

STOCHASTIC CALCULUS OF GENERALIZED DIRICHLET FORMS AND APPLICATIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS IN INFINITE DIMENSIONS

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(Received June 26, 1998)

1. Introduction

In this paper we systematically develop, as a technical tool for our main application below, a stochastic calculus for *generalized* Dirichlet forms (cf. [15]). In particular, we show *Fukushima's decomposition* of additive functionals including its *extended version* to functions not necessarily in the domain \mathcal{F} of the generalized Dirichlet form, cf. Theorem 4.5 (ii), and an *Itô-type formula* in this framework. The extended version of the Fukushima decomposition will be applied in combination with the Itô-type formula in Subsection 6.3. The class of generalized Dirichlet forms is much larger than the well-studied class of symmetric and coercive Dirichlet forms as in [6] resp. [10] and time dependent Dirichlet forms as in [12]. It contains examples of an entirely new kind (cf. Section 6, [15]). Therefore, the results obtained in this paper lead to extensions of the corresponding results in the “classical” theories. In particular the proofs are “locally” completely different (cf. e.g. Theorem 2.3 and Theorem 2.5; though for the reader's convenience we tried to follow the line of argument in [6] as closely as possible). This difference has several reasons: First of all we *do not assume any sector condition*; in certain cases we have to handle \mathcal{E} -quasi-lower-semicontinuous functions instead of \mathcal{E} -quasi-continuous functions (cf. e.g. Remark 2.6 (i)), and finally, the Dirichlet space of the classical situation generalizes to a space \mathcal{F} which is not necessarily stable under composition with normal contractions. In contrast to the classical theory it is not known whether regularity or quasi-regularity alone imply the existence of an associated process. An additional structural assumption on \mathcal{F} is made in [15, IV.2, D3] (i.e. the existence of a nice intermediate space \mathcal{V} has to be assumed) in order to construct explicitly an associated m -tight special standard process M .

In addition to the new theoretical results described above we also present new applications. In Section 6 we construct weak solutions to stochastic differential equations in infinite dimensions of the type

$$(1) \quad dX_t = dW_t + \frac{1}{2} \beta_H^\mu(X_t) dt + \bar{\beta}(X_t) dt, \quad X_0 = z.$$

Here $(X_t)_{t \geq 0}$ takes values in some real separable Banach space E , $z \in E$, $(W_t)_{t \geq 0}$ is an E -valued Brownian motion, $\bar{\beta}$ is some square integrable vector field on E taking values in a real separable Hilbert space $H \subset E$ and $\beta_H^\mu : E \rightarrow E$ is the logarithmic derivative of μ associated with H (cf. Subsection 6.1). In the symmetric case, when $\bar{\beta} = 0$, equation (1) has been studied intensively in [1].

In Subsection 6.2 we give a first application of (1) for more explicit maps β_H^μ and $\bar{\beta}$ using existence results of [2] on invariant probability measures for some given linear operator L , i.e. measures μ solving the equation $\int Lu d\mu = 0$ for all finitely based smooth functions u . We also use results of [16] on the existence of diffusions associated to extensions of such operators. More precisely, in this case we assume E also to be a Hilbert space and that $H \subset E$ densely by a Hilbert-Schmidt map. We then apply (1) with $\bar{\beta} = (1/2)(B - \beta_H^\mu)$ where $B : E \rightarrow E$ is a Borel measurable vector field of the form $B = -id_E + v$, $v : E \rightarrow H$, satisfying (B.1)-(B.3) of Subsection 6.2. Under these assumptions on B there exists an invariant probability measure μ such that the stochastic differential equation

$$(2) \quad dX_t = dW_t - \frac{1}{2}X_t dt + \frac{1}{2}v(X_t)dt, \quad X_0 = z$$

admits a weak solution $\mathbf{M} = (\Omega, (\mathcal{F})_{t \geq 0}, (X_t)_{t \geq 0}, (P_z)_{z \in E})$ for μ -a.e. (even (quasi-)every) $z \in E$. In particular μ is absolutely continuous w.r.t. the Gaussian measure γ on E with Radon-Nikodym derivative φ^2 where φ is in $H^{1,2}(E; \gamma)$, i.e., the Sobolev space over (E, H, γ) . Moreover $\beta_H^\mu = -id_E + 2(\nabla\varphi/\varphi)$. It is known (see [2, Theorem 3.10]) that the generator of \mathbf{M} restricted to the finitely based smooth functions $L = (1/2)\Delta_H + (1/2)B \cdot \nabla$ is μ -symmetric if and only if $v = 2(\nabla\varphi/\varphi)$ or equivalently $B = \beta_H^\mu$. In our general, i.e., non-symmetric situation, $2(\nabla\varphi/\varphi)$ is the orthogonal projection of v on the closure of the set $\{\nabla u \mid u \in \mathcal{F}C_b^\infty\}$ in $L^2(E, H; \mu)$. The diffusion $\widehat{\mathbf{M}} = (\widehat{\Omega}, (\widehat{\mathcal{F}})_{t \geq 0}, (\widehat{X}_t)_{t \geq 0}, (\widehat{P}_z)_{z \in E})$ which is in duality to \mathbf{M} w.r.t. μ weakly solves

$$(3) \quad d\widehat{X}_t = d\widehat{W}_t - \frac{1}{2}\widehat{X}_t dt + 2\frac{\nabla\varphi}{\varphi}(\widehat{X}_t)dt - \frac{1}{2}v(\widehat{X}_t)dt, \quad \widehat{X}_0 = z$$

for μ -a.e. (even (quasi-)every) $z \in E$ where $(\widehat{W}_t)_{t \geq 0}$ is an E -valued $(\widehat{\mathcal{F}}_t)_{t \geq 0}$ -Brownian motion starting at $0 \in E$ with covariance given by the inner product of H . Thus, adding the drifts of (2) and (3) we obtain $2\beta_H^\mu$ as in the symmetric case (cf. e.g. [5]). In Section 6.3 we show that \mathbf{M} also satisfies an Itô-type formula.

Let us now briefly summarize the contents of the remaining Sections 2-5. In Section 2 we introduce the framework and then establish an integral representation theorem for coexcessive functions. As a result we obtain a description of \mathcal{E} -exceptional sets in terms of an appropriate class of measures $\widehat{\mathcal{S}}_{00}$. This is the key-point for the proof of Theorem 4.5(i) and (ii) below. In the symmetric case our class of measures $\widehat{\mathcal{S}}_{00}$ is smaller than the corresponding one in [6, p. 78]. As a consequence the uniform convergence in Lemma 4.1 can be determined (cf. Remark 4.3 below) by a weaker

semi-norm than in [6, Lemma 5.1.2.]. In Section 3 following [13] we associate to every positive continuous additive functional of \mathbf{M} its corresponding *Revuz measure*. Section 4 is devoted to the *Fukushima-decomposition*. Note that for the proof we *only* assume the coresolvent to be sub-Markovian and strongly continuous on \mathcal{V} . *No dual process* is needed. In Theorem 4.5(ii) we give conditions for the extension of the decomposition. In Section 5 similarly to [6], [11] we derive an *Itô-type formula* for the transformation of the martingale part of the decomposition. However, since \mathcal{F} is not necessarily stable under composition with continuously differentiable Lipschitz functions we have to make some assumptions (cf. Theorem 5.6). But these assumptions are easy to check in applications (see Subsection 6.3).

For a large class of further examples and applications as well as for a more detailed presentation we refer to [19] and to forthcoming papers.

2. Framework and supplementary Potential Theory of generalized Dirichlet forms

Let E be a Hausdorff space such that its Borel σ -algebra $\mathcal{B}(E)$ is generated by the set $\mathcal{C}(E)$ of all continuous functions on E . Let m be a σ -finite measure on $(E, \mathcal{B}(E))$ such that $\mathcal{H} = L^2(E, m)$ is a separable (real) Hilbert space with inner product $(\cdot, \cdot)_{\mathcal{H}}$. Let $(\mathcal{A}, \mathcal{V})$ be a real valued coercive closed form on \mathcal{H} . Then \mathcal{V} equipped with inner product $\tilde{\mathcal{A}}_1(u, v) := (1/2)(\mathcal{A}(u, v) + \mathcal{A}(v, u)) + (u, v)_{\mathcal{H}}$ is again a separable real Hilbert space. Let $\|\cdot\|_{\mathcal{V}}$ be the corresponding norm. Identifying \mathcal{H} with its dual \mathcal{H}' we have that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ densely and continuously.

For a linear operator Λ defined on a linear subspace D of one of the Hilbert spaces \mathcal{V}, \mathcal{H} or \mathcal{V}' we will use from now on the notation (Λ, D) . Let $(\Lambda, D(\Lambda, \mathcal{H}))$ be a linear operator on \mathcal{H} satisfying the following conditions:

- D1** (i) $(\Lambda, D(\Lambda, \mathcal{H}))$ generates a C_0 -semigroup of contractions $(U_t)_{t \geq 0}$.
- (ii) $(U_t)_{t \geq 0}$ can be restricted to a C_0 -semigroup on \mathcal{V} .

Denote by $(\Lambda, D(\Lambda, \mathcal{V}))$ the generator corresponding to the restricted semigroup. From [15, Lemma I.2.3., p. 12] we have that if $(\Lambda, D(\Lambda, \mathcal{H}))$ satisfies **D1** then $\Lambda : D(\Lambda, \mathcal{H}) \cap \mathcal{V} \rightarrow \mathcal{V}'$ is closable as an operator from \mathcal{V} into \mathcal{V}' . Let (Λ, \mathcal{F}) denote its closure, then \mathcal{F} is a real Hilbert space with corresponding norm

$$\|u\|_{\mathcal{F}}^2 := \|u\|_{\mathcal{V}}^2 + \|\Lambda u\|_{\mathcal{V}'}^2.$$

By [15, Lemma I.2.4., p. 13] the adjoint semigroup $(\hat{U}_t)_{t \geq 0}$ of $(U_t)_{t \geq 0}$ can be extended to a C_0 -semigroup on \mathcal{V}' and the corresponding generator $(\hat{\Lambda}, D(\hat{\Lambda}, \mathcal{V}'))$ is the dual operator of $(\Lambda, D(\Lambda, \mathcal{V}))$. Let $\hat{\mathcal{F}} := D(\hat{\Lambda}, \mathcal{V}') \cap \mathcal{V}$. Then $\hat{\mathcal{F}}$ is a real Hilbert space with corresponding norm

$$\|u\|_{\hat{\mathcal{F}}}^2 := \|u\|_{\mathcal{V}}^2 + \|\hat{\Lambda} u\|_{\mathcal{V}'}^2.$$

Let the form \mathcal{E} be given by

$$\mathcal{E}(u, v) := \begin{cases} \mathcal{A}(u, v) - \langle \Lambda u, v \rangle & \text{for } u \in \mathcal{F}, v \in \mathcal{V} \\ \mathcal{A}(u, v) - \langle \hat{\Lambda} v, u \rangle & \text{for } u \in \mathcal{V}, v \in \hat{\mathcal{F}} \end{cases}$$

and $\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha(u, v)_{\mathcal{H}}$ for $\alpha > 0$. \mathcal{E} is called the *bilinear form associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$* .

Here, $\langle \cdot, \cdot \rangle$ denotes the dualization between \mathcal{V}' and \mathcal{V} . Note that $\langle \cdot, \cdot \rangle$ restricted to $\mathcal{H} \times \mathcal{V}$ coincides with $(\cdot, \cdot)_{\mathcal{H}}$ and that \mathcal{E} is well-defined. It follows, from [15, Proposition I.3.4., p. 19], that for all $\alpha > 0$ there exist continuous, linear bijections $W_\alpha : \mathcal{V}' \rightarrow \mathcal{F}$ and $\hat{W}_\alpha : \mathcal{V}' \rightarrow \hat{\mathcal{F}}$ such that $\mathcal{E}_\alpha(W_\alpha f, u) = \langle f, u \rangle = \mathcal{E}_\alpha(u, \hat{W}_\alpha f)$, $\forall f \in \mathcal{V}'$, $u \in \mathcal{V}$. Furthermore $(W_\alpha)_{\alpha>0}$ and $(\hat{W}_\alpha)_{\alpha>0}$ satisfy the resolvent equation

$$W_\alpha - W_\beta = (\beta - \alpha)W_\alpha W_\beta \quad \text{and} \quad \hat{W}_\alpha - \hat{W}_\beta = (\beta - \alpha)\hat{W}_\alpha \hat{W}_\beta.$$

Restricting W_α to \mathcal{H} we get a strongly continuous contraction resolvent $(G_\alpha)_{\alpha>0}$ on \mathcal{H} satisfying $\lim_{\alpha \rightarrow \infty} \alpha G_\alpha f = f$ in \mathcal{V} for all $f \in \mathcal{V}$. The resolvent $(G_\alpha)_{\alpha>0}$ is called the *resolvent associated with \mathcal{E}* . Let $(\hat{G}_\alpha)_{\alpha>0}$ be the adjoint of $(G_\alpha)_{\alpha>0}$ in \mathcal{H} . $(\hat{G}_\alpha)_{\alpha>0}$ is called the *coresolvent associated with \mathcal{E}* .

