Matumoto, T. and Nakagawa, N.
Osaka J. Math.
37 (2000), 417–424

EXPLICIT DESCRIPTION OF HOPF SURFACES AND THEIR AUTOMORPHISM GROUPS

TAKAO MATUMOTO and NORIAKI NAKAGAWA

(Received June 29, 1998)

A Hopf surface is a compact complex surface whose universal covering is $\mathbf{C}^2 - (0, 0)$. Hopf surfaces with infinite cyclic fundamental groups are called primary and the others secondary. The holomorphic automorphism groups of primary Hopf surfaces are determined by Namba [5] and Wehler [7]. In this paper we give an explicit description of the covering transformations of secondary Hopf surfaces based on the result of Kato [2], [3] and calculate all the holomorphic automorphism groups. The method of proof is to expand any automorphism into Taylor series at the origin and check the compatibility with the covering transformations.

1. The covering transformation groups of Hopf surfaces

Let G denote the fundamental group of a given secondary Hopf surface. As Kato [2, p. 231] showed, we may assume $G \subset GL(2, \mathbf{C})$ except the case that
(0) G is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}_m$, $m \geq 2$, and generated by $g(z_1, z_2) = (\alpha^n z_1 + \lambda z_2^n, \alpha z_2)$ and $h(z_1, z_2) = (a^n z_1, a z_2)$ where $\alpha, \lambda \in \mathbf{C}$, $0 < |\alpha| < 1$, a = a primitive m-th root of 1, $(m, n) = 1$ and $n \geq 2$.

Due to Kato [2, Prop. 8; 3, Prop. 8'] and the classification of the finite subgroups of $U(2)$ which operate freely on S^3, we shall give an explicit classification of the covering transformation groups G in $GL(2, \mathbf{C})$ modulo conjugate as follows.

We put $H = \{g \in G; |\det g| = 1\}$, $K = \{g \in G; \det g = 1\}$. Kato classified them according to the type of K but we prefer to divide them into decomposable and indecomposable cases. Note that the following sequence

$$1 \to H \to G \to \mathbf{Z} \to 1$$

is exact and G is decomposable if the sequence splits and indecomposable otherwise. We may assume moreover that H is a finite subgroup of $U(2)$.

We take hereafter $\zeta = \exp(\pi i/4)$, $\epsilon = \exp(2\pi i/5)$, $\rho_n = \exp(\pi i/n)$, a = a primitive m-th root of 1 and $\alpha, \beta, \gamma \in \mathbf{C}$ with $0 < |\alpha|, |\beta|, |\gamma| < 1$.

(1) The case when G is decomposable and abelian: G is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}_m$ with $m \geq 2$. A generator g of $\mathbf{Z}$ and a generator h of $\mathbf{Z}_m$ are simultaneously conjugate to

(A) $g(z_1, z_2) = (\alpha z_1, \beta z_2)$ and $h(z_1, z_2) = (a z_1, a^n z_2)$ where $(m, n) = 1$, or

(B) $g(z_1, z_2) = (\alpha z_1 + z_2, \alpha z_2)$ and $h(z_1, z_2) = (az_1, az_2)$.

(2) The case when G is decomposable and not abelian: $G = \mathbf{Z} \times H$ where $\mathbf{Z}$ is generated by $g(z_1, z_2) = (\gamma z_1, \gamma z_2)$, i.e.,

$$\mathbf{Z} = \langle \gamma I \rangle \text{ with } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and $H = H_i$ is a finite subgroup of $U(2)$ which operates freely on S^3 classified by Hopf and Threlfall-Seifert (Cf. Orlik [6, Th.1, p. 111] or Brieskorn[1, p. 347]). In fact we have the following 6 cases where we give a sytem of generators very explicitly. Here a is still a primitive m-th root of 1 and we denote by $\langle g \rangle$ the subgroup generated by g.

