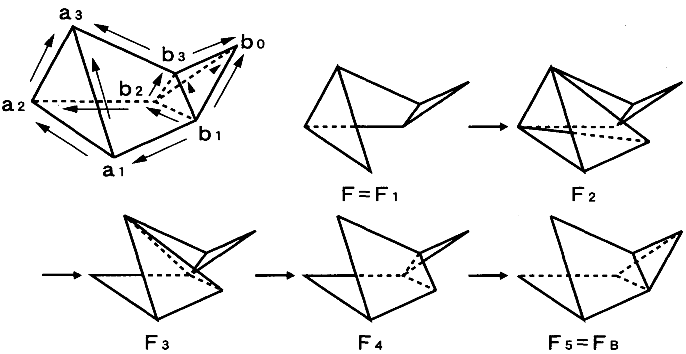

Figure 4.1

vertex	a_1	a_2	a_3	b_0	b_1	b_2	b_3
$d(*;F_2)-d(*;F_1)$	$+1$	0	0	0	$\times$	0	0
$d(*;F_3)-d(*;F_2)$	0	-1	0	0	0	$+1$	0
$d(*;F_4)-d(*;F_3)$	0	0	0	0	0	0	0
$d(*;F_5)-d(*;F_4)$	0	0	0	0	0	0	0

Table 4.1

Since a_1 is an elementary critical point or an ordinary point of F_5, we have $d(a_1;F_1) \leq 1$. If the vertex a_1 is a maximal point or a minimal point of F_1, then a_1 is an ordinary point of F_2, F_3, F_4 and F_5. If a_1 is an ordinary point of F_1, then a_1 is a saddle point of F_2, F_3, F_4 and F_5 by Lemma 3.2.

Similarly, the vertices a_2, a_3, b_0, b_1, b_2 and b_3 are elementary critical points or ordinary points of F_2, F_3, and F_4 (in particular, b_1 is a minimal point). Hence the surfaces F_2, F_3 and F_4 are elementary surfaces, and F and F_B are related by simplex moves on elementary surfaces.

type	(1,1)						(1,2)					
	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}
order	P_1	P_2	P_3	P_1	P_3	P_3	P_1	P_2	P_3	P_1	P_3	P_3

type	(1,3)						(1,4)					
	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}
order	P_4	P_2	P_3	P_5	P_3	P_3	P_4	P_2	P_3	P_5	P_3	P_3

type	(1,5)						(1,6)					
	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}
order	P_4	P_2	P_3	P_1	P_6	P_3	P_4	P_2	P_3	P_1	P_6	P_3

type	(2,1)						(2,2)					
	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}
order	P_1	P_2	P_3	P_1	P_3	P_3	P_1	P_2	P_3	P_1	P_3	P_3

Table 4.2

In Case 1 generally, we use one of the following six kinds of order of simplex moves;

P_1. $F = F_1 \xrightarrow{\Delta_1} F_2 \xrightarrow{\Delta_2} F_3 \xrightarrow{\Delta_3} F_4 \xrightarrow{\Delta_4} F_5 = F_B$,
P_2. $F = F_1 \xrightarrow{\Delta_4} F_2 \xrightarrow{\Delta_1} F_3 \xrightarrow{\Delta_3} F_4 \xrightarrow{\Delta_2} F_5 = F_B$,
P_3. $F = F_1 \xrightarrow{\Delta_4} F_2 \xrightarrow{\Delta_1} F_3 \xrightarrow{\Delta_2} F_4 \xrightarrow{\Delta_3} F_5 = F_B$,
P_4. $F = F_1 \xrightarrow{\Delta_2} F_2 \xrightarrow{\Delta_1} F_3 \xrightarrow{\Delta_3} F_4 \xrightarrow{\Delta_4} F_5 = F_B$,
P_5. $F = F_1 \xrightarrow{\Delta_3} F_2 \xrightarrow{\Delta_2} F_3 \xrightarrow{\Delta_1} F_4 \xrightarrow{\Delta_4} F_5 = F_B$, and
P_6. $F = F_1 \xrightarrow{\Delta_4} F_2 \xrightarrow{\Delta_2} F_3 \xrightarrow{\Delta_1} F_4 \xrightarrow{\Delta_3} F_5 = F_B$.