By [15, Proposition I.4.6., p. 24] we have that $(G_\alpha)_{\alpha>0}$ is sub-Markovian if and only if

D2 $u \in \mathcal{F} \Rightarrow u^+ \wedge 1 \in \mathcal{V} \quad \text{and} \quad \mathcal{E}(u, u - u^+ \wedge 1) \geq 0$

is satisfied.

DEFINITION 2.1. The bilinear form \mathcal{E} associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ is called a *generalized Dirichlet form* if **D2** holds.

An element u of \mathcal{H} is called 1-excessive (resp. 1-coexcessive) if $\beta G_{\beta+1} u \leq u$ (resp. $\beta \hat{G}_{\beta+1} u \leq u$) for all $\beta \geq 0$. Let \mathcal{P} (resp. $\hat{\mathcal{P}}$) denote the 1-excessive (resp. 1-coexcessive) elements of \mathcal{V} . For an arbitrary Borel set $B \in \mathcal{B}(E)$ and an element $u \in \mathcal{H}$ such that $\{v \in \mathcal{H} \mid v \geq u \cdot 1_B\} \cap \mathcal{F} \neq \emptyset$ (resp. $\hat{u} \in \hat{\mathcal{P}}_{\hat{\mathcal{F}}}$ (cf. below for the definition of $\hat{\mathcal{P}}_{\hat{\mathcal{F}}}$)) let $u_B := e_{u \cdot 1_B}$ be the 1-reduced function (resp. $\hat{u}_B := \hat{e}_{\hat{u} \cdot 1_B}$ be the 1-coreduced function) of $u \cdot 1_B$ (resp. $\hat{u} \cdot 1_B$) as defined in [15, Definition III.1.8., p. 65]. Let u_B^α (resp. \hat{u}_B^α), $\alpha > 0$ denote the element $(u \cdot 1_B)_\alpha$ (resp. $(\hat{u} \cdot 1_B)_\alpha$) of [15, Prop. III.1.6., p. 62]. Note that in general only if B is open our definition of reduced function coincides with the one of [6, p. 92], [10, Exercise III.3.10(ii), p. 84]. In particular, if $B \in \mathcal{B}(E)$ is such that $m(B) = 0$, then $u_B = 0$. Note also that by our definition of reduced function [15, III. Lemma 2.1.(ii)] extends to general Borel sets. If $B = E$ we rather use the notation e_u instead of u_E . An increasing sequence of closed subsets $(F_k)_{k \geq 1}$ is called an \mathcal{E} -nest, if for every function $u \in \mathcal{P} \cap \mathcal{F}$ it follows that $u_{F_k} \rightarrow 0$ in \mathcal{H} and weakly in \mathcal{V} . A subset $N \subset E$ is called \mathcal{E} -exceptional if there is

an \mathcal{E} -nest $(F_k)_{k \geq 1}$ such that $N \subset \bigcap_{k \geq 1} E \setminus F_k$. A property of points in E holds \mathcal{E} -quasi-everywhere (\mathcal{E} -q.e.) if the property holds outside some \mathcal{E} -exceptional set. A function f defined up to some \mathcal{E} -exceptional set $N \subset E$ is called \mathcal{E} -quasi-continuous (\mathcal{E} -q.c.) (resp. \mathcal{E} -quasi-lower-semicontinuous (\mathcal{E} -q.l.s.c.)) if there exists an \mathcal{E} -nest $(F_k)_{k \in \mathbb{N}}$, such that $\bigcup_{k \geq 1} F_k \subset E \setminus N$ and $f|_{F_k}$ is continuous (resp. lower-semicontinuous) for all k . For an \mathcal{E} -nest $(F_k)_{k \geq 1}$ let

$$C(\{F_k\}) = \left\{ f : A \rightarrow \mathbb{R} \mid \bigcup_{k \geq 1} F_k \subset A \subset E, f|_{F_k} \text{ is continuous } \forall k \right\}$$

$$C_l(\{F_k\}) = \left\{ f : A \rightarrow \mathbb{R} \mid \bigcup_{k \geq 1} F_k \subset A \subset E, f|_{F_k} \text{ is lower-semicontinuous } \forall k \right\}$$

We denote by \tilde{f} an \mathcal{E} -q.c. m -version of f , conversely f denotes the m -class represented by an \mathcal{E} -q.c. m -version \tilde{f} of f .

DEFINITION 2.2. The generalized Dirichlet form \mathcal{E} associated with (A, \mathcal{V}) and $(\Lambda, D(\Lambda, \mathcal{H}))$ is called *quasi-regular* if:

- (i) There exists an \mathcal{E} -nest $(E_k)_{k \geq 1}$ consisting of compact sets.
- (ii) There exists a dense subset of \mathcal{F} whose elements have \mathcal{E} -q.c. m -versions.
- (iii) There exist $u_n \in \mathcal{F}$, $n \in \mathbb{N}$, having \mathcal{E} -q.c. m -versions \tilde{u}_n , $n \in \mathbb{N}$, and an \mathcal{E} -exceptional set $N \subset E$ such that $\{\tilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

From now on we assume that we have given a quasi-regular generalized Dirichlet form. We remark that by quasi-regularity every element in \mathcal{F} admits an \mathcal{E} -q.c. m -version. Let $\mathcal{C}, \mathcal{D} \subset \mathcal{H}$. We define $\mathcal{D}_{\mathcal{C}} := \{u \in \mathcal{D} \mid \exists f \in \mathcal{C}, u \leq f\}$. For a subset $\mathcal{G} \subset \mathcal{H}$ denote by $\tilde{\mathcal{G}}$ all the \mathcal{E} -q.c. m -versions of elements in \mathcal{G} . In particular $\tilde{\mathcal{P}}_{\mathcal{F}}$ denotes the set of all \mathcal{E} -q.c. m -versions of 1-excessive elements in \mathcal{V} which are dominated by elements of \mathcal{F} . Note that $\widetilde{\mathcal{F} \cap \mathcal{P}} \subset \tilde{\mathcal{P}}_{\mathcal{F}}$ and that $\tilde{\mathcal{P}}_{\mathcal{F}} - \tilde{\mathcal{P}}_{\mathcal{F}}$ is a linear lattice, that is $\tilde{u} \wedge \alpha \in \tilde{\mathcal{P}}_{\mathcal{F}} - \tilde{\mathcal{P}}_{\mathcal{F}}$ for all $\alpha \geq 0$ and all $\tilde{u} \in \tilde{\mathcal{P}}_{\mathcal{F}} - \tilde{\mathcal{P}}_{\mathcal{F}}$. We emphasize that an element in $\mathcal{P}_{\mathcal{F}}$ not necessarily admits an \mathcal{E} -q.c. m -version.

We are now in the situation to state an integral representation theorem for elements in $\widehat{\mathcal{P}}_{\tilde{\mathcal{F}}}$.

Theorem 2.3. *Let $\hat{u} \in \widehat{\mathcal{P}}_{\tilde{\mathcal{F}}}$. Then there exists a unique σ -finite and positive measure $\mu_{\hat{u}}$ on $(E, \mathcal{B}(E))$ charging no \mathcal{E} -exceptional set, such that*

$$\int \tilde{f} d\mu_{\hat{u}} = \lim_{\alpha \rightarrow \infty} \mathcal{E}_1(f, \alpha \widehat{G}_{\alpha+1} \hat{u}) \quad \forall \tilde{f} \in \tilde{\mathcal{P}}_{\mathcal{F}} - \tilde{\mathcal{P}}_{\mathcal{F}}.$$

Proof. Set $I_{\hat{u}}(\tilde{f}) = \lim_{\alpha \rightarrow \infty} \mathcal{E}_1(f, \alpha \widehat{G}_{\alpha+1} \hat{u})$, $\tilde{f} \in \tilde{\mathcal{P}}_{\mathcal{F}} - \tilde{\mathcal{P}}_{\mathcal{F}}$. The limit exists since $\mathcal{E}_1(f, \alpha \widehat{G}_{\alpha+1} \hat{u})$ splits into two parts which are both increasing and bounded. Then $I_{\hat{u}}$ is a nonnegative linear functional on $\tilde{\mathcal{P}}_{\mathcal{F}} - \tilde{\mathcal{P}}_{\mathcal{F}}$. Let $(\tilde{f}_n)_{n \in \mathbb{N}} \subset \tilde{\mathcal{P}}_{\mathcal{F}} - \tilde{\mathcal{P}}_{\mathcal{F}}$ such that $\tilde{f}_n \downarrow 0$

pointwise on E for $n \rightarrow \infty$. Similar to the proof of Theorem 1 in [4] we will show that

$$(4) \quad I_{\hat{u}}(\tilde{f}_n) \downarrow 0 \text{ as } n \rightarrow \infty.$$

Fix $\varphi \in L^1(E; m) \cap \mathcal{B}(E)$ such that $0 < \varphi \leq 1$. By [15, Lemma III.3.10., p. 73] there exists an \mathcal{E} -nest $(F_k)_{k \in \mathbb{N}}$, such that $\widetilde{G}_1 \varphi \geq 1/k$ everywhere on F_k for all $k \in \mathbb{N}$. Since \mathcal{E} is quasi-regular we may assume that F_k , $k \in \mathbb{N}$, is compact. We may further assume by [15, Lemma 3.5., p. 71] that $(\tilde{f}_n)_{n \in \mathbb{N}} \subset C(\{F_k\})$. From Dini's Theorem we know that given $k_0 \in \mathbb{N}$ there exists $n(k_0) \in \mathbb{N}$, such that for all $n \geq n(k_0)$

$$f_n \leq \frac{1}{k_0} G_1 \varphi \quad \text{m-a.e. on } F_{k_0}.$$

Since $f_n \leq f_1 \in \mathcal{P}_{\mathcal{F}} - \mathcal{P}_{\mathcal{F}}$ there exists $f \in \mathcal{F}$ such that $f_n \leq f$ and therefore we have for all $n \in \mathbb{N}$

$$f_n \leq \frac{1}{k_0} G_1 \varphi + f_{F_{k_0}^c} \quad \text{m-a.e.}$$

Let $\hat{f} \in \widehat{\mathcal{F}}$ such that $\hat{u} \leq \hat{f}$. Then

$$\begin{aligned} I_{\hat{u}}(\tilde{f}_n) &= \lim_{\alpha \rightarrow \infty} \mathcal{E}_1(f_n, \alpha \widehat{G}_{\alpha+1} \hat{u}) \\ &\leq \limsup_{\alpha \rightarrow \infty} \mathcal{E}_1 \left(\frac{1}{k_0} G_1 \varphi + f_{F_{k_0}^c}, \alpha \widehat{G}_{\alpha+1} \hat{u} \right) \\ &\leq \mathcal{E}_1 \left(\frac{1}{k_0} G_1 \varphi + f_{F_{k_0}^c}, \hat{f} \right). \end{aligned}$$

Since $\lim_{k \rightarrow \infty} (1/k) G_1 \varphi + f_{F_k^c} = 0$ weakly in \mathcal{V} we conclude that $\lim_{n \rightarrow \infty} I_{\hat{u}}(\tilde{f}_n) = 0$ and (4) is shown. By the Theorem of Daniell-Stone there exists a unique measure say $\mu_{\hat{u}}$ on $\sigma(\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}})$ such that $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}} \subset L^1(\mu_{\hat{u}})$. By [15, Proposition IV.1.9., p. 77] we know that $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ separates the points of $E \setminus N$ where N is an \mathcal{E} -exceptional set and consequently $\sigma(\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}) \supset \mathcal{B}(E \setminus N)$. Since $\mu_{\hat{u}}(\widetilde{N}) = \lim_{\alpha \rightarrow \infty} \mathcal{E}_1(1_{\widetilde{N}}, \alpha \widehat{G}_{\alpha+1} \hat{u}) = 0$ for every \mathcal{E} -exceptional set \widetilde{N} we may assume that $\mu_{\hat{u}}$ is a Borel-measure. Finally $\int_E \widetilde{G}_1 \varphi d\mu_{\hat{u}} \leq \mathcal{E}_1(G_1 \varphi, \hat{f}) < \infty$ implies that $\mu_{\hat{u}}$ is σ -finite. \square

From now on we fix an m -tight special standard process

$$\mathbf{M} = (\Omega, (\mathcal{F}_t)_{t \geq 0}, (Y_t)_{t \geq 0}, (P_z)_{z \in E_{\Delta}})$$

with lifetime ζ such that the resolvent $R_{\alpha} f$ of \mathbf{M} is an \mathcal{E} -q.c. m -version of $G_{\alpha} f$ for all $\alpha > 0$, $f \in \mathcal{H} \cap \mathcal{B}_b(E)$. Note that in addition to quasi-regularity a structural assumption on \mathcal{F} is made in [15, IV.2, D3] in order to construct explicitly an associated

m -tight special standard process. Since we make no use of this technical assumption and since it may be subject to some further progress we instead prefer to assume the existence of \mathbf{M} . We remark that we use the resolvent of \mathbf{M} in the proofs of Lemma 2.4 and Theorem 2.5 below but that the corresponding statements are independent of \mathbf{M} and only depend on the generalized Dirichlet form.

We remark that whenever we fix a filtration in this paper, it will be the natural filtration.