(C1) $G = \langle \gamma I \rangle \times H_1$ where $H_1 = \langle aI \rangle \times B'_{2^k(2\ell+1)}$ and $K = A_{2(2\ell+1)}$ with $(2^k(2\ell+1), m) = 1$, $2\ell + 1 \geq 3$ and $k \geq 3$. Note that $B'_{2^k(2\ell+1)} = \left\langle \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \right\rangle$ where s and d^2 have finite order $2\ell + 1$ and 2^{k-1} and that $A_{2(2\ell+1)} = \left\langle \begin{pmatrix} -s & 0 \\ 0 & -s^{-1} \end{pmatrix} \right\rangle$ is the cyclic group of order $2(2\ell + 1)$. Note also that $n^2 \equiv 1 \pmod{p^j}$ implies $n \equiv \pm 1 \pmod{p^j}$ for odd prime p. We may consider $B'_{4(2\ell+1)} = B_{2\ell+1}$ for $k = 2$.

(C2) $G = \langle \gamma I \rangle \times H_2$ where $H_2 = \langle aI \rangle \times B_n$ and $K = B_n$ with $(m, 4n) = 1$ and $n \geq 2$. Note that $B_n = \left\langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} \rho_n & 0 \\ 0 & \rho_n^{-1} \end{pmatrix} \right\rangle$ is the binary dihedral group of order $4n$.

(C3) $G = \langle \gamma I \rangle \times H_3$ where $H_3 = \langle aI \rangle \times C$ and $K = C$ with $(m, 6) = 1$. Note that $C = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix} \right\rangle$ is the binary tetrahedral group of order 24.

(C4) $G = \langle \gamma I \rangle \times H_4$ where $H_4 = \langle aI \rangle \times C'_{8\cdot 3^k}$ and $K = B_2$ with $(m, 6) = 1$ and $k \geq 1$. Note that $C'_{8\cdot 3^k} = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{\omega}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix} \right\rangle$ is a group of order $8 \cdot 3^k$ where ω is a primitive 3^k-th root of 1. Note also that C'_{24} is abstractly isomorphic to C but not conjugate.

(C5) $G = \langle \gamma I \rangle \times H_5$ where $H_5 = \langle aI \rangle \times D$ and $K = D$ with $(m, 6) = 1$ and the binary octahedral group $D = \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix} \right\rangle$ of order 48.

(C6) $G = \langle \gamma I \rangle \times H_6$ where $H_6 = \langle aI \rangle \times E$ and $K = E$ with $(m, 30) = 1$ and the binary icosahedral group $E = \left\langle \begin{pmatrix} \epsilon^3 & 0 \\ 0 & \epsilon^2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} \epsilon^4 - \epsilon & \epsilon^2 - \epsilon^3 \\ \epsilon^2 - \epsilon^3 & \epsilon - \epsilon^4 \end{pmatrix} \right\rangle$ of order 120.

(3) The case when G is indecomposable: In the following cases from (D1) to (D6)

$$G = G_0 \cup gG_0, \quad G_0 = \langle \gamma^2 I \rangle \times H \text{ and } g = \gamma u$$

and in the case (D7)

$$G = G_0 \cup gG_0 \cup g^2G_0,\ \ G_0 = \langle \gamma^3 I\rangle \times H \text{ and } g = \gamma u.$$

(D1) $H = \left\langle \begin{pmatrix} a & 0 \\ 0 & a^n \end{pmatrix} \right\rangle$, $K = \left\langle \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \right\rangle$ and $u = \begin{pmatrix} 0 & t^{-1} \\ t & 0 \end{pmatrix}$ where s has the finite order $m_K \geq 3$ with $n \not\equiv n^2 \equiv 1 \pmod m$. Let $m = 2^k(2\ell + 1)$ and $b = a^{2\ell+1}$. Then we note that any solution of $n \not\equiv n^2 \equiv 1 \pmod{2^k}$ gives one of $b^n = b^{-1}$, $-b$ or $-b^{-1}$. Note also that $t \in \mathbf{C}^*$ and the conjugacy class is independent of the value of t.

(D2) $H = \left\langle \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right\rangle$, $K = \{\pm I\}$ and $u = \begin{pmatrix} 0 & t^{-1} \\ t & 0 \end{pmatrix}$ with $m = 2(2\ell + 1) \geq 6$ and $t \in \mathbf{C}^*$. Note that we may assume $t = 1$ in the conjugate class.

(D3) $H = H_1$ as in (C1) and $u = \begin{pmatrix} 0 & t^{-1} \\ t & 0 \end{pmatrix}$ where $t^2 = (-s)^k$ for some integer k and $t \neq 1$. This condition not mentioned in [2] is necessary.