For each type in Case 1, we give an example of order such that F and F_B are related by a sequence of simplex moves on elementary surfaces; see Table 4.2.

Case 2. Suppose that B^3 are of type (3,1), (3,3), (4,1), (4,2), (4,3), (4,4), (4,5) or (4,6).

As the division of B^3, we take four 3-simplices $\Delta_4^3, \Delta_5^3, \Delta_6^3, \Delta_7^3$, where

$$\Delta_5^3 = |a_1a_2a_3b_3|, \Delta_6^3 = |a_1a_2b_2b_3|, \text{ and } \Delta_7^3 = |a_1b_1b_2b_3|.$$

We use one of the following four kinds of order of simplex moves;

Q_1. $F = F_1 \xrightarrow{\Delta_5} F_2 \xrightarrow{\Delta_6} F_3 \xrightarrow{\Delta_7} F_4 \xrightarrow{\Delta_4} F_5 = F_B$,

Q_2. $F = F_1 \xrightarrow{\Delta_4} F_2 \xrightarrow{\Delta_5} F_3 \xrightarrow{\Delta_7} F_4 \xrightarrow{\Delta_6} F_5 = F_B$,

Q_3. $F = F_1 \xrightarrow{\Delta_4} F_2 \xrightarrow{\Delta_7} F_3 \xrightarrow{\Delta_6} F_4 \xrightarrow{\Delta_5} F_5 = F_B$, and

Q_4. $F = F_1 \xrightarrow{\Delta_4} F_2 \xrightarrow{\Delta_5} F_3 \xrightarrow{\Delta_6} F_4 \xrightarrow{\Delta_7} F_5 = F_B$.

For each type in Case 2, we give an example of order such that F and F_B are related by a sequence of simplex moves for elementary surfaces; see Table 4.3.

type	(3,1)						(3,3)					
	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}
order	Q_1	Q_2	Q_3	Q_1	Q_4	Q_2	Q_1	Q_2	Q_3	Q_1	Q_4	Q_2

type	(4,1)						(4,2)					
	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}
order	Q_1	Q_2	Q_3	Q_1	Q_4	Q_2	Q_1	Q_2	Q_3	Q_1	Q_4	Q_3

type	(4,3)						(4,4)					
	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}
order	Q_1	Q_2	Q_3	Q_1	Q_4	Q_2	Q_1	Q_2	Q_3	Q_1	Q_3	Q_2

type	(4,5)						(4,6)					
	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}	T_1	T_2	T_3	T_{12}	T_{13}	T_{23}
order	Q_1	Q_2	Q_3	Q_1	Q_4	Q_3	Q_1	Q_2	Q_3	Q_1	Q_3	Q_2

Table 4.3

This completes the proof of Lemma 4.3.

For a surface F which satisfies (2.1), we denote the surface obtained by the Λ-moves at all the points of F with their degrees ≥ 2 by $\widehat{F}$. Then $\widehat{F}$ is elementary (cf. Lemma 4.1). By Lemma 4.2, we have the following.

Corollary 4.4. *For any elementary surface F in R^4, F and $\widehat{F}$ are related by a sequence of simplex moves on elementary surfaces.*

5. Proof of Theorem 1.1

To prove Theorem 1.1, we prepare three more lemmas.

Let σ^3 be a 3-simplex $|a_0a_1a_2a_3|$ in R^4 which satisfies (2.1). We take a 2-simplex $\rho_0^2 = |a_{01}a_{02}a_{03}|$ which satisfies (2.1), where a_{0i} $(i = 1, 2, 3)$ is an interior point of

$|a_0a_i|$ and close to a_0 and the 3-simplex $a_0 * \rho_0^2$ is similar to σ^3. Similarly we take 2-simplices ρ_1^2, ρ_2^2 and ρ_3^2 near a_1, a_2 and a_3 respectively.