For $u \in \mathcal{P}$ we denote by \bar{u} some \mathcal{E} -q.l.s.c. regularization, i.e. $\bar{u} := \sup_{n \geq 1} nR_{n+1}u$.

Lemma 2.4. *Let $F \subset E$ be closed, $g \in L^2(E; m) \cap \mathcal{B}(E)^+$.*

Then $\text{supp}(\mu_{(\widehat{G}_1 g)_F}) \subset F$.

Proof. Fix $\varphi \in L^1(E; m) \cap \mathcal{B}(E)^+$ such that $0 < \varphi \leq 1$. Define $\varphi^F := E.[\int_{D_F}^\infty e^{-t} \varphi(Y_t) dt]$ where $D_F = \inf\{t \geq 0 \mid Y_t \in F\}$. Using the strong Markov property of \mathbf{M} it is easy to see that φ^F is a Borel measurable m -version of an 1-excessive element in \mathcal{V} and that $\varphi^F \geq (G_1 \varphi)_F$. In particular φ^F is 1-supermedian for $(R_\alpha)_{\alpha > 0}$ and

$$(5) \quad \overline{(G_1 \varphi)_F} = \sup_{n \geq 1} nR_{n+1}(G_1 \varphi)_F \leq \sup_{n \geq 1} nR_{n+1} \varphi^F \leq \varphi^F \quad \mathcal{E}\text{-q.e.},$$

furthermore

$$R_1 \varphi - \varphi^F \text{ is } \begin{cases} > 0 & \mathcal{E}\text{-q.e. on } F^c \\ 0 & \text{on } F. \end{cases}$$

Since $\mu_{(\widehat{G}_1 g)_F}$ does not charge \mathcal{E} -exceptional sets it follows from (5) that

$$\int (R_1 \varphi - \varphi^F) d\mu_{(\widehat{G}_1 g)_F} \leq \int (R_1 \varphi - \overline{(G_1 \varphi)_F}) d\mu_{(\widehat{G}_1 g)_F}$$

but the expression on the right hand side is equal to zero since by $G_1 \varphi = (G_1 \varphi)_F$ m -a.e. on F

$$\begin{aligned} \int R_1 \varphi d\mu_{(\widehat{G}_1 g)_F} &= \lim_{\alpha \rightarrow \infty} \alpha(G_1 \varphi, ((\widehat{G}_1 g)_F^\alpha - \widehat{G}_1 g \cdot 1_F)^-)_\mathcal{H} \\ &= \lim_{\alpha \rightarrow \infty} \alpha((G_1 \varphi)_F, ((\widehat{G}_1 g)_F^\alpha - \widehat{G}_1 g \cdot 1_F)^-)_\mathcal{H} \\ &= \lim_{\alpha \rightarrow \infty} \lim_{\beta \rightarrow \infty} \mathcal{E}_1(\beta R_{\beta+1}(G_1 \varphi)_F, (\widehat{G}_1 g)_F^\alpha) \\ &\leq \sup_{\beta \geq 1} \int \beta R_{\beta+1}(G_1 \varphi)_F d\mu_{(\widehat{G}_1 g)_F} = \int \overline{(G_1 \varphi)_F} d\mu_{(\widehat{G}_1 g)_F}. \end{aligned}$$

For intermediate steps cf. [15, III.1, p. 60ff]. Now $\mu_{(\widehat{G}_1 g)_F}(F^c) = 0$ follows by a standard argument, because $\mu_{(\widehat{G}_1 g)_F}$ is σ -finite. □

As a generalization of [6, p. 78] we introduce the following class of measures

$$\widehat{S}_{00} := \{\mu_{\widehat{u}} \mid \widehat{u} \in \widehat{\mathcal{P}}_{\widehat{G}_1 \mathcal{H}_b^+} \text{ and } \mu_{\widehat{u}}(E) < \infty\}$$

where \mathcal{H}_b^+ are the positive and bounded elements in \mathcal{H} . Then we have

Theorem 2.5. *For $B \in \mathcal{B}(E)$ the following conditions are equivalent:*

- (i) B is \mathcal{E} -exceptional
- (ii) $\mu(B) = 0 \ \forall \mu \in \widehat{S}_{00}$

Proof. (i) \Rightarrow (ii) is clear. Next we will show $\neg(\text{i}) \Rightarrow \neg(\text{ii})$. Fix $\varphi \in L^1(E; m) \cap \mathcal{B}(E)$ such that $0 < \varphi \leq 1$. If B is not \mathcal{E} -exceptional then

$$\text{cap}_{\varphi}(B) = \inf\{((\widehat{G}_1 \varphi)_U, \varphi)_{\mathcal{H}} \mid U \supset B, U \text{ open}\} > 0.$$

Since cap_{φ} is regular there exists a compact $K \subset B$ with $\text{cap}_{\varphi}(K) > 0$. Let $D_0^+ - D_0^+ = \{f_n; n \in \mathbb{N}\}$ be a countable dense subset of bounded functions in \mathcal{F} with \mathcal{E} -q.c. m -versions $\overline{D}_0^+ - \overline{D}_0^+ = \{\tilde{f}_n; n \in \mathbb{N}\} \subset \widetilde{D}_0^+ - \widetilde{D}_0^+ \subset \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ which separate the points of $E \setminus N$ where N is an \mathcal{E} -exceptional set (cf. [15, Proposition IV.1.9.(ii), p. 77] for the existence). There exists further (cf. [15, Lemma IV.1.10., p. 77]) an \mathcal{E} -nest $(F_k)_{k \in \mathbb{N}}$ consisting of compact metrizable sets such that $\{R_1 \varphi, \tilde{f}_n; n \in \mathbb{N}\} \subset C(\{F_k\})$ and such that $R_1 \varphi \geq 1/k$ \mathcal{E} -q.e. on F_k for all $k \geq 1$. We may further assume, that $N \subset \bigcap_{k \geq 1} F_k^c$. Choose $n_0 \in \mathbb{N}$ such that $\text{cap}_{\varphi}(K \cap F_k) > 0$ for all $k \geq n_0$. Since $\text{cap}_{\varphi}(F_k^c) \xrightarrow[k \rightarrow \infty]{} 0$, there exists $k_0 \geq n_0$ with

$$(6) \quad \text{cap}_{\varphi}(K \cap F_{k_0}) - \text{cap}_{\varphi}(F_{k_0}^c) > 0.$$

Let $\rho_{F_{k_0}}$ be a metric on F_{k_0} which is compatible with the relative topology on F_{k_0} inherited from E . Define for $n \in \mathbb{N}$

$$B_n := \left\{ z \in F_{k_0} \mid \rho_{F_{k_0}}(z, K \cap F_{k_0}) < \frac{1}{n} \right\},$$

$$\overline{B}_n := \left\{ z \in F_{k_0} \mid \rho_{F_{k_0}}(z, K \cap F_{k_0}) \leq \frac{1}{n} \right\}.$$

Then $B_n \cup F_{k_0}^c \subset E$ is open for all n and thus

$$\begin{aligned} \text{cap}_{\varphi}(K \cap F_{k_0}) &= \inf_{\substack{K \cap F_{k_0} \subset U \\ U \text{ open}}} \text{cap}_{\varphi}(U) \\ &\leq \inf_{n \geq 1} \text{cap}_{\varphi}(B_n \cup F_{k_0}^c) \\ &= \inf_{n \geq 1} \mathcal{E}_1(G_1 \varphi, (\widehat{G}_1 \varphi)_{B_n \cup F_{k_0}^c}) \\ &\leq \inf_{n \geq 1} \int R_1 \varphi \, d\mu_{(\widehat{G}_1 \varphi)_{\overline{B}_n}} + \text{cap}_{\varphi}(F_{k_0}^c) \end{aligned}$$

since $(\widehat{G}_1\varphi)_{B_n \cup F_{k_0}^c} \leq (\widehat{G}_1\varphi)_{\overline{B}_n} + (\widehat{G}_1\varphi)_{F_{k_0}^c}$. It now follows from (6) that

$$(7) \quad 0 < \int R_1\varphi d\mu_{(\widehat{G}_1\varphi)_\infty}$$

where $(\widehat{G}_1\varphi)_\infty$ is defined to be the weak limit of $((\widehat{G}_1\varphi)_{\overline{B}_n})_{n \in \mathbb{N}}$ in \mathcal{V} . Note that $(\widehat{G}_1\varphi)_\infty \in \widehat{\mathcal{P}}_{\widehat{G}_1\mathcal{H}_t^+}$ and thus there exists a unique $\mu_{(\widehat{G}_1\varphi)_\infty}$ by Theorem 2.3. For convenience we set $\hat{\mu}_n := \mu_{(\widehat{G}_1\varphi)_{\overline{B}_n}}$, $n \in \mathbb{N}$, and $\hat{\mu}_\infty := \mu_{(\widehat{G}_1\varphi)_\infty}$. We have to show $\text{supp}(\hat{\mu}_\infty) \subset K \cap F_{k_0}$, because then by (7)

$$0 < \int R_1\varphi d\hat{\mu}_\infty = \int R_1\varphi 1_{K \cap F_{k_0}} d\hat{\mu}_\infty \leq \hat{\mu}_\infty(K \cap F_{k_0}) \leq \hat{\mu}_\infty(B).$$

Clearly $\text{supp}(\hat{\mu}_\infty) \subset K \cap F_{k_0}$ implies $\hat{\mu}_\infty(E) < \infty$ and thus $\hat{\mu}_\infty \in \hat{\mathcal{S}}_{00}$. We will proceed in several steps.

Step 1. There exists a subsequence such that $\hat{\mu}_{n_k}$ converges weakly to some μ : By Lemma 2.4 we know that $\text{supp}(\hat{\mu}_n) \subset F_{k_0}$ for all $n \in \mathbb{N}$ and thus we have for all $f \in C_0(F_{k_0})$

$$\begin{aligned} \sup_{n \in \mathbb{N}} \left| \int f d\hat{\mu}_n \right| &\leq \|f\|_\infty \sup_{n \in \mathbb{N}} \hat{\mu}_n(F_{k_0}) \\ &\leq \|f\|_\infty \sup_{n \in \mathbb{N}} \mathcal{E}_1(k_0 G_1\varphi, (\widehat{G}_1\varphi)_{\overline{B}_n}) \\ &\leq \|f\|_\infty \mathcal{E}_1(k_0 G_1\varphi, \widehat{G}_1\varphi) \\ &< \infty. \end{aligned}$$

It follows that $\{\hat{\mu}_n; n \in \mathbb{N}\}$ is relatively compact for the vague topology. Let us choose a subsequence $(\hat{\mu}_{n_k})_{k \in \mathbb{N}}$ which is convergent to some μ with respect to the vague topology. Since F_{k_0} is compact it follows that $(\hat{\mu}_{n_k})_{k \in \mathbb{N}}$ is weakly convergent to μ .

Step 2. μ is finite and $\text{supp}(\mu) \subset K \cap F_{k_0}$: Since $1_{F_{k_0}} \in C_b(F_{k_0})$ it follows that

$$\mu(F_{k_0}) = \lim_{k \rightarrow \infty} \hat{\mu}_{n_k}(F_{k_0}) \leq k_0 \mathcal{E}_1(G_1\varphi, \widehat{G}_1\varphi) < \infty.$$

Further, since $\overline{B}_j^c \uparrow (K \cap F_{k_0})^c$ as $j \rightarrow \infty$ (the complements are taken in F_{k_0}) we conclude by the Porte-manteau-Theorem and Lemma 2.4 that

$$\mu((K \cap F_{k_0})^c) = \lim_{j \rightarrow \infty} \mu(\overline{B}_j^c) \leq \lim_{j \rightarrow \infty} \liminf_{n_k \geq j} \hat{\mu}_{n_k}(\overline{B}_j^c) = 0.$$

Step 3. μ does not charge \mathcal{E} -exceptional sets: Setting $\hat{\mu}(A) = \mu(A \cap F_{k_0})$ for $A \in \mathcal{B}(E)$ we may interpret μ as a Borel measure on E . We will make no distinction between μ and $\hat{\mu}$ in the following. Let $(E_k)_{k \in \mathbb{N}}$ be an

arbitrary \mathcal{E} -nest. Then (with the complements in E)

$$\begin{aligned} \mu\left(\bigcap_{k \geq 1} E_k^c\right) &= \lim_{k \rightarrow \infty} \mu(E_k^c) \\ &\leq \lim_{k \rightarrow \infty} \liminf_{j \rightarrow \infty} \hat{\mu}_{n_j}(E_k^c) \\ &\leq \lim_{k \rightarrow \infty} \liminf_{j \rightarrow \infty} \int_{E_k^c} k_0 R_1 \varphi d\hat{\mu}_{n_j} \\ &\leq k_0 \lim_{k \rightarrow \infty} \liminf_{j \rightarrow \infty} \int E. \left[\int_{\sigma_{E_k^c}} e^{-t} \varphi(Y_t) dt \right] d\hat{\mu}_{n_j} \\ &\leq k_0 \lim_{k \rightarrow \infty} \text{cap}_\varphi(E_k^c) \\ &= 0 \end{aligned}$$

implies that $\mu(N) = 0$ for every \mathcal{E} -exceptional set $N \in \mathcal{B}(E)$.