(D4) $H = H_2$ as in (C2) and $u = \begin{pmatrix} \rho_{2n} & 0 \\ 0 & \rho_{2n}^{-1} \end{pmatrix}$ with $n \geq 3$.

(D5) $H = \langle aI \rangle \times \left\langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\rangle$ with $(m, 2) = 1$ and $u = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$. Note that $H = H_2$ with $n = 2$.

(D6) $H = H_3$ as in (C3) and $u = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$.

(D7) H is the same as in the case (D5) and $u = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix}$.

2. Automorphism groups of secondary Hopf surfaces

According to the above classification we will prove the following theorem.

Theorem 1. *The holomorphic automorphism group* $\mathrm{Aut}(X) = \widetilde{\mathrm{Aut}}(X)/G$ *of each secondary Hopf surface* $X = \{\mathbf{C}^2 - (0,0)\}/G$ *is described as follows.*

(0) *The case G is not conjugate to any subgroup of $GL(2, \mathbf{C})$:*
$\widetilde{\mathrm{Aut}}(X) = \{f(z_1, z_2) = (a^n z_1 + bz_2^n, az_2); a \in \mathbf{C}^*, b \in \mathbf{C}\}$ *with* $n \geq 2$.

(1) *The case G is contained in $GL(2, \mathbf{C})$ and abelian:*
The case (A) *is divided into the following 5 families.*

(A1) *In the case when $\alpha = \beta$ and $n \equiv 1 \pmod m$,*
$\widetilde{\mathrm{Aut}}(X) = GL(2, \mathbf{C})$.

(A2) *In the case when $\alpha = \beta$ and $n \not\equiv n^2 \equiv 1 \pmod m$,*
$\widetilde{\mathrm{Aut}}(X) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}; a, b, c, d \in \mathbf{C}^* \right\}$.

(A3) *In the case when $\alpha^k = \beta$ for some integer $k \geq 2$ and $k \equiv n \pmod m$,*
$\widetilde{\mathrm{Aut}}(X) = \{f(z_1, z_2) = (az_1, cz_1^k + bz_2); a, b \in \mathbf{C}^*, c \in \mathbf{C}\}$.

(A4) *In the case when $\alpha = \beta^\ell$ for some integer $\ell \geq 2$ and $n\ell \equiv 1 \pmod m$,*

$\widetilde{\mathrm{Aut}}(X) = \{f(z_1, z_2) = (az_1 + cz_2^{\ell}, bz_2); a, b \in \mathbf{C}^*, c \in \mathbf{C}\}.$

(A5) *In the other cases than* (A1), (A2), (A3) *and* (A4),

$$\widetilde{\mathrm{Aut}}(X) = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}; a, b \in \mathbf{C}^*\right\}.$$

(B) $\widetilde{\mathrm{Aut}}(X) = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}; a \in \mathbf{C}^*, b \in \mathbf{C}\right\}.$

(2) *The case G is decomposable and not abelian*:

(C1) $\widetilde{\mathrm{Aut}}(X) = \mathbf{C}^* I \left\langle \begin{pmatrix} is & 0 \\ 0 & -is^{-1} \end{pmatrix}, \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \right\rangle = \mathbf{C}^* I \cdot B_{2(2\ell+1)}$ *with* $\ell \geq 1$.

(C2) $\widetilde{\mathrm{Aut}}(X) = \mathbf{C}^* I \left\langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} \rho_{2n} & 0 \\ 0 & \rho_{2n}^{-1} \end{pmatrix} \right\rangle = \mathbf{C}^* I \cdot B_{2n}$ *if* $n \geq 3$, *and*

$\widetilde{\mathrm{Aut}}(X) = \mathbf{C}^* I \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix} \right\rangle = \mathbf{C}^* I \cdot D$ *if* $n = 2$.