Let F be an elementary surface and F' a surface in R^4 obtained from F by a 3-simplex move associated with σ^3. For the set U of vertices of σ^3 which are in $F \cap F'$, we take a 3-ball $C^3 = \mathrm{cl}(\sigma^3 - \bigcup_{a_i \in U} a_i * \rho_i^2)$. Let F_C be a surface obtained from F by the 3-cellular move associated with C^3. We notice that F_C satisfies (2.1). Then we have the following.

Lemma 5.1. (1) *F_C is an elementary surface.*
(2) *F and F_C are related by a sequence of simplex moves on elementary surfaces.*

Proof. Let τ_i^2 be a 2-face of σ^3 with $a_i * \tau_i^2 = \sigma^3$ $(i = 0, 1, 2, 3)$.

(1) If $F \cap \sigma^3 = \tau_0^2 = |a_1a_2a_3|$, then the new vertices $a_0, a_{10}, a_{12}, a_{13}, a_{20}, a_{21}, a_{23}, a_{30}, a_{31}$ and a_{32} are generated in F_C by the 3-cellular move. The edges in F_C around a_{10} are $|a_{10}a_0|, |a_{10}a_{12}|$, and $|a_{10}a_{13}|$. Then a_0 is an elementary critical point or an ordinary point of F_C by Lemma 2.3. We see that the rest of the vertices of F_C are also elementary critical points or ordinary points, and hence F_C is elementary. The other types are similarly examined.
(2) We may assume that $t(a_0) < t(a_1) < t(a_2) < t(a_3)$. We divide the proof into 14 cases according to $F \cap \sigma^3$.

Type (0). $F \cap \sigma^3$ consists of $\tau_0^2 = |a_1a_2a_3|$.

As the division of C^3, we take seven 3-simplices:

$$|a_0a_{12}a_{23}a_{31}|,\ |a_0a_{10}a_{12}a_{31}|,\ |a_{10}a_{12}a_{13}a_{31}|,\ |a_{12}a_{20}a_{21}a_{23}|,$$
$$|a_0a_{12}a_{20}a_{23}|,\ |a_{23}a_{30}a_{31}a_{32}|,\ |a_0a_{23}a_{30}a_{31}|.$$

We apply 3-simplex moves associated with these 3-simplices in this order to obtain a sequence of simplex moves on surfaces which satisfy (2.1)

$$F = F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_8 = F_C,$$

see Figure 5.1. We notice that the levels of the vertices of C^3 are

$$t(a_0) < t(a_{10}) < t(a_{12}) < t(a_{13}) < t(a_{20})$$
$$< t(a_{21}) < t(a_{23}) < t(a_{30}) < t(a_{31}) < t(a_{32}).$$

Then the difference of the degrees of these vertices is shown in Table 5.1.