Step 4. $\mu = \hat{\mu}_\infty$:

Let $f \in \mathcal{F}$. There exists $(f_{m_k})_{k \in \mathbb{N}} \subset D_0^+ - D_0^+$ such that $\lim_{k \rightarrow \infty} f_{m_k} = f$ in \mathcal{F} . By [15, Corollary III.3.8., p. 73] we may assume that $(\tilde{f}_{m_k})_{k \in \mathbb{N}} \subset \overline{D_0^+} - \overline{D_0^+}$ converges \mathcal{E} -q.e. to some \mathcal{E} -q.c m -version \tilde{f} of f . We will show that $(\tilde{f}_{m_k})_{k \in \mathbb{N}}$ is $L^1(\mu)$ -Cauchy. Since $|\tilde{f}_{m_k} - \tilde{f}_{m_l}| \in C_b(F_{k_0})$ and $|\tilde{f}_{m_k} - \tilde{f}_{m_l}| \leq \bar{e}_{f_{m_k} - f_{m_l}} + \bar{e}_{f_{m_l} - f_{m_k}}$ \mathcal{E} -q.e. where $\bar{e}_{f_{m_k} - f_{m_l}}, \bar{e}_{f_{m_l} - f_{m_k}}$ are some \mathcal{E} -q.l.s.c. regularizations we have

$$\begin{aligned} \int |\tilde{f}_{m_k} - \tilde{f}_{m_l}| d\mu &\leq \lim_{j \rightarrow \infty} \int (\bar{e}_{f_{m_k} - f_{m_l}} + \bar{e}_{f_{m_l} - f_{m_k}}) d\hat{\mu}_{n_j} \\ &\leq \mathcal{E}_1(e_{f_{m_k} - f_{m_l}} + e_{f_{m_l} - f_{m_k}}, \hat{G}_1 \varphi) \\ &\leq \|e_{f_{m_k} - f_{m_l}} + e_{f_{m_l} - f_{m_k}}\|_{\mathcal{H}} \|\varphi\|_{\mathcal{H}} \end{aligned}$$

and we conclude by [15, Lemma III.2.2.(i), p. 66]. Then for a new subsequence eventually

$$\begin{aligned} \int \tilde{f} d\mu &= \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \int \tilde{f}_{m_k} d\hat{\mu}_{n_j} \\ &= \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \mathcal{E}_1(f_{m_k}, (\hat{G}_1 \varphi)_{\overline{B}_j}) \\ &= \mathcal{E}_1(f, (\hat{G}_1 \varphi)_\infty), \end{aligned}$$

but since μ does not charge \mathcal{E} -exceptional sets by Step 3 the equality holds for every \mathcal{E} -q.c. m -version \tilde{f} of f . From this it is easy to see that $\mu = \hat{\mu}_\infty$. \square

REMARK 2.6. (i) Note that if every element in $\mathcal{P}_{G_1 \mathcal{H}_b^+}$ admits an \mathcal{E} -q.c. m -version then before Step 1 in the proof of Theorem 2.5 one can show directly $\text{supp}(\hat{\mu}_\infty) \subset K$. Indeed we may assume that $G_1 \varphi, (G_1 \varphi)_{\overline{B}_j}, \alpha R_{\alpha+1}(G_1 \varphi)_{\overline{B}_j}$ are continuous on \overline{B}_j for every $j, \alpha \in \mathbb{N}$. We may also assume that $G_1 \varphi \geq 1/k_0$ on each \overline{B}_j .

We then have for each $j \in \mathbb{N}$

$$\begin{aligned} \int G_1\varphi - (G_1\varphi)_{\overline{B}_j} d\hat{\mu}_\infty &= \lim_{\alpha \rightarrow \infty} \lim_{n \geq j} \lim_{\beta \rightarrow \infty} \int ((G_1\varphi)_{\overline{B}_j} - \alpha R_{\alpha+1}(G_1\varphi)_{\overline{B}_j}) d\mu_{(\hat{G}_1\varphi)_{\overline{B}_n}^\beta} \\ &\leq \lim_{\alpha \rightarrow \infty} \|(G_1\varphi)_{\overline{B}_j} - \alpha R_{\alpha+1}(G_1\varphi)_{\overline{B}_j}\|_{\infty, \overline{B}_j} \int k_0 G_1\varphi d\hat{\mu}_\infty \end{aligned}$$

and the last expression is zero by Dini's Theorem (here $\|\cdot\|_{\infty, \overline{B}_j}$ denotes the sup norm on the compact space \overline{B}_j). We then conclude as in the proof of Lemma 2.4.

(ii) The assertion of Theorem 2.5 remains true if we replace \hat{S}_{00} by $\{\mu \in \hat{S}_{00} \mid \mu(E) = 1\}$. Note also that if $(\hat{G}_\alpha)_{\alpha>0}$ is sub-Markovian we may replace \hat{S}_{00} by the larger class $\{\mu_{\hat{u}} \mid \|\hat{u}\|_\infty < \infty \text{ and } \mu_{\hat{u}}(E) < \infty\}$ and then our definition coincides with the one of [6, p. 78].

(iii) Similar to [6], [12] it is possible to define the measures of finite (co-)energy integral and to show that these measures have properties similar to those in [6], [12]. We will also use the notation $\hat{U}_1\mu_{\hat{u}}$ for \hat{u} .

3. Positive continuous additive functionals and Revuz measure

A family $(A_t)_{t \geq 0}$ of functions on Ω is called an *additive functional* (abbreviated AF) of $\mathbf{M} = (\Omega, (\mathcal{F}_t)_{t \geq 0}, (Y_t)_{t \geq 0}, (P_z)_{z \in E_\Delta})$, if:

- (i) $A_t(\cdot)$ is \mathcal{F}_t -measurable for all $t \geq 0$.
- (ii) There exists a *defining* set $\Lambda \in \mathcal{F}_\infty$ and an \mathcal{E} -exceptional set $N \subset E$, such that $P_z[\Lambda] = 1$ for all $z \in E \setminus N$, $\theta_t(\Lambda) \subset \Lambda$ for all $t > 0$ and for each $\omega \in \Lambda$, $t \mapsto A_t(\omega)$ is right continuous on $[0, \infty[$ and has left limits on $]0, \zeta(\omega)[$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_\zeta(\omega)$ for $t \geq \zeta(\omega)$ and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \geq 0$.

An AF A is called a *continuous additive functional* (abbreviated CAF), if $t \mapsto A_t(\omega)$ is continuous on $[0, \infty)$, a *positive, continuous additive functional* (abbreviated PCAF) if $A_t(\omega) \geq 0$ and a finite AF, if $|A_t(\omega)| < \infty$ for all $t \geq 0, \omega \in \Lambda$. Two AF's A, B are said to be equivalent (in notation $A = B$) if for each $t > 0$ $P_z(A_t = B_t) = 1$ for \mathcal{E} -q.e. $z \in E$. For a Borel measure ν on E and $B \in \mathcal{B}(E)$ let $P_\nu(B) := \int P_z(B)\nu(dz)$ and E_ν be the expectation w.r.t. P_ν . The energy of an AF A of \mathbf{M} is then defined by

$$(8) \quad e(A) = \frac{1}{2} \lim_{\alpha \rightarrow \infty} \alpha^2 E_m \left[\int_0^\infty e^{-\alpha t} A_t^2 dt \right],$$

whenever this limit exists in $[0, \infty]$. We will set $\bar{e}(A)$ for the same expression but with $\overline{\lim}$ instead of \lim .

From now on let us assume that the coresolvent $(\hat{G}_\alpha)_{\alpha>0}$ associated with \mathcal{E} is sub-Markovian.

Theorem 3.1. *Let A be a PCAF of \mathbf{M} . Then there exists a unique positive measure μ_A on $(E, \mathcal{B}(E))$, charging no \mathcal{E} -exceptional set and called the Revuz measure of*

A, such that

$$(9) \quad \int_E f \, d\mu_A = \lim_{\alpha \rightarrow \infty} \alpha E_m \left[\int_0^\infty e^{-\alpha t} f(Y_t) \, dA_t \right] \quad \text{for all } f \in \mathcal{B}(E)^+.$$

Furthermore, there exists an \mathcal{E} -nest of compact sets $(\tilde{F}_n)_{n \geq 1}$ such that $\mu_A(\tilde{F}_n) < \infty$ and such that $U_A^1 1_{\tilde{F}_n}$ is an \mathcal{E} -q.l.s.c. m -version of some element in $\mathcal{P}_{G_1 \mathcal{H}_1^*}$ for each n .

Proof. As usual we set $U_\alpha^A f(z) := E_z[\int_0^\infty e^{-\alpha t} f(Y_t) \, dA_t]$ for a PCAF A of \mathbf{M} and $f \in \mathcal{B}(E)^+$. Then we have the following resolvent equations (cf. e.g. [14, 36.16]) for $0 \leq \alpha < \beta$ and $f \in \mathcal{B}(E)^+$

$$(10) \quad U_\alpha^A f = U_\alpha^\beta f + (\beta - \alpha)R_\alpha U_\alpha^\beta f = U_\alpha^\beta f + (\beta - \alpha)R_\beta U_\alpha^\alpha f,$$

but one has to be careful not to subtract because we make no finiteness assumptions on $U_\alpha^\beta f$. As usual the sub-Markovianity of $(\tilde{G}_\alpha)_{\alpha > 0}$ then implies the existence of μ_A . Clearly μ_A does not charge \mathcal{E} -exceptional sets. Fix $\varphi \in L^1(E; m) \cap \mathcal{B}(E)$ such that $0 < \varphi \leq 1$. For $z \in E$ set

$$\Phi(z) := E_z \left[\int_0^\zeta e^{-s} \varphi(Y_s) e^{-A_s} \, ds \right].$$

By the original proof of D. Revuz (cf. [13, p. 509]) we have

$$U_A^1 \Phi = R_1 \varphi - \Phi \quad \mathcal{E}\text{-q.e.}$$

and since $U_A^1 \Phi$ is \mathcal{E} -q.l.s.c. $-\Phi$ is also \mathcal{E} -q.l.s.c. Let $(F_n)_{n \geq 1}$ be an \mathcal{E} -nest of compact sets such that $-\Phi \in C_l(\{F_n\})$. Then

$$\tilde{F}_n := \left\{ \Phi \geq \frac{1}{n} \right\} \cap F_n, \quad n \geq 1,$$

are compact subsets of E and similar to the proof of [6, Lemma 5.1.7.] one can show that $(\tilde{F}_n)_{n \geq 1}$ is an \mathcal{E} -nest. Finally (as in [13, Lemme II.2, p. 508]) $U_A^1 1_{\tilde{F}_n} \leq n U_A^1 \Phi \leq n R_1 \varphi$ \mathcal{E} -q.e. implies that $\mu_A(\tilde{F}_n) < \infty$, $n \geq 1$. Indeed, by the resolvent equation (10) and the sub-Markovianity of $(\tilde{G}_\alpha)_{\alpha > 0}$ we have for all $\beta \geq 1$, $n \geq 1$

$$\begin{aligned} & \int \beta(R_\beta n \varphi - U_A^\beta 1_{\tilde{F}_n}) \, dm \\ &= \int (\beta(R_1 n \varphi - U_A^1 1_{\tilde{F}_n}) - (\beta - 1)\beta R_\beta (R_1 n \varphi - U_A^1 1_{\tilde{F}_n})) \, dm \\ &\geq \int (R_1 n \varphi - U_A^1 1_{\tilde{F}_n}) \, dm \geq 0, \end{aligned}$$

hence $\int \beta R_{\beta} n \varphi \, dm \geq \int \beta U_A^{\beta} 1_{\tilde{F}_n} \, dm$ for all $\beta \geq 1$, and therefore $\mu_A(\tilde{F}_n) \leq \int n \varphi \, dm < \infty$, $n \geq 1$. Clearly $U_A^1 1_{\tilde{F}_n}$ is \mathcal{E} -q.l.s.c. and by [15, Lemma III.2.1.(i), p. 65] $U_A^1 1_{\tilde{F}_n}$ is an m -version of some element in $\mathcal{P}_{G_1 \mathcal{H}_b^+}$ for each n . \square

Let A be a PCAF and let μ_A be the associated Revuz measure of Theorem 3.1. From its proof we know that there exists an \mathcal{E} -nest of compact sets $(\tilde{F}_n)_{n \in \mathbb{N}}$ such that $\mu_A(\tilde{F}_n) < \infty$ and such that $U_A^1 1_{\tilde{F}_n}$ is an \mathcal{E} -q.l.s.c. m -version of some element in $\mathcal{P}_{G_1 \mathcal{H}_b^+}$. Since $E[A_t] \leq e^t \sup_{n \geq 1} U_A^1 1_{\tilde{F}_n}$ \mathcal{E} -q.e. it is straightforward to see by (10) that for any $\nu \in \hat{S}_{00}$, $t > 0$

$$(11) \quad E_{\nu}[A_t] \leq e^t \|\hat{U}_1 \nu\|_{\infty} \mu_A(E).$$

4. Fukushima's decomposition of AF's

For the proof of Theorem 4.5 below we follow the same strategy as in [6, Chapter 5]. Let \tilde{u} be an \mathcal{E} -q.c. function. Then $(\tilde{u}(Y_t) - \tilde{u}(Y_0))_{t \geq 0}$ is an AF of \mathbf{M} , and independent (up to equivalence) of the special choice \tilde{u} . We then set

$$(12) \quad A^{[\tilde{u}]} = (\tilde{u}(Y_t) - \tilde{u}(Y_0))_{t \geq 0}.$$

It follows from the sub-Markovianity of $(\hat{G}_{\alpha})_{\alpha > 0}$ that for $\tilde{u} \in \tilde{\mathcal{H}}$

$$(13) \quad \bar{e}(A^{[\tilde{u}]}) \leq \overline{\lim}_{\alpha \rightarrow \infty} \alpha(u - \alpha G_{\alpha} u, u)_{\mathcal{H}}.$$

Note that in the case where $(\hat{G}_{\alpha})_{\alpha > 0}$ is strongly continuous on \mathcal{V} the right hand side above is equal to $\mathcal{E}(u, u)$ for all $u \in \mathcal{F}$. Define

$$\mathcal{M} := \{M \mid M \text{ is a finite AF, } E_z[M_t^2] < \infty, E_z[M_t] = 0 \\ \text{for } \mathcal{E}\text{-q.e. } z \in E \text{ and all } t \geq 0\}.$$

$M \in \mathcal{M}$ is called a *martingale additive functional* (MAF). Furthermore define

$$(14) \quad \overset{\circ}{\mathcal{M}} = \{M \in \mathcal{M} \mid e(M) < \infty\}.$$

The elements of $\overset{\circ}{\mathcal{M}}$ are called MAF's of finite energy.

