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1-CONFORMALLY FLAT STATISTICAL SUBMANIFOLDS

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Introduction

We study 1-conformally flat statistical submanifolds of flat statistical manifolds. Let φ be a function on a domain Ω in an affine space $\mathbf{A}^{n+1}$. Denoting by $\tilde{D}$ the canonical flat affine connection on $\mathbf{A}^{n+1}$, we can consider a Hessian domain $(\Omega, \tilde{D}, \tilde{g} = \tilde{D}d\varphi)$ a flat statistical manifold. In this paper, we show that, if $\tilde{g}$ is positive definite, n-dimensional level surfaces of φ are 1-conformally flat statistical submanifolds of $(\Omega, \tilde{D}, \tilde{g})$, and that a 1-conformally flat statistical manifold with a Riemannian metric can be locally realized as a submanifold of a flat statistical manifold.

The concept of α-conformally equivalence was first treated in [9] with respect to sequential estimation theory. On the realization of statistical manifolds in the affine space, see [5][6][7].

1. Theorems

Let $\tilde{D}$ be the canonical flat affine connection on an $(n+1)$-dimensional real affine space $\mathbf{A}^{n+1}$, and $\{x^1, \dots, x^{n+1}\}$ the canonical affine coordinate on it, i.e., $\tilde{D}dx^i = 0$. If the Hessian $\tilde{D}d\varphi = \sum_{i,j}(\partial^2\varphi)/(\partial x^i \partial x^j)dx^i dx^j$ of a function φ on a domain Ω in $\mathbf{A}^{n+1}$ is non-degenerate, we call $(\Omega, \tilde{D}, \tilde{g} = \tilde{D}d\varphi)$ a Hessian domain.

For a torsion-free affine connection ∇ and a pseudo-Riemannian metric h on a manifold N, the triple (N, ∇, h) is called a statistical manifold if ∇h is symmetric. If the curvature tensor R of ∇ vanishes, (N, ∇, h) is said to be flat. A Hessian domain $(\Omega, \tilde{D}, \tilde{g} = \tilde{D}d\varphi)$ is a flat statistical manifold. Conversely, a flat statistical manifold is locally a Hessian domain [1][10].

For $\alpha \in \mathbf{R}$, statistical manifolds (N, ∇, h) and $(N, \bar{\nabla}, \bar{h})$ are said to be α-conformally equivalent if there exists a function ϕ on N such that

$$\begin{aligned}
\bar{h}(X, Y) &= e^{\phi}h(X, Y), \\
h(\bar{\nabla}_X Y, Z) &= h(\nabla_X Y, Z) - \frac{1+\alpha}{2}d\phi(Z)h(X, Y) \\
&\quad + \frac{1-\alpha}{2}\{d\phi(X)h(Y, Z) + d\phi(Y)h(X, Z)\}
\end{aligned}$$

for $X, Y, Z \in \mathcal{X}(N)$, where $\mathcal{X}(N)$ is the set of all tangent vector fields on N. A statistical manifold (N, ∇, h) is called α-conformally flat if (N, ∇, h) is locally α-conformally equivalent to a flat statistical manifold [6].

For a pseudo-Riemannian manifold $(\tilde{N}, \tilde{h})$ and a submanifold N of $\tilde{N}$, we call (N, ∇, h) *a statistical submanifold* of $(\tilde{N}, \tilde{h})$ if (N, ∇, h) is a statistical manifold, where ∇ is an affine connection on N and h the induced pseudo-Riemannian metric for $\tilde{h}$. Let $\tilde{\nabla}$ be an affine connection on $\tilde{N}$. We denote by $T_xN \oplus T_xN^{\perp}$ the orthogonal decomposition of $T_x\tilde{N}$ with respect to $\tilde{h}$, where $T_x\tilde{N}$ and T_xN are the set of all tangent vectors at x on $\tilde{N}$ and on N, respectively. If $(\nabla_X Y)_x$ is the T_xN-component of $(\tilde{\nabla}_X Y)_x$ for $X, Y \in \mathcal{X}(N)$ and an arbitrary x in N, we call (N, ∇, h) *the statistical submanifold realized in* $(\tilde{N}, \tilde{\nabla}, \tilde{h})$.

Amari said that, if $(\tilde{N}, \tilde{\nabla}, \tilde{h})$ is a statistical manifold for a Riemannian metric $\tilde{h}$ and a submanifold N of $\tilde{N}$, (N, ∇, h) is a statistical manifold for the above induced connection ∇ and the induced metric h [1]. For a pseudo-Riemannian metric $\tilde{h}$, (N, ∇, h) is a statistical manifold if h is non-degenerate. Then, through this paper, we call a statistical submanifold realized in a statistical manifold $(\tilde{N}, \tilde{\nabla}, \tilde{h})$, simply, a statistical submanifold of $(\tilde{N}, \tilde{\nabla}, \tilde{h})$.