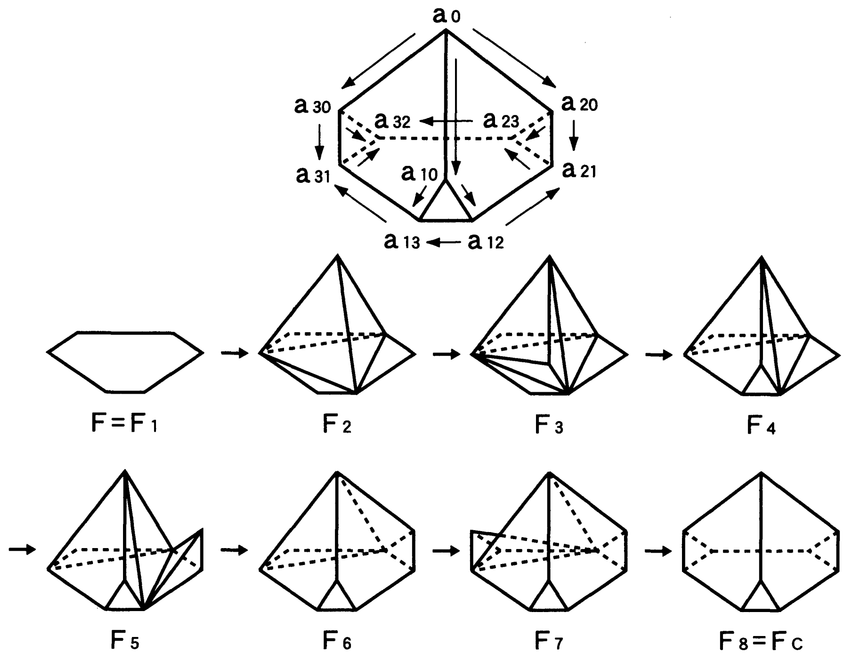

Figure 5.1

vertex	a_0	a_{10}	a_{12}	a_{13}	a_{20}	a_{21}	a_{23}	a_{30}	a_{31}	a_{32}
$d(*;F_2)-d(*;F_1)$	$\times$	$\times$	$+1$	0	$\times$	0	0	$\times$	0	0
$d(*;F_3)-d(*;F_2)$	0	$\times$	0	0	$\times$	0	0	$\times$	0	0
$d(*;F_4)-d(*;F_3)$	0	0	0	0	$\times$	0	0	$\times$	0	0
$d(*;F_5)-d(*;F_4)$	0	0	0	0	$\times$	0	0	$\times$	0	0
$d(*;F_6)-d(*;F_5)$	0	0	0	0	0	0	0	$\times$	0	0
$d(*;F_7)-d(*;F_6)$	0	0	0	0	0	0	0	$\times$	0	0
$d(*;F_8)-d(*;F_7)$	0	0	0	0	0	0	0	0	0	0

Table 5.1

We see that $F_2, \cdots, F_6$ and F_7 are elementary surfaces and that F and F_C are related by a sequence of simplex moves on elementary surfaces.

The other 13 cases are similarly examined. The following is an example of a division of C^3 and an order of simplex moves for each case so that F and F_C are related by a sequence of simplex moves on elementary surfaces.

Type (1); $F \cap \sigma^3 = \tau_1^2$.

$$|a_{03}a_1a_{20}a_{32}|,\ |a_{01}a_{02}a_{03}a_{20}|,\ |a_{01}a_{03}a_1a_{20}|,\ |a_{03}a_{30}a_{31}a_{32}|,$$
$$|a_{03}a_1a_{31}a_{32}|,\ |a_{20}a_{21}a_{23}a_{32}|,\ |a_1a_{20}a_{21}a_{32}|.$$

Type (2); $F \cap \sigma^3 = \tau_2^2$.

$$|a_{03}a_{10}a_2a_{31}|,\ |a_{01}a_{02}a_{03}a_{10}|,\ |a_{02}a_{03}a_{10}a_2|,\ |a_{03}a_{30}a_{31}a_{32}|,$$
$$|a_{03}a_2a_{31}a_{32}|,\ |a_{10}a_{12}a_{13}a_{31}|,\ |a_{10}a_{12}a_2a_{31}|.$$

Type (3); $F \cap \sigma^3 = \tau_3^2$.

$$|a_{02}a_{10}a_{21}a_3|,\ |a_{01}a_{02}a_{03}a_{10}|,\ |a_{02}a_{03}a_{10}a_3|,\ |a_{02}a_{21}a_{23}a_3|,$$
$$|a_{02}a_{20}a_{21}a_{23}|,\ |a_{10}a_{12}a_{13}a_{21}|,\ |a_{10}a_{13}a_{21}a_3|.$$

Type (01); $F \cap \sigma^3 = \tau_0^2 \cup \tau_1^2$.