(C3) $\widetilde{\mathrm{Aut}}(X) = \mathbf{C}^* I \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix} \right\rangle = \mathbf{C}^* I \cdot D.$

(C4) $\widetilde{\mathrm{Aut}}(X) = \mathbf{C}^* I \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix} \right\rangle = \mathbf{C}^* I \cdot D.$

(C5) $\widetilde{\mathrm{Aut}}(X) = \mathbf{C}^* I \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix} \right\rangle = \mathbf{C}^* I \cdot D.$

(C6) $\widetilde{\mathrm{Aut}}(X) = \mathbf{C}^* I \left\langle \begin{pmatrix} \epsilon^3 & 0 \\ 0 & \epsilon^2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} \epsilon^4 - \epsilon & \epsilon^2 - \epsilon^3 \\ \epsilon^2 - \epsilon^3 & \epsilon - \epsilon^4 \end{pmatrix} \right\rangle = \mathbf{C}^* I \cdot E.$

(3) *The case G is indecomposable*:

(D1) $\widetilde{\mathrm{Aut}}(X) = \mathbf{C}^* I \left\langle \begin{pmatrix} s^{1/2} & 0 \\ 0 & s^{-1/2} \end{pmatrix}, \begin{pmatrix} 0 & t^{-1} \\ t & 0 \end{pmatrix} \right\rangle \cong \mathbf{C}^* I \cdot B_{m_K}$ *with* $m_K \geq 3$.

(D2) $\widetilde{\mathrm{Aut}}(X) = \left\{\begin{pmatrix} a & bt^{-1} \\ \pm bt & \pm a \end{pmatrix}; a, b \in \mathbf{C} \text{ and } a^2 - b^2 \neq 0\right\}.$

(D3) $\widetilde{\mathrm{Aut}}(X) = \mathbf{C}^* I \left\langle \begin{pmatrix} is & 0 \\ 0 & -is^{-1} \end{pmatrix}, \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \right\rangle = \mathbf{C}^* I \cdot B_{2(2\ell+1)}$ *with* $\ell \geq 1$.

(D4) $\widetilde{\mathrm{Aut}}(X) = \mathbf{C}^* I \left\langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} \rho_{2n} & 0 \\ 0 & \rho_{2n}^{-1} \end{pmatrix} \right\rangle = \mathbf{C}^* I \cdot B_{2n}$ *with* $n \geq 3$.

(D5) $\widetilde{\mathrm{Aut}}(X) = \mathbf{C}^* I \left\langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \right\rangle = \mathbf{C}^* I \cdot B_4.$

(D6) $\widetilde{\mathrm{Aut}}(X) = \mathbf{C}^* I \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix} \right\rangle = \mathbf{C}^* I \cdot D.$

(D7) $\widetilde{\mathrm{Aut}}(X) = \mathbf{C}^* I \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix} \right\rangle = \mathbf{C}^* I \cdot C.$

3. Automorphism groups for the decomposable cases

When G is torsion free, that is, in the case of the primary Hopf surface, the holomorphic automorphism group $\mathrm{Aut}(X)$ is calculated in Namba [5, §2] except for the case (0) and Wehler [7, p. 24] for the cases including the case (0).

In the case (0) all the generators of the automorphism group given by Wheler are compatible with the torsion elements of G and we get the result.

In the case (1)(A) or (B) all the generators of the automorphism group are given by Namba and we have only to check the compatibility with the torsion elements of G and get the result. Note that $\begin{pmatrix} a^n & 0 \\ 0 & a \end{pmatrix}$ is contained in H if and only if $a = a^{n^2}$ in the case (A).

In the other cases any automorphism φ not only should have the form

$$\varphi(z_1, z_2) = \left(\sum_{i,j \geq 0} a_{ij} z_1^i z_2^j, \sum_{i,j \geq 0} b_{ij} z_1^i z_2^j \right)$$

by Hartogs theorem, but also should satisfy

$$\varphi^{-1} \circ \gamma I \circ \varphi = \gamma I$$

because $d\varphi^{-1} \circ \gamma I \circ d\varphi = \gamma I$. Hence

$$\begin{aligned} \varphi(\gamma z_1, \gamma z_2) &= \left(\sum_{i,j \geq 0} a_{ij} (\gamma z_1)^i (\gamma z_2)^j, \sum_{i,j \geq 0} b_{ij} (\gamma z_1)^i (\gamma z_2)^j \right) \\ &= \left(\sum_{i,j \geq 0} a_{ij} \gamma^{i+j} z_1^i z_2^j, \sum_{i,j \geq 0} b_{ij} \gamma^{i+j} z_1^i z_2^j \right) \\ &= \left(\sum_{i,j \geq 0} \gamma a_{ij} z_1^i z_2^j, \sum_{i,j \geq 0} \gamma b_{ij} z_1^i z_2^j \right). \end{aligned}$$