In this paper we aim to prove the next theorems.

Theorem 1. *Let M be a simply connected n-dimensional level surface of φ on an $(n+1)$-dimensional Hessian domain $(\Omega, \tilde{D}, \tilde{g} = \tilde{D}d\varphi)$ with a Riemannian metric $\tilde{g}$, and suppose that $n \geq 2$. If we consider $(\Omega, \tilde{D}, \tilde{g})$ a flat statistical manifold, (M, D, g) is a* 1*-conformally flat statistical submanifold of* $(\Omega, \tilde{D}, \tilde{g})$*, where we denote by D and g the connection and the Riemannian metric on M induced by $\tilde{D}$ and $\tilde{g}$.*

Theorem 2. *An arbitrary* 1*-conformally flat statistical manifold of* $\dim n \geq 2$ *with a Riemannian metric can be locally realized as a submanifold of a flat statistical manifold of* $\dim(n+1)$.

We shall show a corollary of Theorem 1 with relation to projectively flat connections and dual-projectively flat connections in the last section.

2. Statistical Manifolds and Affine Differential Geometry

In this section, we study a level surface M of φ on an $(n+1)$-dimensional Hessian domain $(\Omega, \tilde{D}, \tilde{g})$, using affine differential geometry and the concept of statistical submanifold. A level surface M of φ is an n-dimensional submanifold of Ω if and only if $d\varphi_x \neq 0$ for all $x \in M$. Henceforward, we suppose that $n \geq 2$, that $\tilde{g}$ is a Riemannian metric, and that $d\varphi_x \neq 0$ for all $x \in M$.

Let $\tilde{E}$ be the gradient vector field on Ω defined by

$$\tilde{g}(\tilde{X}, \tilde{E}) = d\varphi(\tilde{X}) \quad \text{for } \tilde{X} \in \mathcal{X}(\Omega).$$

Since $\tilde{g}$ is positive definite and $d\varphi_x \neq 0$ for all $x \in M$, $d\varphi(\tilde{E})$ does not vanish on M and a vector $\tilde{E}_x$ is vertical to T_xM with respect to $\tilde{g}$, where T_xM is the set of all tangent vectors at x on M. We set $E = -d\varphi(\tilde{E})^{-1}\tilde{E}$ on M. Then the vector field $\tilde{E}$ is transversal to M, and so is E.

Let ι be a canonical immersion of M into Ω. For $\tilde{D}$ and an affine immersion (ι, E), we can define the induced affine connection D^E, the fundamental form g^E, the shape operator S^E and the transversal connection form τ^E on M by

$$\tilde{D}_X Y = D^E_X Y + g^E(X, Y)E \tag{1}$$

$$\tilde{D}_X E = S^E(X) + \tau^E(X)E \quad \text{for } X, Y \in \mathcal{X}(M). \tag{2}$$

We denote by (M, D, g) the statistical submanifold of $(\Omega, \tilde{D}, \tilde{g})$, considering $(\Omega, \tilde{D}, \tilde{g})$ a statistical manifold. Then the next holds.

Lemma 2.1. *A statistical submanifold (M, D, g) coincides with a manifold (M, D^E, g^E) induced by an affine immersion (ι, E), i.e.,*

$$D = D^E, \ \ g = g^E \quad \textit{on } M.$$

Proof. Let $D^{\tilde{E}}$ be the induced affine connection, $g^{\tilde{E}}$ the fundamental form, $S^{\tilde{E}}$ the shape operator, and $\tau^{\tilde{E}}$ the transversal connection form, for $\tilde{D}$ and $\tilde{E}$. Since E_x and $\tilde{E}_x$ are vertical to T_xM for $x \in M$ with respect to $\tilde{g}$, $D = D^E = D^{\tilde{E}}$ holds. From (1) and $\tilde{D}_X Y = D^{\tilde{E}}_X Y + g^{\tilde{E}}(X, Y)\tilde{E}$, we have

$$g^{\tilde{E}} = -d\varphi(\tilde{E})^{-1} g^E. \tag{3}$$

By [3] we know that

$$g^{\tilde{E}} = -d\varphi(\tilde{E})^{-1} g. \tag{4}$$

From (3) and (4) $g = g^E$ holds. □

Since g is non-degenerate, so is g^E. Then (ι, E) is called a non-degenerate immersion. Moreover, the immersion (ι, E) has the following property.