Let $M \in \mathcal{M}$. There exists an \mathcal{E} -exceptional set N_M , such that $(M_t, \mathcal{F}_t, P_z)_{t \geq 0}$ is a square integrable martingale for all $z \in E \setminus N_M$. Analogous to [9, III. Théorème 3] or see also [6, A. 3] there exists after a method of perfection a unique (up to equivalence) PCAF $\langle M \rangle$, called the sharp bracket of M , such that $(M_t^2 - \langle M \rangle_t, \mathcal{F}_t, P_z)_{t \geq 0}$ is a martingale for all $z \in E \setminus N_M$. It then follows that one half of the total mass of the Revuz measure $\mu_{\langle M \rangle}$ associated to the quadratic variation of $M \in \mathcal{M}$ is equal to the

energy of M , i.e.

$$(15) \quad e(M) = \frac{1}{2} \int_E d\mu_{\langle M \rangle}.$$

For $M, L \in \overset{\circ}{\mathcal{M}}$ let

$$\langle M, L \rangle_t = \frac{1}{2} (\langle M + L \rangle_t - \langle M \rangle_t - \langle L \rangle_t).$$

Then $(\langle M, L \rangle_t)_{t \geq 0}$ is a CAF of bounded variation on each finite interval of t and satisfies

$$E_z(M_t L_t) = E_z(\langle M, L \rangle_t) \quad \forall t \geq 0, \mathcal{E}\text{-q.e. } z \in E.$$

Furthermore the finite signed measure $\mu_{\langle M, L \rangle}$ defined by $\mu_{\langle M, L \rangle} = (1/2)(\mu_{\langle M+L \rangle} - \mu_{\langle M \rangle} - \mu_{\langle L \rangle})$ is related to $\langle M, L \rangle$ in the sense of Theorem 3.1. If $f \in \mathcal{B}_b(E)^+$, then $f \bullet \mu_{\langle \cdot, \cdot \rangle}$ is symmetric, bilinear and positive on $\overset{\circ}{\mathcal{M}} \times \overset{\circ}{\mathcal{M}}$, where $f \bullet \mu_{\langle M, L \rangle}(A) := \int_A f d\mu_{\langle M, L \rangle}$ for every $A \in \mathcal{B}(E)$ and every pair $(M, L) \in \overset{\circ}{\mathcal{M}} \times \overset{\circ}{\mathcal{M}}$. Define

$$\mathcal{N}_c = \{N \mid N \text{ is a finite CAF, } e(N) = 0, E_z[|N_t|] < \infty \text{ for } \mathcal{E}\text{-q.e. } z \in E \text{ and all } t \geq 0\}.$$

The ‘‘isometry’’ (15) and the continuity statement (13) are fundamental for the stochastic calculus related to \mathcal{E} .

We set $\mathcal{A} = \overset{\circ}{\mathcal{M}} \oplus \mathcal{N}_c$. Namely \mathcal{A} consists of AF’s such that

$$A_t = M_t + N_t \quad M \in \overset{\circ}{\mathcal{M}}, N \in \mathcal{N}_c.$$

\mathcal{A} is a linear space of AF’s of finite energy. Furthermore by (11) this decomposition is unique. We define the mutual energy of $A, B \in \mathcal{A}$ by

$$e(A, B) = \frac{1}{2} \lim_{\alpha \rightarrow \infty} \alpha^2 E_m \left[\int_0^\infty e^{-\alpha t} A_t B_t dt \right].$$

By the Cauchy-Schwarz inequality we know that $e(A, B) = 0$ when either A or B is in \mathcal{N}_c . Therefore

$$(16) \quad e(A) = e(M) \quad \text{if } A = M + N, \quad M \in \overset{\circ}{\mathcal{M}}, \quad N \in \mathcal{N}_c.$$

Using Theorem 2.5 and the Lemma of Borel-Cantelli the proof of the following lemma is similar to the proof of [6, Lemma 5.1.2.(i), p. 182]

Lemma 4.1. *Let $(F_k)_{k \geq 1}$ be an \mathcal{E} -nest. Let $\tilde{u}, \tilde{u}_n \in C(\{F_k\})$, $n \in \mathbb{N}$. Let $(S_n)_{n \in \mathbb{N}} \subset \mathbb{R}$, such that $\lim_{n \rightarrow \infty} S_n = 0$. Suppose that there exists for each $\mu \in \widehat{S}_{00}$*

and $T > 0$ a constant $C^{T,\mu}$, such that

$$P_\mu \left(\sup_{0 \leq t \leq T} |\tilde{u}(Y_t) - \tilde{u}_n(Y_t)| > \varepsilon \right) \leq \frac{C^{T,\mu}}{\varepsilon} S_n.$$

Then there exists a subsequence $(\tilde{u}_{n_k})_{k \in \mathbb{N}}$, such that for \mathcal{E} -q.e. $z \in E$

$$P_z(\tilde{u}_{n_k}(Y_t) \text{ converges to } \tilde{u}(Y_t) \text{ uniformly in } t \text{ on each compact interval of } [0, \infty)) = 1.$$

In contrast to [6], [12] in the following lemma we determine convergence in a weaker semi-norm (cf. Remark 4.3 below).

Lemma 4.2. *Let $\tilde{u} \in \tilde{\mathcal{H}}_{\mathcal{F}^\pm}$ where $\mathcal{H}_{\mathcal{F}^\pm} := \{u \in \mathcal{H} \mid u, -u \in \mathcal{H}_{\mathcal{F}}\}$ and let $\varepsilon > 0$. Then we have for any $\mu \in \widehat{\mathcal{S}}_{00}$ and $T > 0$*

$$P_\mu \left(\sup_{0 \leq t \leq T} |\tilde{u}(Y_t)| > \varepsilon \right) \leq \frac{e^T}{\varepsilon} \|h\|_{\mathcal{H}} \|e_u + e_{-u}\|_{\mathcal{H}},$$

where h is in \mathcal{H}_b^+ such that $\widehat{U}_1\mu \leq \widehat{G}_1h$.

Proof. Set $U = \{|\tilde{u}| > \varepsilon\}$. Then $\{\sup_{0 \leq t \leq T} |\tilde{u}(Y_t)| > \varepsilon\} \subset \{\sigma_U \leq T; \sigma_U < \zeta\}$ where σ_U is the first hitting time of U , i.e. $\sigma_U = \inf\{t > 0 \mid Y_t \in U\}$. By the right-continuity of the associated process we have P_z -a.s. for \mathcal{E} -q.e. $z \in E$

$$\frac{e^{T-\sigma_U}}{\varepsilon} |\tilde{u}(Y_{\sigma_U})| \text{ is } \begin{cases} \geq 1 & \text{on } \{\sigma_U \leq T; \sigma_U < \zeta\} \\ \geq 0 & \text{elsewhere.} \end{cases}$$

Let \bar{e}_u, \bar{e}_{-u} be \mathcal{E} -q.l.s.c. regularizations of e_u, e_{-u} . Since $|\tilde{u}| \leq \bar{e}_u + \bar{e}_{-u}$ \mathcal{E} -q.e. by making use of our smaller class of measures $\widehat{\mathcal{S}}_{00}$ it is just straightforward to conclude. □

REMARK 4.3. Let us define a semi-norm on $\tilde{\mathcal{H}}_{\mathcal{F}^\pm}$ by $\|v\|_e := \|e_v + e_{-v}\|_{\mathcal{H}}$. Let $(\tilde{u}_n)_{n \in \mathbb{N}} \subset \tilde{\mathcal{H}}_{\mathcal{F}^\pm}$ be $\|\cdot\|_e$ -convergent to $\tilde{u} \in \tilde{\mathcal{H}}_{\mathcal{F}^\pm}$. Then, using Lemma 4.2 we see that Lemma 4.1 applies. Since for $f \in \mathcal{F}$, $\|f\|_e \leq 6K\|f\|_{\mathcal{F}}$ we have in particular, that if $u, u_n \in \mathcal{F}$, $n \in \mathbb{N}$ such that $u_n \xrightarrow[n \rightarrow \infty]{} u$ in \mathcal{F} then $u_n \xrightarrow[n \rightarrow \infty]{} u$ w.r.t. $\|\cdot\|_e$.

Using Theorem 2.5 and (11) the proof of the following theorem is similar to [6, p. 203].

Theorem 4.4. *Let $(M^n)_{n \in \mathbb{N}} \subset \overset{\circ}{\mathcal{M}}$ be e -Cauchy. Then there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ and a unique $M \in \overset{\circ}{\mathcal{M}}$, such that $\lim_{n \rightarrow \infty} e(M^n - M) = 0$ and for \mathcal{E} -q.e. $z \in E$*

$$P_z(\lim_{k \rightarrow \infty} M_t^{n_k} = M_t \text{ uniformly in } t \text{ on each compact interval of } [0, \infty)) = 1.$$

Let us assume from now on that $(\widehat{G}_\alpha)_{\alpha>0}$ is strongly continuous on \mathcal{V} .

Theorem 4.5. (i) *Let $u \in \mathcal{F}$. There exists a unique $M^{[u]} \in \overset{\circ}{\mathcal{M}}$ and a unique $N^{[u]} \in \mathcal{N}_c$ such that*

$$(17) \quad A^{[u]} = M^{[u]} + N^{[u]}.$$

(ii) *Let $(F_k)_{k \geq 1}$ be an \mathcal{E} -nest. Let $\tilde{u}, \tilde{u}_n \in C(\{F_k\})$, $n \in \mathbb{N}$, such that we have (17) for $A^{[u_n]}$, $n \in \mathbb{N}$ and such that $\bar{e}(A^{[u_n - u]}) \xrightarrow{n \rightarrow \infty} 0$. Assume furthermore that the conditions of Lemma 4.1 are satisfied for \tilde{u}, \tilde{u}_n , $n \in \mathbb{N}$. Then (17) extends to $A^{[u]}$.*

REMARK 4.6. (i) In general it is not possible to find a decomposition of the additive functional $(\bar{u}(Y_t) - \bar{u}(Y_0))_{t \geq 0}$ for all $u \in \mathcal{P}$ of type (17) where $N_t^{[u]}$ is of zero energy. Here \bar{u} denotes a 1-excessive regularization of u . As an example consider the uniform motion (to the right) on the real line and $\bar{u}(x) = e^x 1_{(-\infty, 0)}(x)$, $x \in \mathbb{R}$.

(ii) If we do not require the strong continuity of $(\widehat{G}_\alpha)_{\alpha>0}$ on \mathcal{V} Theorem 4.5(i) applies to all $u \in \mathcal{F}$, such that $\alpha \widehat{G}_\alpha u \rightarrow u$ in \mathcal{V} as $\alpha \rightarrow \infty$.

Proof (of Theorem 4.5). After all preparations (among others Theorem 2.3, Theorem 2.5, Theorem 3.1, (11), Lemma 4.2) we can finally show (i) similar to the proof of the corresponding statement in [6, Theorem 5.2.2., p. 203ff]. Therefore we omit the proof of (i) and only show (ii).