$$|a_{02}a_{12}a_{23}a_{32}|,\ |a_{02}a_{12}a_{21}a_{23}|,\ |a_{02}a_{20}a_{21}a_{23}|,\ |a_{02}a_{10}a_{12}a_{32}|,$$
$$|a_{10}a_{12}a_{13}a_{32}|,\ |a_{10}a_{13}a_{31}a_{32}|,\ |a_{02}a_{10}a_{30}a_{32}|,$$
$$|a_{10}a_{30}a_{31}a_{32}|,\ |a_{01}a_{02}a_{10}a_{30}|,\ |a_{01}a_{02}a_{03}a_{30}|.$$

Type (02); $F \cap \sigma^3 = \tau_0^2 \cup \tau_2^2$.

$$|a_{03}a_{13}a_{23}a_{31}|,\ |a_{03}a_{23}a_{31}a_{32}|,\ |a_{03}a_{30}a_{31}a_{32}|,\ |a_{03}a_{13}a_{20}a_{23}|,$$
$$|a_{13}a_{20}a_{21}a_{23}|,\ |a_{12}a_{13}a_{20}a_{21}|,\ |a_{03}a_{10}a_{13}a_{20}|,$$
$$|a_{10}a_{12}a_{13}a_{20}|,\ |a_{01}a_{02}a_{03}a_{10}|,\ |a_{02}a_{03}a_{10}a_{20}|.$$

Type (03); $F \cap \sigma^3 = \tau_0^2 \cup \tau_3^2$.

$$|a_{02}a_{12}a_{21}a_{32}|,\ |a_{02}a_{21}a_{23}a_{32}|,\ |a_{02}a_{20}a_{21}a_{23}|,\ |a_{02}a_{12}a_{30}a_{32}|,$$
$$|a_{12}a_{30}a_{31}a_{32}|,\ |a_{12}a_{13}a_{30}a_{31}|,\ |a_{02}a_{10}a_{12}a_{30}|,$$
$$|a_{10}a_{12}a_{13}a_{30}|,\ |a_{02}a_{03}a_{10}a_{30}|,\ |a_{01}a_{02}a_{03}a_{10}|.$$

Type (12); $F \cap \sigma^3 = \tau_1^2 \cup \tau_2^2$.

$$|a_{03}a_{13}a_{23}a_{30}|,\ |a_{13}a_{23}a_{30}a_{32}|,\ |a_{13}a_{30}a_{31}a_{32}|,\ |a_{03}a_{13}a_{21}a_{23}|,$$
$$|a_{03}a_{20}a_{21}a_{23}|,\ |a_{02}a_{03}a_{20}a_{21}|,\ |a_{01}a_{03}a_{13}a_{21}|,$$
$$|a_{01}a_{02}a_{03}a_{21}|,\ |a_{01}a_{12}a_{13}a_{21}|,\ |a_{01}a_{10}a_{12}a_{13}|.$$

Type (13); $F \cap \sigma^3 = \tau_1^2 \cup \tau_3^2$.

$$|a_{02}a_{12}a_{20}a_{32}|,\ |a_{12}a_{20}a_{23}a_{32}|,\ |a_{12}a_{20}a_{21}a_{23}|,\ |a_{02}a_{12}a_{31}a_{32}|,$$
$$|a_{02}a_{30}a_{31}a_{32}|,\ |a_{02}a_{03}a_{30}a_{31}|,\ |a_{01}a_{02}a_{12}a_{31}|,$$
$$|a_{01}a_{02}a_{03}a_{31}|,\ |a_{01}a_{12}a_{13}a_{31}|,\ |a_{01}a_{10}a_{12}a_{13}|.$$

Type (23); $F \cap \sigma^3 = \tau_2^2 \cup \tau_3^2$.