Lemma 2.2. *An affine immersion (ι, E) is equiaffine, i.e.,*

$$\tau^E = 0 \quad \textit{on } M.$$

Proof. We have

$$\tau^{\tilde{E}} = (d \log |d\varphi(\tilde{E})|)(X) \tag{5}$$

by [3]. Calculating the right-hand side of (5), we have

$$\tau^{\tilde{E}} = d\varphi(\tilde{E})^{-1}X(d\varphi(\tilde{E})).$$

Thus, we obtain

$$\begin{aligned}\tilde{D}_X E &= -\tilde{D}_X(d\varphi(\tilde{E})^{-1}\tilde{E}) \\ &= -X(d\varphi(\tilde{E})^{-1})\tilde{E} - d\varphi(\tilde{E})^{-1}D_X\tilde{E} \\ &= d\varphi(\tilde{E})^{-2}X(d\varphi(\tilde{E}))\tilde{E} - d\varphi(\tilde{E})^{-1}\{S^{\tilde{E}}(X) + \tau^{\tilde{E}}(X)\tilde{E}\} \\ &= -d\varphi(\tilde{E})^{-1}S^{\tilde{E}}(X).\end{aligned}$$

Hence $S^E = -d\varphi(\tilde{E})^{-1}S^{\tilde{E}}$ and $\tau^E = 0$ hold. □

It is known that the structure induced by a non-degenerate equiaffine immersion is the statistical manifold structure. Conversely, Kurose proved the next proposition.

Proposition 2.3 ([6]). *A simply connected statistical manifold can be realized in* $\mathbf{A}^{n+1}$ *by a non-degenerate equiaffine immersion if and only if it is* 1*-conformally flat. Such an immersion is uniquely determined up to affine transformations of* $\mathbf{A}^{n+1}$.

Proposition 2.3 can be proved by projectively flatness of the dual connection of a given connection [2]. Finally, let us show Theorem 1.

Proof of Theorem 1. By Lemma 2.2 and Proposition 2.3 a statistical manifold (M, D^E, g^E) is 1-conformally flat. Thus Theorem 1 holds by Lemma 2.1. □

3. Proof of Theorem 2

Let (N, ∇, h) be a 1-conformally flat statistical manifold of $\dim n \geq 2$ with a Riemannian metric h. By Proposition 2.3 (N, ∇, h) can be realized by a non-degenerate equiaffine immersion. We denote by (ι, E) a non-degenerate equiaffine immersion into $\mathbf{A}^{n+1}$ which realizes (N, ∇, h). Then we can immerse (N, ∇, h) into a flat statistical manifold as the next lemma.

Lemma 3.1. *For a simply connected open subset U of N and a small $\varepsilon > 0$, we define a function ϕ on $\tilde{U} = \bigcup_{q\in U}\{\iota(q) \oplus (-\varepsilon, \varepsilon)\cdot E_q\}$ by*

$$\phi(p) = e^{-t} \quad \text{for } p = \iota(p_0) + tE_{p_0},\ \ p_0 \in U, t \in (-\varepsilon, \varepsilon).$$

Then (U, ∇, h) is a statistical submanifold of a flat statistical manifold $(\tilde{U}, \tilde{D}, \tilde{D}d\phi)$.

Proof. For $X, Y \in \mathcal{X}(U)$, we have

$$d\phi(X) = 0, \ \ d\phi(E) = -1,$$

and

$$\begin{aligned}(\tilde{D}_X d\phi)(Y) &= X(d\phi(Y)) - d\phi(\tilde{D}_X Y) \\ &= -d\phi(\nabla_X Y + h(X, Y)E) \\ &= -h(X, Y)d\phi(E) \\ &= h(X, Y).\end{aligned}$$

Thus, (U, ∇, h) is a submanifold of $(\tilde{U}, \tilde{D}, \tilde{D}d\phi)$.

We also denote by E a vector field on $\tilde{U}$ whose value is E_{p_0} at $p = \iota(p_0) + tE_{p_0}$. On $\iota(U)$ we have

$$E(d\phi(E)) = 1, \ \ \tilde{D}_E E = 0,$$

and

$$(\tilde{D}_E d\phi)(E) = E(d\phi(E)) - d\phi(\tilde{D}_E E) = 1.$$

Thus $(\tilde{D}d\phi)_{\iota(p_0)}$ is positive definite for $p_0 \in U$. From continuity of a function ϕ, $\tilde{D}d\phi$ is a Riemannian metric on $\tilde{U}$ for a small ε. Hence $(\tilde{U}, \tilde{D}, \tilde{D}d\phi)$ is a flat statistical manifold. □

4. Dual-Projectively Flat Connections

In this section, we describe dual-projectively flatness of an affine connection D on a level surface M and projectively flatness of the dual-connection D' of D.