As $0 < |\gamma| < 1$, we have

$$a_{ij} = \begin{cases} \text{arbitrary} & \text{if } i + j = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } b_{ij} = \begin{cases} \text{arbitrary} & \text{if } i + j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular $\varphi \in GL(2, \mathbf{C})$. So $\widetilde{\mathrm{Aut}}(X)$ is the normalizer $N_{GL(2,\mathbf{C})}(G)$ of G in $GL(2, \mathbf{C})$. Moreover, since any element of G whose absolute value of determinent is one should be contained in H, we see that φ is contained in $N_{GL(2,\mathbf{C})}(H)$.

In the case (2) we see that $\mathbf{C}^* I \times H \subset \widetilde{\mathrm{Aut}}(X) = N_{GL(2,\mathbf{C})}(H) \subset N_{GL(2,\mathbf{C})}(K)$. Kato [2, Lemma 5; 3, p. 222] determined $N_{GL(2,\mathbf{C})}(K)$ as follows.

Let $A_m = \left\langle \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\rangle$. We put $g_b = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$ and $\overline{g}_c = \begin{pmatrix} 0 & c^{-1} \\ c & 0 \end{pmatrix}$. Then,

$$N_{GL(2,\mathbf{C})}(A_m) = \{e g_b, f \overline{g}_c; b, c, e, f \in \mathbf{C}^*\} \ (m \geq 3),$$
$$N_{GL(2,\mathbf{C})}(B_n) = \mathbf{C}^* I \cdot B_{2n} \ (n \geq 3),$$

$$N_{GL(2,\mathbf{C})}(B_2) = N_{GL(2,\mathbf{C})}(C) = N_{GL(2,\mathbf{C})}(D) = \mathbf{C}^* I \cdot D \text{ and}$$
$$N_{GL(2,\mathbf{C})}(E) = \mathbf{C}^* I \cdot E.$$

So, we have only to prove in the cases (C1) and (C4).

In the case (C1) where $K = A_{2(2\ell+1)}$ we have $N_{GL(2,\mathbf{C})}(K) = \{eg_b, f\overline{g}_c; b, c, e, f \in \mathbf{C}^*\}$. Each element of the group $B'_{2^k(2\ell+1)} = \langle g_s, d\overline{g}_1 \rangle$ has the form $d^{2p}g_{s^q}$ or $d^{2p+1}\overline{g}_{s^q}$. Note that $d^{2^{k-1}} = -1$ and $-g_s = g_{-s}$. Since $g_b d\overline{g}_1 = d\overline{g}_{b^{-2}}g_b$, $\overline{g}_c g_s = g_{s^{-1}}\overline{g}_c$ and $\overline{g}_c d\overline{g}_1 = d\overline{g}_{c^2}\overline{g}_c$, we have $g_b \in N_{GL(2,\mathbf{C})}(\langle g_s, d\overline{g}_1\rangle)$ if $b^{-2} = (-s)^p$ for some integer p. Clearly $\overline{g}_1 \in N_{GL(2,\mathbf{C})}(H)$ and we get the result.

In the case (C4) where $K = B_2$ we have $N_{GL(2,\mathbf{C})}(K) = \mathbf{C}^* I \cdot D$. We put $h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $u = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ and $v = \frac{1}{\sqrt{2}}\begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix}$. Then $h = v^{-1}u^2v$, $C'_{8\cdot 3^k} = \langle u^2, \omega v\rangle$ and $D = \langle u, v\rangle$ for a primitive 3^k-th root ω of 1. Put $N = N_{GL(2,\mathbf{C})}(C'_{8\cdot 3^k})$. It is easy to see $u^2, v \in N$. Moreover, $uvu^{-1} = u^{-1}vhu^{-1}$ implies $u \in N$. So, the result follows.