$$|a_{01}a_{10}a_{21}a_{31}|,\ |a_{10}a_{13}a_{21}a_{31}|,\ |a_{10}a_{12}a_{13}a_{21}|,\ |a_{01}a_{21}a_{31}a_{32}|,$$
$$|a_{01}a_{30}a_{31}a_{32}|,\ |a_{01}a_{03}a_{30}a_{32}|,\ |a_{01}a_{02}a_{21}a_{32}|,$$
$$|a_{01}a_{02}a_{03}a_{32}|,\ |a_{02}a_{21}a_{23}a_{32}|,\ |a_{02}a_{20}a_{21}a_{23}|.$$

Type (012); $F \cap \sigma^3 = \tau_0^2 \cup \tau_1^2 \cup \tau_2^2$.

$$|a_{10}a_{13}a_{21}a_{3}|,\ |a_{10}a_{12}a_{13}a_{21}|,\ |a_{02}a_{03}a_{10}a_{3}|,\ |a_{01}a_{02}a_{03}a_{10}|,$$
$$|a_{02}a_{10}a_{21}a_{3}|,\ |a_{02}a_{21}a_{23}a_{3}|,\ |a_{02}a_{20}a_{21}a_{23}|.$$

Type (013); $F \cap \sigma^3 = \tau_0^2 \cup \tau_1^2 \cup \tau_3^2$.

$$|a_{10}a_{12}a_{2}a_{31}|,\ |a_{10}a_{12}a_{13}a_{31}|,\ |a_{03}a_{2}a_{31}a_{32}|,\ |a_{03}a_{30}a_{31}a_{32}|,$$
$$|a_{02}a_{03}a_{10}a_{2}|,\ |a_{01}a_{02}a_{03}a_{10}|,\ |a_{03}a_{10}a_{2}a_{31}|.$$

Type (023); $F \cap \sigma^3 = \tau_0^2 \cup \tau_2^2 \cup \tau_3^2$.

$$|a_{1}a_{20}a_{21}a_{32}|,\ |a_{20}a_{21}a_{23}a_{32}|,\ |a_{03}a_{1}a_{31}a_{32}|,\ |a_{03}a_{30}a_{31}a_{32}|,$$
$$|a_{01}a_{03}a_{1}a_{20}|,\ |a_{01}a_{02}a_{03}a_{20}|,\ |a_{03}a_{1}a_{20}a_{32}|.$$

Type (123); $F \cap \sigma^3 = \tau_1^2 \cup \tau_2^2 \cup \tau_3^2$.

$$|a_{0}a_{23}a_{30}a_{31}|,\ |a_{23}a_{30}a_{31}a_{32}|,\ |a_{0}a_{12}a_{20}a_{23}|,\ |a_{12}a_{20}a_{21}a_{23}|,$$
$$|a_{0}a_{12}a_{23}a_{31}|,\ |a_{0}a_{10}a_{12}a_{31}|,\ |a_{10}a_{12}a_{13}a_{31}|.$$

This completes the proof.

Let F and F' be two surfaces in R^4 such that they satisfy (2.1) and that F' is obtained from F by a 3-simplex move associated with $p * \rho^2$, where p is a vertex of $F \cap F'$ and ρ^2 is a 2-simplex in F. Suppose that all the critical points of F and F' except p are elementary. Let F_p (resp. F'_p) be a surface obtained from F (resp. F') by the Λ-move at p.

For the cylindrical neighborhood $N[a,b]$ of p in R^4 and the point $\widehat{p} \in \mathrm{int}N[b]$ associated with the Λ-move at p, we take a 2-ball $D^2 = (p * \rho^2) \cap (\partial N)[a,b]$ and a 3-ball $B^3 = (p * \rho^2 - p * D^2) \cup (\widehat{p} * D^2)$. By Lemma 4.1, F_p and F'_p are elementary surfaces and differ by B^3.