Let (N, h) be a pseudo-Riemannian manifold. Torsion free affine connections ∇ and $\bar{\nabla}$ on N are projectively equivalent if there exists a 1-form κ such that

$$\bar{\nabla}_X Y = \nabla_X Y + \kappa(X)Y + \kappa(Y)X$$

for $X, Y \in \mathcal{X}(N)$. An affine connection ∇ is called projectively flat if ∇ is locally projectively equivalent to a flat affine connection. Torsion free affine connections ∇ and $\bar{\nabla}$ on N are dual-projectively equivalent if there exists a 1-form κ such that

$$h(\bar{\nabla}_X Y, Z) = h(\nabla_X Y, Z) - \kappa(Z)h(X, Y)$$

for $X, Y, Z \in \mathcal{X}(N)$. An affine connection ∇ is called dual-projectively flat if ∇ is locally dual-projectively equivalent to a flat affine connection [4].

For a statistical manifold (N, ∇, h) there exists the torsion free affine connection ∇' on N such that

$$Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \nabla'_X Z)$$

The connection ∇' is said to be the dual connection of ∇, and (N, ∇', h) the dual statistical manifold of (N, ∇, h). If (N^s, ∇^s, h^s) and $(N^s, \nabla^{s\prime}, h^s)$ are statistical submanifolds of (N, ∇, h) and (N, ∇', h), respectively, $(N^s, \nabla^{s\prime}, h^s)$ is the dual statistical manifold of (N^s, ∇^s, h^s) [1].

Statistical manifolds (N, ∇, h) and $(N, \bar{\nabla}, \bar{h})$ are α-conformally equivalent if and only if the dual statistical manifolds (N, ∇', h) and $(N, \bar{\nabla}', \bar{h})$ are $(-\alpha)$-conformally equivalent. Especially, a statistical manifold (N, ∇, h) is 1-conformally flat if and only if the dual statistical manifold (N, ∇', h) is (-1)-conformally flat [6].

Moreover, Kurose showed that, by Proposition 9.1 in [8], a statistical manifold (N, ∇', h) is (-1)-conformally flat if and only if ∇' is a projectively flat connection with symmetric Ricci tensor, and that

Proposition 4.1 ([6]). *A statistical manifold (N, ∇, h) is 1-conformally flat if and only if the dual connection ∇' is a projectively flat connection with symmetric Ricci tensor.*

On projectively flatness, Ivanov described the next proposition on section 2 in [4].

Proposition 4.2 ([4]). *A statistical manifold (N, ∇, h) is 1-conformally flat if and only if ∇ is a dual-projectively flat connection with symmetric Ricci tensor.*

For a level surface of a Hessian domain, we obtain the next corollary of Theorem 1 by Proposition 4.1 and 4.2.

Corollary 4.3. *Let M be a simply connected n-dimensional level surface of φ on an $(n+1)$-dimensional Hessian domain $(\Omega, \tilde{D}, \tilde{g} = \tilde{D}d\varphi)$ with a Riemannian metric $\tilde{g}$, and suppose that $n \geq 2$. Let (M, D, g) be a statistical submanifolds of $(\Omega, \tilde{D}, \tilde{g})$ and D' the dual connection of D. Then, D is a dual-projectively flat connection with symmetric Ricci tensor and D' is a projectively flat connection with symmetric Ricci tensor.*

References

[1] S. Amari: Differential-Geometrical Methods in Statistics, Springer-Verlag, 1985.
[2] F. Dillen, K. Nomizu and L. Vrancken: *Conjugate connections and Radon's theorem in affine*

differential geometry, Monatsh. Math. **109** (1990), 221–235.
[3] J.H. Hao and H. Shima: *Level surfaces of non-degenerate functions in* $\mathbf{R}^{n+1}$, Geom. Dedicata, **50** (1994), 193–204.
[4] S. Ivanov: *On dual-projectively flat affine connections*, J. Geom. **53** (1995), 89–99.
[5] T. Kurose: *Dual connections and affine geometry*, Math. Zeitschrift, **203** (1990), 115–121.
[6] T. Kurose: *On the divergence of 1-conformally flat statistical manifolds*, Tôhoku Math. J. **46** (1994), 427–433.
[7] H. Matsuzoe: *On realization of conformally-projectively flat statistical manifolds and the divergences*, Hokkaido Math. J. **27** (1998), 409–421.
[8] K. Nomizu and U. Simon: *Notes on conjugate connections*, Geometry and Topology of Submanifolds, **IV** (F.Dillen and L.Verstraelen eds.), World Scientific, 1992, 152–173.
[9] I. Okamoto, S. Amari and K. Takeuchi: *Asymptotic theory of sequential estimation: differential geometrical approach*, Ann. Statist. **19** (1991), 961–981.
[10] H. Shima: *Geometry of Hessian domains*, Differential Geometry and its Applications, **7** (1997), 277–290.

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