Let (17) be valid for \tilde{u}_n , $n \in \mathbb{N}$. By the uniqueness of the decomposition we know that $M^{[u_n]} - M^{[u_m]} = M^{[u_n - u_m]}$. With (13) and (16) we have

$$e(M^{[u_n - u_m]}) = e(A^{[u_n - u_m]}) \leq 2\bar{e}(A^{[u - u_n]}) + 2\bar{e}(A^{[u - u_m]}).$$

It follows that $(M^{[u_n]})_{n \in \mathbb{N}} \subset \overset{\circ}{\mathcal{M}}$ is e -Cauchy and then by Theorem 4.4 it makes sense to set

$$\begin{aligned} M^{[u]} &= \lim_{n \rightarrow \infty} M^{[u_n]} \quad \text{in } (\overset{\circ}{\mathcal{M}}, e) \\ N^{[u]} &= A^{[u]} - M^{[u]}. \end{aligned}$$

It only remains to show $N^{[u]} \in \mathcal{N}_c$. Note that there exists a subsequence n_k such that for \mathcal{E} -q.e. $z \in E$

$$P_z(N_t^{[u_{n_k}]}) \text{ converges uniformly in } t \text{ on each compact interval of } [0, \infty) = 1$$

since the same is true for $A^{[u]}$ and $M^{[u]}$ by Lemma 4.1 and Theorem 4.4. Therefore $N^{[u]}$ is a CAF. Finally

$$\bar{e}(N^{[u]}) = \bar{e}(A^{[u - u_n]} - (M^{[u]} - M^{[u_n]}) + N^{[u_n]})$$

$$\leq 3\bar{e}(A^{[u-u_n]}) + 3e(M^{[u]} - M^{[u_n]})$$

implies that $N^{[u]}$ is of zero energy. □

5. An Itô-type formula

Lemma 5.1. *Let $f \in \mathcal{B}_b(E)$ and $M \in \overset{\circ}{\mathcal{M}}$. Then there exists a unique element denoted by $f \bullet M \in \overset{\circ}{\mathcal{M}}$, such that*

$$(18) \quad \frac{1}{2} \int_E f d\mu_{(M,L)} = e(f \bullet M, L) \quad \text{for all } L \in \overset{\circ}{\mathcal{M}}.$$

Proof. Let first $f \in \mathcal{B}_b(E)^+$ and $M, L \in \overset{\circ}{\mathcal{M}}$. Since

$$\begin{aligned} \left| \frac{1}{2} \int_E f d\mu_{(M,L)} \right| &\leq \left(\frac{1}{2} \int_E f d\mu_{(M)} \right)^{1/2} \left(\frac{1}{2} \int_E f d\mu_{(L)} \right)^{1/2} \\ &\leq \|f\|_\infty e(M)^{1/2} e(L)^{1/2} \end{aligned}$$

there exists a unique bounded linear map $T_f : \overset{\circ}{\mathcal{M}} \rightarrow \overset{\circ}{\mathcal{M}}$, such that $(1/2) \int_E f d\mu_{(M,L)} = e(T_f(M), L)$. Finally we set $f \bullet M := T_f(M)$ and $T_f(M) := T_{f^+}(M) - T_{f^-}(M)$ for $f = f^+ - f^- \in \mathcal{B}_b(E)$. □

Let us assume from now on that in **D1(ii)** the adjoint semigroup $(\hat{U}_t)_{t \geq 0}$ of $(U_t)_{t \geq 0}$ can also be restricted to a C_0 -semigroup on \mathcal{V} . Let $(\hat{\Lambda}, D(\hat{\Lambda}, \mathcal{H}))$ denote the generator of $(\hat{U}_t)_{t \geq 0}$ on \mathcal{H} , $\hat{A}(u, v) := \mathcal{A}(v, u)$, $u, v \in \mathcal{V}$ and let the coform $\hat{\mathcal{E}}$ be defined as the bilinear form associated with (\hat{A}, \mathcal{V}) and $(\hat{\Lambda}, D(\hat{\Lambda}, \mathcal{H}))$. Note that since $(\hat{G}_\alpha)_{\alpha > 0}$ was assumed to be sub-Markovian the corresponding statement of **D2** holds for the coform. Let us further assume from now on that the coform $\hat{\mathcal{E}}$ is quasi-regular too. We fix an m -tight special standard process $\hat{\mathbf{M}} = (\hat{\Omega}, (\hat{\mathcal{F}}_t)_{t \geq 0}, (\hat{Y}_t)_{t \geq 0}, (\hat{P}_z)_{z \in E_\Delta})$ such that $\hat{R}_\alpha f = \hat{E} \cdot [\int_0^\infty e^{-\alpha t} f(\hat{Y}_t) dt]$ is an $\hat{\mathcal{E}}$ -q.c. (= \mathcal{E} -q.c.) m -version of $\hat{G}_\alpha f$ for all $f \in \mathcal{H} \cap \mathcal{B}_b(E)$. Necessary and sufficient conditions for the existence of such a process are given in [15]. $\hat{\mathbf{M}}$ is then in duality to \mathbf{M} w.r.t. m .

REMARK 5.2. (i) Let $(\hat{\mathcal{P}}_{\hat{\mathcal{F}}})_b$ denote the set of bounded elements of $\hat{\mathcal{P}}_{\hat{\mathcal{F}}}$. By quasi-regularity of $\hat{\mathcal{E}}$ we know that $\widetilde{(\hat{\mathcal{P}}_{\hat{\mathcal{F}}})_b} - \widetilde{(\hat{\mathcal{P}}_{\hat{\mathcal{F}}})_b}$ separates the points of $E \setminus N$, where N is an $\hat{\mathcal{E}}$ -exceptional set. However, note that an element in $(\hat{\mathcal{P}}_{\hat{\mathcal{F}}})_b$ not necessarily admits an $\hat{\mathcal{E}}$ -q.c. m -version. Let μ, ν be finite measures on $(E, \mathcal{B}(E))$ charging no $\hat{\mathcal{E}}$ -exceptional set. Then by [14, Appendices A.0.8] it follows that $\mu = \nu$ if $\int \hat{R}_\alpha \tilde{f} d\mu = \int \hat{R}_\alpha \tilde{f} d\nu$ for all $\tilde{f} \in \widetilde{(\hat{\mathcal{P}}_{\hat{\mathcal{F}}})_b}$.

(ii) Let A, B be two PCAF's. If $\mu_A = \mu_B$ then $A = B$. Indeed if $\mu_A = \mu_B$ then by (10) and (19) for every \mathcal{E} -nest $(F_k)_{k \in \mathbb{N}}$ of compact sets $U_A^\beta 1_{F_k} = U_B^\beta 1_{F_k}$ \mathcal{E} -q.e. for

each $k \in \mathbb{N}$ and $\beta > 0$. We may even choose $(F_k)_{k \in \mathbb{N}}$ (cf. the proof of Theorem 3.1) such that $U_B^1 1_{F_k} \leq kR_1\varphi$ for every $k \in \mathbb{N}$. Then with $A^{(1)} := \int_0^\cdot 1_{F_k}(Y_s) dA_s$, $A^{(2)} := \int_0^\cdot 1_{F_k}(Y_s) dB_s$ for fixed k remarking that $\int v_{ij} dv \leq \int 2kR_1\varphi dv < \infty$ for any $v \in \hat{S}_{00}$ (where v_{ij} is as in [6, Theorem 5.1.2.]) we conclude as in the proof of [6, Theorem 5.1.2.]. Since $k \in \mathbb{N}$ was arbitrary we get $A = B$.

Let μ_A be the measure defined in Theorem 3.1. Then as in [11, Lemma 4.1.7., p. 91] one shows that for every $f \in \mathcal{B}(E)^+$, $g \in L^2(E; m) \cap \mathcal{B}_b(E)^+$, $\beta > 0$

$$(19) \quad \int f \widehat{R}_\beta g d\mu_A = \lim_{\alpha \rightarrow \infty} \alpha(U_A^{\alpha+\beta} f, \widehat{R}_\beta g)_\mathcal{H}.$$

Lemma 5.3. *For $f \in \mathcal{B}_b(E)$ and $M, L \in \overset{\circ}{\mathcal{M}}$. Then we have*

$$\mu_{(f \bullet M, L)} = f \bullet \mu_{(M, L)}.$$

Proof. Let $M \in \overset{\circ}{\mathcal{M}}$ and $f \in \mathcal{B}_b(E)$. Analogous to [9, Théorème 4, p. 127] or [6, Theorem A.3.19.] there exists (after perfection) a unique $\tilde{M} \in \overset{\circ}{\mathcal{M}}$, such that for all $L \in \overset{\circ}{\mathcal{M}}$, $t > 0$ and \mathcal{E} -q.e. $z \in E$

$$E_z[\langle \tilde{M}, L \rangle_t] = E_z \left[\int_0^t f(Y_s) d\langle M, L \rangle_s \right].$$

For all $g \in L^2(E; m) \cap \mathcal{B}_b(E)$, $\alpha > 0$, we then have

$$\int_E \widehat{R}_\alpha g d\mu_{(\tilde{M}, L)} = \int_E (\widehat{R}_\alpha g) f d\mu_{(M, L)},$$

and consequently

$$\mu_{(f \bullet M, L)} = f \bullet \mu_{(M, L)},$$

since $e(\tilde{M} - f \bullet M, L) = 0$, for all $L \in \overset{\circ}{\mathcal{M}}$ by the preceding Lemma. □

The proof of the next Theorem is based on the proof of Theorem 5.3.2. in [11, p.160]. For convenience we write $\mu_{(u, v)}$ instead of $\mu_{(M^{[u]}, M^{[v]})}$ and $\mu_{(u)}$ instead of $\mu_{(M^{[u]})}$.

Theorem 5.4 (Product rule). *Let $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_n)$ be an n -tuple of \mathcal{E} -q.c. m -versions of bounded elements in \mathcal{H} such that $A^{[\Phi(f)]}$ admits the decomposition of Theorem 4.5 for all $\Phi \in C^1(\mathbb{R}^n)$ with $\Phi(0) = 0$ and let $\bar{e}(A^{[f_i - kR_k f_i]}) \rightarrow 0$, as $k \rightarrow \infty$, $1 \leq i \leq n$. Let further the martingale part $M^{[u]}$ of the decomposition of Theorem 4.5*

be continuous for all u in $G_1\mathcal{H}_b$. Let $\Phi, \Psi \in C^1(\mathbb{R}^n)$, $\Phi(0) = \Psi(0) = 0$ and $\tilde{w} \in \tilde{\mathcal{H}}$, w bounded such that we have the decomposition of Theorem 4.5 for $A^{[w]}$, then:

$$(20) \quad \mu_{(\Phi(f), \Psi(f), w)} = \Phi(\tilde{f}_1, \dots, \tilde{f}_n) \bullet \mu_{(\Psi(f), w)} + \Psi(\tilde{f}_1, \dots, \tilde{f}_n) \bullet \mu_{(\Phi(f), w)}.$$

Proof. It is enough to show

$$\int_E h d\mu_{(\Phi^2(f), w)} = 2 \int_E h\Phi(\tilde{f}_1, \dots, \tilde{f}_n) d\mu_{(\Phi(f), w)}$$

for $h = \widehat{R}_\beta g$, $g \in \mathcal{B}_b(E)^+ \cap L^1(E; m)$, $\beta > 0$, because then we will consider $\mu_{((\Phi(f)+\Psi(f))^2, w)}$. We may furthermore assume that $\int h dm = 1$. Then by (19)

$$\begin{aligned} & \int_E h d\mu_{(\Phi^2(f), w)} \\ &= \lim_{\alpha \rightarrow \infty} \alpha(\alpha + \beta) E_{hm} \left[\int_0^\infty e^{-(\alpha+\beta)t} M_t^{[\Phi^2(f)]} M_t^{[w]} dt \right] \\ &= \lim_{\alpha \rightarrow \infty} \alpha(\alpha + \beta) E_{hm} \left[\int_0^\infty e^{-(\alpha+\beta)t} (\Phi^2(\tilde{f})(Y_t) - \Phi^2(\tilde{f})(Y_0))(\tilde{w}(Y_t) - \tilde{w}(Y_0)) dt \right] \\ &= 2 \lim_{\alpha \rightarrow \infty} \alpha(\alpha + \beta) E_{h\Phi(\tilde{f})m} \left[\int_0^\infty e^{-(\alpha+\beta)t} (\Phi(\tilde{f})(Y_t) - \Phi(\tilde{f})(Y_0))(\tilde{w}(Y_t) - \tilde{w}(Y_0)) dt \right] \\ &\quad + \lim_{\alpha \rightarrow \infty} \alpha(\alpha + \beta) E_{hm} \left[\int_0^\infty e^{-(\alpha+\beta)t} (\Phi(\tilde{f})(Y_t) - \Phi(\tilde{f})(Y_0))^2(\tilde{w}(Y_t) - \tilde{w}(Y_0)) dt \right] \\ &= 2 \lim_{\alpha \rightarrow \infty} I_\alpha + \lim_{\alpha \rightarrow \infty} II_\alpha. \end{aligned}$$