Lemma 5.2. *F_p and F'_p are related by a sequence of simplex moves on elementary surfaces.*

Proof. Let ℓ_p (resp. ℓ'_p) be a polygonal curve $F \cap (\partial N)[a,b]$ (resp. $F' \cap (\partial N)[a,b]$) which satisfies (2.1). Then ℓ_p and ℓ'_p differ by D^2. We take a division of D^2 into 2-simplices $\tau_1^2, \tau_2^2, \cdots, \tau_{n-1}^2$ such that the 2-simplex moves associated with $\tau_1^2, \tau_2^2, \cdots,$ τ_{n-1}^2 are applied to ℓ_p in this order to obtain ℓ'_p.

Let $p * \rho_i^2$ be a 3-simplex with $(p * \rho_i^2) \cap (\partial N)[a,b] = \tau_i^2$ and $\rho_i^2 \subset \rho^2$ ($i = 1, 2, \cdots, n-1$). Notice that $p * \rho^2$ is divided into $\{p * \rho_1^2, p * \rho_2^2, \cdots, p * \rho_{n-1}^2\}$. Let B_i^3 be a 3-ball $(p * \rho_i^2 - p * \tau_i^2) \cup (\widehat{p} * \tau_i^2)$ ($i = 1, 2, \cdots, n-1$). We may assume that B_i^3 satisfies (2.1). Then there exists a sequence of cellular moves on surfaces

$$F_p = F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_n = F'_p$$

such that F_{i+1} is obtained from F_i by the 3-cellular move associated with B_i^3 and that F_i satisfies (2.1). By Lemma 4.3, two surfaces F_i and F_{i+1} are related by a sequence of simplex moves on elementary surfaces. This completes the proof.

Suppose that F and F' are surfaces in R^4 which satisfy (2.1) and that F' is obtained from F by a 3-simplex move associated with σ^3.

Lemma 5.3. *$\widehat{F}$ and $\widehat{F'}$ are related by a sequence of simplex moves on elementary surfaces.*

Proof. Let σ^3 be $|a_0 a_1 a_2 a_3|$ with $t(a_0) < t(a_1) < t(a_2) < t(a_3)$. We use the notations in Lemma 5.1. For the 3-ball C^3 obtained by cutting the corners off from σ^3, we have a sequence of surfaces

$$F \longrightarrow F_C \longrightarrow F'.$$

We note that F' is obtained from F_C by the composition of the 3-simplex moves associated with $a_i * \rho_i^2$ ($a_i \in U$); see Figure 5.2.

Let S be the set of vertices of F with their degrees≥ 2 except the vertices of σ^3. We classify the vertices in U into four (possibly empty) sets:

$U_{11} = \{v | d(v; F) \leq 1, d(v; F') \leq 1\},$
$U_{12} = \{v | d(v; F) = 1, d(v; F') = 2\},$

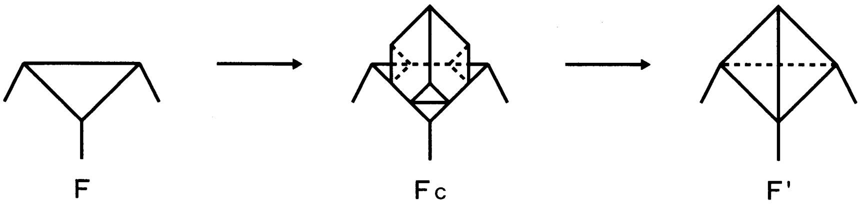

Figure 5.2

$U_{21} = \{v | d(v; F) = 2, d(v; F') = 1\}$, and
$U_{22} = \{v | d(v; F) \geq 2, d(v; F') \geq 2\}$.

Then we obtain a sequence of surfaces between $\widehat{F}$ and $\widehat{F'}$

$$\widehat{F} = F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow F_4 \longrightarrow F_5 = \widehat{F'}$$

such that

(1) $\widehat{F} = F_1$ is obtained from F by the composition of the Λ-moves at the vertices in $S \cup U_{21} \cup U_{22}$,
(2) F_2 is obtained from F_C by the composition of the Λ-moves at the vertices in $S \cup U_{21} \cup U_{22}$,
(3) F_3 is obtained from F_C by the composition of the Λ-moves at the vertices in $S \cup U_{12} \cup U_{21} \cup U_{22}$,
(4) F_4 is obtained from F' by the composition of the Λ-moves at the vertices in $S \cup U_{12} \cup U_{21} \cup U_{22}$, and
(5) $F_5 = \widehat{F'}$ is obtained from F' by the composition of the Λ-moves at the vertices in $S \cup U_{12} \cup U_{22}$; see Figure 5.3.