4. Automorphism groups for the indecomposable cases

With the discussion in §3 we see that any automorphism φ is contained in $N_{GL(2,\mathbf{C})}(H)$. Since the inner-automorphism induced by φ preserves the value of the determinant, it keeps the subset $\gamma u G_0$ and in the case (D7) the subset $\gamma^2u^2G_0$, too. So, $\varphi \in \widetilde{\mathrm{Aut}}(X)$ if and only if $[u, \varphi] = u\varphi u^{-1}\varphi^{-1} \in H$ in the cases from (D1) to (D6). Also $\varphi \in \widetilde{\mathrm{Aut}}(X)$ if and only if $[u, \varphi], [u^2, \varphi] \in H$ in the case (D7). Note also that $[v, \varphi_1], [v, \varphi_2] \in H$ implies $[v, \varphi_1^{-1}\varphi_2] \in H$ if $\varphi_1, \varphi_2 \in N_{GL(2,\mathbf{C})}(H)$.

Now we will verify the above condition for each case.

In the case (D1) we have $N_{GL(2,\mathbf{C})}(K) = N_{GL(2,\mathbf{C})}(H) = \{eg_b, f\overline{g}_c; b, c, e, f \in \mathbf{C}^*\}$. Note that $\overline{g}_c^2 = I$, $g_b\overline{g}_c = \overline{g}_c g_{b^{-1}}$ and $\overline{g}_c\overline{g}_{c'} = g_{c^{-1}c'}$. Then,

$$[u, eg_b] = \begin{pmatrix} b^{-2} & 0 \\ 0 & b^2 \end{pmatrix} \text{ and } [u, f\overline{g}_c] = \begin{pmatrix} t^{-2}c^2 & 0 \\ 0 & t^2c^{-2} \end{pmatrix} \text{ for } u = \begin{pmatrix} 0 & t^{-1} \\ t & 0 \end{pmatrix}.$$

These elements are contained in H if $b^2 = s^k, c^2 = s^\ell t^2$ for some integers k, ℓ. Therefore we get the result. Note that $g_{-1} \in H$ and hence $g_{\sqrt{-1}} \in \widetilde{\mathrm{Aut}}(X)$ when $b^n = -b$ or $-b^{-1}$.

In the case (D2) we have $N_{GL(2,\mathbf{C})}(K) = GL(2, \mathbf{C})$. The commutator of u and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is equal to

$$\frac{1}{ad - bc}\begin{pmatrix} d^2 - c^2t^{-2} & -bd + act^{-2} \\ bdt^2 - ac & a^2 - b^2t^2 \end{pmatrix}$$

and should be contained in H. So, $ac = bdt^2$ and $d^2 = \pm a^2$. Since H does not contain $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, we have $d = \pm a$. Hence we get the result.

In the case (D3) we put $v = \begin{pmatrix} is & 0 \\ 0 & -is^{-1} \end{pmatrix}$ and $w = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$. Then, $\widetilde{\mathrm{Aut}}(X)$ is contained in the automorphism group $\mathbf{C}^* I\langle v, w\rangle$ of the case (C1). $[u, v] = v^{-2} \in H$ implies $v \in \widetilde{\mathrm{Aut}}(X)$. Since $w \in H$, we should have $[u, w] = \begin{pmatrix} t^{-2} & 0 \\ 0 & t^2 \end{pmatrix} \in H$. This means $t^2 = (-s)^k$ for some integer k and we get the result.

In the case (D4) we put $h_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $h_2 = \begin{pmatrix} \rho_n & 0 \\ 0 & \rho_n^{-1} \end{pmatrix}$ with $u = \begin{pmatrix} \rho_{2n} & 0 \\ 0 & \rho_{2n}^{-1} \end{pmatrix}$. Then we know $N_{GL(2,\mathbf{C})}(H) = \mathbf{C}^* I\langle h_1, u\rangle$ and $[u, h_1] = u^2 = h_2 \in H$. Clearly $[u, u] = I \in H$. Therefore $\widetilde{\mathrm{Aut}}(X) = \mathbf{C}^* I\langle h_1, u\rangle$.