By (19) and Lebesgue's Theorem we have

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} I_\alpha &= \lim_{\alpha \rightarrow \infty} \alpha(U_{(\Phi(f), w)}^{\alpha+\beta} 1, h\Phi(\tilde{f})) \\ &= \lim_{\alpha \rightarrow \infty} \alpha \lim_{\gamma \rightarrow \infty} (\gamma R_{\gamma+\alpha+\beta} U_{(\Phi(f), w)}^{\alpha+\beta} 1, h\Phi(\tilde{f})) \\ &= \lim_{\alpha \rightarrow \infty} \alpha \lim_{\gamma \rightarrow \infty} (\gamma U_{(\Phi(f), w)}^{\gamma+\alpha+\beta} 1, \widehat{R}_{\alpha+\beta}(h\Phi(\tilde{f}))) \\ &= \lim_{\alpha \rightarrow \infty} \alpha \int_E \widehat{R}_{\alpha+\beta}(h\Phi(\tilde{f})) d\mu_{(\Phi(f), w)} \\ &= \int_E h\Phi(\tilde{f}) d\mu_{(\Phi(f), w)} \end{aligned}$$

and for some constant $L > 0$ $\lim_{\alpha \rightarrow \infty} II_\alpha$ is dominated by

$$nL \sum_{i=1}^n \overline{\lim}_{\alpha \rightarrow \infty} \alpha(\alpha + \beta) E_{hm} \left[\int_0^\infty e^{-(\alpha+\beta)t} (\tilde{f}_i(Y_t) - \tilde{f}_i(Y_0))^2 |\tilde{w}(Y_t) - \tilde{w}(Y_0)| dt \right].$$

For $1 \leq i \leq n$, $k \in \mathbb{N}$, we set $f_{ki} = kR_k \tilde{f}_i$. Since we assumed that $\bar{e}(A^{[f_i - f_{ki}]}) \rightarrow 0$, as

$k \rightarrow \infty$, $1 \leq i \leq n$, it is enough to show that for $1 \leq i \leq n$

$$\overline{\lim}_{\alpha \rightarrow \infty} \alpha(\alpha + \beta) E_{hm} \left[\int_0^\infty e^{-(\alpha+\beta)t} (f_{ki}(Y_t) - f_{ki}(Y_0))^2 |\tilde{w}(Y_t) - \tilde{w}(Y_0)| dt \right]$$

tends to zero as $k \rightarrow \infty$. By our assumption $(M_t^{[f_{ki}]}, \mathcal{F}_t, P_{hm})_{t \geq 0}$ is a continuous square integrable martingale and consequently by the Burkholder-Davis-Gundy inequality

$$E_{hm} \left[(M_t^{[f_{ki}]})^4 \right] \leq C E_{hm} \left[(M^{[f_{ki}]})_t^2 \right]$$

where $C > 0$ is a constant independent of f_{ki} . Then

$$\begin{aligned} & \overline{\lim}_{\alpha \rightarrow \infty} \alpha(\alpha + \beta) E_{hm} \left[\int_0^\infty e^{-(\alpha+\beta)t} (f_{ki}(Y_t) - f_{ki}(Y_0))^2 |\tilde{w}(Y_t) - \tilde{w}(Y_0)| dt \right] \\ & \leq 2 \overline{\lim}_{\alpha \rightarrow \infty} \left(\alpha(\alpha + \beta) E_{hm} \left[\int_0^\infty e^{-(\alpha+\beta)t} (M_t^{[f_{ki}]})^4 dt \right] \right)^{1/2} \left(\int h d\mu_{\langle w \rangle} \right)^{1/2} \\ & \leq 2 \overline{\lim}_{\alpha \rightarrow \infty} \left(\alpha(\alpha + \beta) E_{hm} \left[\int_0^\infty e^{-(\alpha+\beta)t} \langle M^{[f_{ki}]} \rangle_t^2 dt \right] \right)^{1/2} \left(\int h d\mu_{\langle w \rangle} \right)^{1/2}. \end{aligned}$$

Then

$$\begin{aligned} & \overline{\lim}_{\alpha \rightarrow \infty} \alpha(\alpha + \beta) E_{hm} \left[\int_0^\infty e^{-(\alpha+\beta)t} (\langle M^{[f_{ki}]} \rangle_t)^2 dt \right] \\ & = \overline{\lim}_{\alpha \rightarrow \infty} \alpha(\alpha + \beta) E_{hm} \left[\int_0^\infty e^{-(\alpha+\beta)t} 2 \int_0^t (\langle M^{[f_{ki}]} \rangle_t - \langle M^{[f_{ki}]} \rangle_s) d \langle M^{[f_{ki}]} \rangle_s dt \right] \\ & = 2 \overline{\lim}_{\alpha \rightarrow \infty} \alpha(\alpha + \beta) E_{hm} \left[\int_0^\infty \int_s^\infty e^{-(\alpha+\beta)t} \langle M^{[f_{ki}]} \rangle_{t-s} \circ \vartheta_s dt d \langle M^{[f_{ki}]} \rangle_s \right] \\ & = 2 \overline{\lim}_{\alpha \rightarrow \infty} \alpha(\alpha + \beta) E_{hm} \left[\int_0^\infty e^{-(\alpha+\beta)s} E_{Y_s} \left[\int_0^\infty e^{-(\alpha+\beta)t} \langle M^{[f_{ki}]} \rangle_t dt \right] d \langle M^{[f_{ki}]} \rangle_s \right] \\ & = 2 \overline{\lim}_{\alpha \rightarrow \infty} \alpha \int h U_{\langle M^{[f_{ki]} \rangle} }^{\alpha+\beta} U_{\langle M^{[f_{ki]} \rangle} }^{\alpha+\beta} 1 dm \\ & \leq 2 \int h U_{\langle M^{[f_{ki]} \rangle} }^\gamma 1 d\mu_{\langle M^{[f_{ki]} \rangle} } \end{aligned}$$

for every $\gamma > 0$. Now since $U_{\langle M^{[f_{ki]} \rangle} }^1 1(z)$ is bounded by the same constant for \mathcal{E} -q.e. z in E and since

$$U_{\langle M^{[f_{ki]} \rangle} }^{\gamma+1} 1(z) = U_{\langle M^{[f_{ki]} \rangle} }^1 1(z) - \gamma R_{\gamma+1} U_{\langle M^{[f_{ki]} \rangle} }^1 1(z) \downarrow 0$$

for \mathcal{E} -q.e. z in E as $\gamma \rightarrow \infty$ we conclude by Lebesgue's Theorem. □

Theorem 5.5 (Chain rule). *Let $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_n)$ be an n -tuple of \mathcal{E} -q.c. m -versions of bounded elements in \mathcal{H} such that $A^{[\Phi(\tilde{f})]}$ admits the decomposition of*

Theorem 4.5 for all $\Phi \in C^1(\mathbb{R}^n)$ with $\Phi(0) = 0$ and let $\bar{e}(A^{[f_i - kR_k f_i]}) \rightarrow 0$, as $k \rightarrow \infty$, $1 \leq i \leq n$. Let further the martingale part $M^{[u]}$ of the decomposition of Theorem 4.5 be continuous for all u in $G_1\mathcal{H}_b$. Let $\Phi \in C^1(\mathbb{R}^n)$, $\Phi(0) = 0$ and $\tilde{w} \in \tilde{\mathcal{H}}$, w bounded such that we have the decomposition of Theorem 4.5 for $A^{[w]}$, then

$$(21) \quad \mu_{(\Phi(f), w)} = \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i}(\tilde{f}) \bullet \mu_{(f_i, w)}.$$

Proof. We first observe that any powers of the coordinate functions satisfy (21) by the product rule and then by the product rule again all polynomials of n variables vanishing at the origin.

Let $K \subset \mathbb{R}^n$ be a compact set such that $\tilde{f}(z) \in K$ for \mathcal{E} -q.e. $z \in E$. Let $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi = 1$ on $K \subset K' = \text{supp}(\chi)$. There exists (cf. [3, II.4.2, 4.3, p. 57]) a sequence of polynomials vanishing at the origin $(p_j)_{j \in \mathbb{N}}$, such that $p_j \xrightarrow{j \rightarrow \infty} \Phi$, $(\partial p_j / \partial x_i) \xrightarrow{j \rightarrow \infty} (\partial \Phi / \partial x_i)$, $1 \leq i \leq n$, uniformly on K' . Note that $[(\Phi - p_j)\chi](\tilde{f})(z) = (\Phi - p_j)(\tilde{f})(z)$ for \mathcal{E} -q.e. $z \in E$. Then we have for all $g \in L^2(E; m) \cap \mathcal{B}_b(E)$, $\alpha > 0$

$$\begin{aligned} \sum_{i=1}^n \int \widehat{R}_\alpha g \frac{\partial \Phi}{\partial x_i}(\tilde{f}) d\mu_{(f_i, w)} &= \sum_{i=1}^n \lim_{j \rightarrow \infty} \int \widehat{R}_\alpha g \frac{\partial p_j}{\partial x_i}(\tilde{f}) d\mu_{(f_i, w)} \\ &= \lim_{j \rightarrow \infty} \int \widehat{R}_\alpha g d\mu_{(p_j(f), w)} \\ &= \int \widehat{R}_\alpha g d\mu_{(\Phi(f), w)} \end{aligned}$$

where the second identity followed from the product rule. Because of (15) the third identity follows since we assumed to have the decomposition of Theorem 4.5 for all $\Phi(f)$ like above and therefore

$$\begin{aligned} e(M^{[\Phi(f)]} - M^{[p_j(f)]}) &= e(A^{[(\Phi - p_j)\chi](f)}) \\ &\leq n \sum_{i=1}^n \left\| \frac{\partial((\Phi - p_j)\chi)}{\partial x_i} \right\|_\infty^2 \bar{e}(A^{[f_i]}). \end{aligned}$$

The last expression tends to zero as $j \rightarrow \infty$. □

Summarizing we get the following

Theorem 5.6 (Itô's Formula). *Let $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_n)$ be an n -tuple of \mathcal{E} -q.c. m -versions of bounded elements in \mathcal{H} such that $A^{[\Phi(f)]}$ admits the decomposition of Theorem 4.5 for all $\Phi \in C^1(\mathbb{R}^n)$ with $\Phi(0) = 0$ and let $\bar{e}(A^{[f_i - kR_k f_i]}) \rightarrow 0$, as $k \rightarrow \infty$, $1 \leq i \leq n$. Let further the martingale part $M^{[u]}$ of the decomposition of Theorem 4.5*

be continuous for all u in $G_1\mathcal{H}_b$. Then we have

$$\Phi(\tilde{f}_1, \dots, \tilde{f}_n)(Y_t) - \Phi(\tilde{f}_1, \dots, \tilde{f}_n)(Y_0) = \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i}(\tilde{f}_1, \dots, \tilde{f}_n) \bullet M_t^{[f_i]} + N_t^{[\Phi(f_1, \dots, f_n)]}$$

for all Φ like above and this decomposition is orthogonal w.r.t. $e(\cdot, \cdot)$.

Proof. The assertion follows by Lemma 5.1, Lemma 5.3 and Theorem 5.5 because

$$e\left(M^{[\Phi(f)]} - \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i}(\tilde{f}) \bullet M^{[f_i]}\right) = \frac{1}{2} \int d\mu_{(M^{[\Phi(f)]} - \sum_{i=1}^n (\partial \Phi / \partial x_i)(\tilde{f}) \bullet M^{[f_i]})} = 0. \quad \square$$

6. An example

6.1. Weak solutions of SDE's in infinite dimensions Let E be a separable real Banach space and $(H, \langle \cdot, \cdot \rangle_H)$ a separable real Hilbert space such that $H \subset E$ densely and continuously. Identifying H with its topological dual H' we obtain that $E' \subset H \subset E$ densely and continuously. Define the linear space of finitely based smooth functions on E by

$$\mathcal{FC}_b^\infty := \{f(l_1, \dots, l_m) \mid m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m), l_1, \dots, l_m \in E'\}.$$

Here $C_b^\infty(\mathbb{R}^m)$ denotes the set of all infinitely differentiable (real-valued) functions on \mathbb{R}^m with all partial derivatives bounded. For $u \in \mathcal{FC}_b^\infty$, $k \in E$ let

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds} u(z + sk) \Big|_{s=0}, \quad z \in E.$$

It follows that if $u = f(l_1, \dots, l_m)$ and $k \in H$ we have that

$$\frac{\partial u}{\partial k}(z) = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(l_1(z), \dots, l_m(z)) \langle l_i, k \rangle_H, \quad z \in E.$$

Consequently, $k \mapsto (\partial u / \partial k)(z)$ is continuous on H and we can define $\nabla u(z) \in H$ by

$$\langle \nabla u(z), k \rangle_H = \frac{\partial u}{\partial k}(z).$$

Let μ be finite positive measure on $(E, \mathcal{B}(E))$. Assume for simplicity $\text{supp}(\mu) \equiv E$. An element k in E is called *well- μ -admissible* if there exist $\beta_k^\mu \in L^2(E; \mu)$ such that for all u, v in \mathcal{FC}_b^∞

$$\int \frac{\partial u}{\partial k} d\mu = - \int u \beta_k^\mu d\mu.$$

Let us assume

(A.1) There exists a dense linear subspace K of E' consisting of well- μ -admissible elements.