We notice that F_2, F_3, and F_4 are elementary surfaces by Lemma 4.1. Then we have the following.

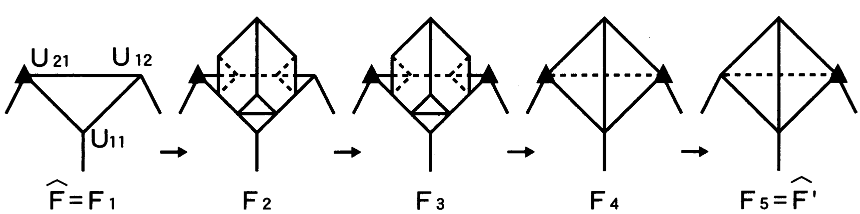

Figure 5.3

(6) Since F_2 is obtained from F_1 by the 3-cellular move associated with C^3, two surfaces F_1 and F_2 are related by a sequence of simplex moves on elementary surfaces by Lemma 5.1(2).
(7) Since F_3 is obtained from F_2 by the composition of the Λ-moves at the ordinary points in U_{12}, two surfaces F_2 and F_3 are related by a sequence of simplex moves on elementary surfaces by Lemma 4.2.
(8) Since F_4 is obtained from F_3 by the composition of the 3-simplex moves associated with $a_i * \rho_i^2$ $(a_i \in U_{11})$ and the 3-cellular moves associated with the 3-balls constructed by picking the vertex a_i of $a_i * \rho_i^2$ $(a_i \in U_{12} \cup U_{21} \cup U_{22})$, two surfaces F_3 and F_4 are related by a sequence of simplex moves on elementary surfaces by Lemma 5.2.
(9) Since F_5 is obtained from F_4 by the composition of the inverse Λ-moves at the ordinary points in U_{21}, two surfaces F_4 and F_5 are related by a sequence of simplex moves on elementary surfaces by Lemma 4.2.

Therefore $\widehat{F}$ and $\widehat{F'}$ are related by a sequence of simplex moves on elementary surfaces and we have the conclusion.

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. It is well-known that $(2) \Rightarrow (1)$ (cf. [4]). We may prove that $(1) \Rightarrow (2)$. Let F and F' be two elementary surfaces in R^4 which are ambient isotopic. By Theorem 3.3, there exists a sequence of simplex moves on surfaces in R^4 between F and F'. Rotating the surfaces and the 3-simplices in this sequence slightly, we obtain a sequence of simplex moves on surfaces in R^4 which satisfy (2.1)

$$\varphi_\theta(F) = F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_n = \varphi_\theta(F').$$

Deforming the surfaces in this sequence by Λ-moves at all the points with their degrees$\geq$ 2, we have a sequence of elementary surfaces

$$\varphi_\theta(F) = F_1 \longrightarrow \widehat{F_1} \longrightarrow \widehat{F_2} \longrightarrow \cdots \longrightarrow \widehat{F_n} \longrightarrow F_n = \varphi_\theta(F').$$

Then F_1 and $\widehat{F_1}$, $\widehat{F_n}$ and F_n are related by a sequence of simplex moves on elementary surfaces by Corollary 4.4, respectively. Moreover, $\widehat{F_i}$ and $\widehat{F_{i+1}}$ are also related by a sequence of simplex moves on elementary surfaces by Lemma 5.3 $(i = 1, 2, \cdots, n-1)$. Hence we obtain a required sequence of simplex moves on elementary surfaces between $\varphi_\theta(F)$ and $\varphi_\theta(F')$.

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