In the case (D5) we put $h_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $h_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $v = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix}$. We know $N_{GL(2,\mathbf{C})}(H) = \mathbf{C}^* I\langle u, v\rangle$ in the case (C2) with $n = 2$ for $u = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$. Note that $u^4 = (u^{-1}v)^2 = -I$, $v^3 = I$ and $u^3 vuv = h_1$. Note also that H is the direct product of $\langle aI\rangle$ and $B_2 = \{\pm I, \pm h_1, \pm h_2, \pm h_1 h_2\}$. So

$$[u, v] = \frac{1}{2} \begin{pmatrix} 1+i & 1+i \\ i-1 & 1-i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta & \zeta \\ \zeta^3 & -\zeta^3 \end{pmatrix} = -h_1 h_2 v (h_1 h_2)^{-1} \notin H$$

and we see that $v \notin \widetilde{\mathrm{Aut}}(X)$. Clearly $u \in \widetilde{\mathrm{Aut}}(X)$. Also $h_1 \in \widetilde{\mathrm{Aut}}(X)$ because $[u, h_1] = u^2 = h_2 \in H$. Therefore $\mathbf{C}^* I\langle h_1, u\rangle \subset \widetilde{\mathrm{Aut}}(X)$. But the result follows, because $\langle h_1, u\rangle = B_4$ is a maximal proper subgroup of $\langle u, v\rangle = D$ of index 3.

In the case (D6) we use the same elements h_1 and h_2 as in the case (D5) and $h_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix}$. We know $N_{GL(2,\mathbf{C})}(H) = \mathbf{C}^* I\langle u, h_3\rangle$ in the case (C3). Clearly $u \in \widetilde{\mathrm{Aut}}(X)$. Since $[u, h_3] = h_1 h_2 h_3 h_1 h_2 \in H$, we have $h_3 \in \widetilde{\mathrm{Aut}}(X)$. So $\widetilde{\mathrm{Aut}}(X) = \mathbf{C}^* I\langle u, h_3\rangle$.

In the case (D7) we know $N_{GL(2,\mathbf{C})}(H) = \mathbf{C}^* I\left\langle u_1 = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, u = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix} \right\rangle$. Let h_1 and h_2 denote the same elements as in the case (D5). Note that $h_1^2 = h_2^2 = -I$, $h_1^{-1} h_2 h_1 = h_2^{-1}$ and H is the direct product of $\langle aI\rangle$ and $B_2 = \{\pm I, \pm h_1, \pm h_2, \pm h_1 h_2\}$. Note also $u_1^2 = h_2$, $[u, u_1] = h_1 h_2 u^{-1} h_1 h_2 \notin H$, $[u, u_1^2] = -h_1 \in H$, $[u^2, u_1^2] = -h_1 h_2 \in H$ and $[u, u] = [u^2, u] = I \in H$. Therefore $\mathbf{C}^* I\langle u_1^2, u\rangle \subset \widetilde{\mathrm{Aut}}(X)$ and $u_1 \notin \widetilde{\mathrm{Aut}}(X)$. The result follows, because $\langle u_1^2, u\rangle = C$ is a maximal proper subgroup of $\langle u_1, u\rangle = D$. The proof of Theorem 1 is completed.

References

[1] E. Brieskorn: *Rationale Singularitäten komplexer Flächen*, Invent. Math. **4** (1968), 336–358.
[2] M. Kato: *Topology of Hopf surfaces*, J. Math. Soc. Japan, **27** (1975), 222–238.
[3] M. Kato: *Erratum to "Topology of Hopf surfaces"*, J. Math. Soc. Japan, **41** (1989), 173–174.
[4] K. Kodaira: *On the structure of compact complex analytic surfaces II*, Amer. J. Math. **88** (1966), 682–721.
[5] M. Namba: *Automorphism groups of Hopf surfaces*, Tôhoku Math. J. **26** (1974), 133–157.
[6] P. Orlik: *Seifert manifolds*, Lecture Notes in Math. **291**, Springer-Verlag, 1972.
[7] J. Wehler: *Versal deformations for Hopf surfaces*, J. Reine Angew. Math. **328** (1982), 22–32.

T. Matumoto
Department of Mathematics
Hiroshima University
Higashi-Hiroshima 739-8526
Japan

N. Nakagawa
Osaka Toin High School
Daito-City, Osaka 574-0013
Japan