Then it is well known that the densely defined positive definite symmetric bilinear form

$$\mathcal{E}^0(u, v) := \frac{1}{2} \int \langle \nabla u, \nabla v \rangle_H d\mu \quad u, v \in \mathcal{FC}_b^\infty$$

is closable on $L^2(E; \mu)$ and that the closure $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a symmetric quasi-regular Dirichlet form. Let $(L^0, D(L^0))$ be the associated generator. Let $\bar{\beta} \in L^2(E, H; \mu)$ (i.e. $\bar{\beta} : E \rightarrow E$ is $\mathcal{B}(E)/\mathcal{B}(E)$ -measurable, $\bar{\beta}(E) \subset H$ and $\|\bar{\beta}\|_H \in L^2(E; \mu)$) be such that

$$(22) \quad \int \langle \bar{\beta}, \nabla u \rangle_H d\mu = 0 \quad \text{for all } u \in \mathcal{FC}_b^\infty.$$

Since \mathcal{FC}_b^∞ is dense in $D(\mathcal{E}^0)$ (22) implies that $\int \langle \bar{\beta}, \nabla u \rangle_H d\mu = 0$ for all $u \in D(\mathcal{E}^0)$ and thus $\int \langle \bar{\beta}, \nabla u \rangle_H v d\mu = - \int \langle \bar{\beta}, \nabla v \rangle_H u d\mu$ for all $u, v \in D(\mathcal{E}^0)_b$. Let

$$Lu := L^0 u + \langle \bar{\beta}, \nabla u \rangle_H, \quad u \in D(L^0)_b.$$

It then follows from [16, Proposition 4.1.] that $(L, D(L^0)_b)$ is closable on $L^1(E; \mu)$ and that the closure $(\bar{L}, D(\bar{L}))$ generates a Markovian C_0 -semigroup of contractions. Furthermore $D(\bar{L})_b \subset D(\mathcal{E}^0)$ and for $u \in D(\bar{L})_b, v \in D(\mathcal{E}^0)_b$

$$(23) \quad \mathcal{E}^0(u, v) - \int \langle \bar{\beta}, \nabla u \rangle_H v d\mu = - \int \bar{L} u v d\mu.$$

Let $(L, D(L))$ with associated resolvent $(G_\alpha)_{\alpha>0}$ be the part of $(\bar{L}, D(\bar{L}))$ on $L^2(E; \mu)$, $(L', D(L'))$ with associated resolvent $(G'_\alpha)_{\alpha>0}$ be the adjoint of $(L, D(L))$ in $L^2(E; \mu)$. According to [15, I.4.9. (ii)] $(L, D(L))$ is associated with the *generalized Dirichlet form*

$$\mathcal{E}(u, v) := \begin{cases} (-Lu, v) & \text{for } u \in D(L), v \in L^2(E; \mu) \\ (u, -L'v) & \text{for } u \in L^2(E; \mu), v \in D(L') \end{cases}$$

where (\cdot, \cdot) is the inner product in $L^2(E; \mu)$. There exists (cf. [16, Th.4.6., Prop. 4.7.]) a μ -tight special standard process $\mathbf{M} = (\Omega, \mathcal{F}_\infty, (X_t)_{t \geq 0}, (P_z)_{z \in E_\Delta})$ with life time ζ that is associated with $(L, D(L))$ in the sense that $R_\alpha f(\cdot) := E. [\int_0^\infty e^{-\alpha t} f(X_t) dt]$ is an \mathcal{E}^0 -q.c. m -version of $G_\alpha f$ for all $f \in \mathcal{B}_b(E) \cap L^2(E; \mu), \alpha > 0$. Furthermore $P_z[\zeta = +\infty] = 1, P_z[t \mapsto X_t \text{ is continuous on } [0, \infty)] = 1$ for \mathcal{E}^0 -q.e. $z \in E$. Note that by [16, Lemma 4.5] L -nests and \mathcal{E}^0 -nests coincide. Therefore \mathcal{E} -exceptional and \mathcal{E}^0 -exceptional sets coincide.

Since $-\bar{\beta}$ satisfies the same assumptions as $\bar{\beta}$ the closure $(\bar{L}', D(\bar{L}'))$ of $L'u := L^0u - \langle \bar{\beta}, \nabla u \rangle_H$, $u \in D(L^0)_b$ on $L^1(E; \mu)$ generates a Markovian C_0 -semigroup of contractions too, $D(\bar{L}')_b \subset D(\mathcal{E}^0)$ and for $u \in D(\bar{L}')_b$, $v \in D(\mathcal{E}^0)_b$ $\mathcal{E}^0(u, v) + \int \langle \bar{\beta}, \nabla u \rangle_H v d\mu = - \int \bar{L}'uv d\mu$. It is easy to see that the part of $(\bar{L}', D(\bar{L}'))$ on $L^2(E; \mu)$ is $(L', D(L'))$. Let $(R'_\alpha)_{\alpha>0}$ denote the resolvent of the associated coprocess. Since $(G'_\alpha)_{\alpha>0}$ is sub-Markovian and strongly continuous on $\mathcal{V} = L^2(E; \mu)$, Theorem 4.5 applies for $u \in D(L)$ with $N_t^{[u]} = \int_0^t Lu(X_s) ds$. Let $v \in D(L)_b$, $g \in L^2(E; \mu) \cap \mathcal{B}_b(E)^+$, $\gamma > 0$, then by (19)

$$\begin{aligned} \int R'_\gamma g d\mu_{(v)} &= 2\mathcal{E}^0(v, vR'_\gamma g) - \mathcal{E}^0(v^2, R'_\gamma g) \\ &= \int R'_\gamma g \langle \nabla v, \nabla v \rangle_H d\mu. \end{aligned}$$

Now let $u_n \in D(L)_b$ such that $u_n \xrightarrow{n \rightarrow \infty} u$ in $D(L)$. Since by (23) $u_n \xrightarrow{n \rightarrow \infty} u$ in $D(\mathcal{E}^0)$ we have $\int R'_\gamma g d\mu_{(u)} = \int R'_\gamma g \langle \nabla u, \nabla u \rangle_H d\mu$. In particular $\int R'_\gamma g d\mu_{(u)} = \int R'_\gamma g d\mu_A$ where $A = \int_0^t \langle \nabla u(X_s), \nabla u(X_s) \rangle_H ds$. Therefore by Remark 5.2(i) and (ii) it follows that $\langle M^{[u]} \rangle_t = \int_0^t \langle \nabla u(X_s), \nabla u(X_s) \rangle_H ds$. Note that $\langle M^{[u]} \rangle$ is finite since $\langle \nabla u, \nabla u \rangle_H \in L^1(E; \mu)$. Assume

(A.2) $u_k(\cdot) :=_{E'} \langle k, \cdot \rangle_E \in L^2(E; \mu)$ for all $k \in K$.

Here $_{E'} \langle \cdot, \cdot \rangle_E$ denotes the dualization between E and E' . Then clearly $u_k \in D(L)$,

(24)
$$Lu_k = \frac{1}{2} \beta_k^\mu + \langle \bar{\beta}, k \rangle_H, \quad k \in K,$$

and

(25)
$$\langle M^{[u_k]}, M^{[u_{k'}]} \rangle_t = t \langle k, k' \rangle_H, \quad k, k' \in K.$$

Choosing an ONB $K_0 \subset K$ of H which separates the points of E by Theorem 4.5 applied to u_k , $k \in K_0$ we get a countable system of 1-dimensional SDE 's with independent 1-dimensional Brownian motions according to (25) and drifts given according to (24). If we assume

(A.3) For one (and hence all) $t > 0$ there exists a probability measure μ_t on $(E, \mathcal{B}(E))$, such that

$$\int e^{i_{E'} \langle k, z \rangle_E} \mu_t(dz) = e^{-(1/2)t \|k\|_H^2} \quad \text{for all } k \in E'$$

similar to [1, Theorem 6.6] it is then possible to lift the countable system of 1-dimensional equations to a single equation on E , namely we have

Theorem 6.1. *There exist maps $W, N^0 : \Omega \rightarrow C([0, \infty), E)$ with the following properties:*

- (i) $\omega \mapsto W_t(\omega) := W(\omega)(t)$ and $\omega \mapsto N_t^0(\omega) := N^0(\omega)(t)$ are both $\mathcal{F}_t/\mathcal{B}(E)$ measurable for $t \geq 0$.
- (ii) *There exists an \mathcal{E}^0 -exceptional set $S \subset E$ such that under each $P_z, z \in E \setminus S$, $W = (W_t)_{t \geq 0}$ is an E -valued $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion starting at $0 \in E$ with covariance $\langle \cdot, \cdot \rangle_H$ (i.e. under each $P_z, z \in E \setminus S$, for all $0 \leq s < t$ $W_t - W_s$ is independent of \mathcal{F}_s and $\langle k, W_t - W_s \rangle_E$ is mean zero Gaussian with variance $(t - s)\|k\|_H^2$).*
- (iii) *For each $k \in K, t \geq 0$ and \mathcal{E}^0 -q.e. $z \in E$ we have P_z -a.s.*

$$E' \langle k, W_t \rangle_E = M_t^{[u_k]} \text{ and } E' \langle k, N_t^0 \rangle_E = \frac{1}{2} \int_0^t \beta_k^\mu(X_s) ds.$$

- (iv) *For \mathcal{E}^0 -q.e. $z \in E$ we have P_z -a.s.*

$$(26) \quad X_t = z + W_t + N_t^0 + \int_0^t \bar{\beta}(X_s) ds$$

where the last integral is in the sense of Bochner (cf. the following Remark 6.2) and where for $k \in K$ $\langle k, \int_0^t \bar{\beta}(X_s) ds \rangle_H = \int_0^t \langle \bar{\beta}(X_s), k \rangle_H ds$.

REMARK 6.2. (i) The assumption that the Gaussian measures satisfying (A.3) exist is of course, necessary. It just means that there exists (cf. [7, p. 74]) a Brownian semigroup on E with covariance $\langle \cdot, \cdot \rangle_H$, i.e., there is a Brownian motion on E over H . Hence (A.3) is the best one could hope for.

(ii) In the above general situation there is no guarantee that $k \mapsto \beta_k^\mu(z), k \in K$, is represented by an element in E for μ -a.e. $z \in E$. But if we assume

(A.4) There exists a $\mathcal{B}(E)/\mathcal{B}(E)$ -measurable map $\beta_H^\mu : E \rightarrow E$ such that

- (a) $E' \langle k, \beta_H^\mu \rangle_E = \beta_k^\mu$ μ -a.s. for each $k \in K$,
- (b) $\|\beta_H^\mu\|_E \in L^1(E; \mu)$

then we may define the process N^0 in Theorem 6.1 as a Bochner integral. In fact, it is easy to see that $\|\beta_H^\mu\|_E \in L^1(E; \mu) \cap \mathcal{B}(E)$ implies the finiteness of the AF $\int_0^t \|\beta_H^\mu\|_E(X_s) ds$. Hence, by [20, Theorem 1, p. 133, Corollary 2, p. 134], $N_t^0 := (1/2) \int_0^t \beta_H^\mu(X_s) ds, t \geq 0$ (where the integral is in the sense of Bochner P_z -a.s for \mathcal{E}^0 -q.e. $z \in E$) has the desired properties.

(iii) It is easy to see that (A.4) is equivalent to the following assumption:

(A.4') There exists a $\mathcal{B}(E)/\mathcal{B}(E)$ -measurable map $B : E \rightarrow E$ such that

- (a) $(1/2)E' \langle k, B \rangle_E = Lu_k$ μ -a.s. for each $k \in K$,
- (b) $\|B\|_E \in L^1(E; \mu)$.

Analogous to (ii) we may then replace $N_t^0 + \int_0^t \bar{\beta}(X_s) ds$ in (26) by the Bochner integral $(1/2) \int_0^t B(X_s) ds$.

6.2. Applications In this subsection we assume that E is a separable real Hilbert space with inner product $\|\cdot\|_E := \langle \cdot, \cdot \rangle_E^{1/2}$ and that $H \subset E$ densely by a Hilbert-Schmidt map. Then there exists a nonnegative definite injective self-adjoint Hilbert-Schmidt operator T on E such that $H = T(E)$ and $\|\cdot\|_H = \|T^{-1} \cdot\|_E$. Analogous to [7, Theorem 4.4 Step 3.] we see that $\|\cdot\|_E$ is measurable over H , hence (A.3) holds. Let $B : E \rightarrow E$ be a Borel measurable vector field satisfying the following conditions:

- (B.1) $\lim_{\|z\|_E \rightarrow \infty} \langle B(z), z \rangle_E = -\infty$,
- (B.2) ${}_{E'} \langle l, B \rangle_E : E \rightarrow \mathbb{R}$ is weakly continuous for all $l \in E'$.
- (B.3) There exist $C_1, C_2, d \in (0, \infty)$, such that $\|B(z)\|_E \leq C_1 + C_2 \|z\|_E^d$ for all $z \in E$.

Then by [2, Theorem 5.2.] there exists a probability measure μ on $(E, \mathcal{B}(E))$ such that ${}_{E'} \langle l, B \rangle_E \in L^2(E; \mu)$ for all $l \in E'$ and such that

$$(27) \quad \int \left(\frac{1}{2} \Delta_H u + \frac{1}{2} {}_{E'} \langle \nabla u, B \rangle_E \right) d\mu = 0 \quad \text{for all } u \in \mathcal{FC}_b^\infty$$

where Δ_H is the Gross-Laplacian, i.e.,

$$\Delta_H u = \sum_{i,j=1}^m \frac{\partial f}{\partial x_i \partial x_j} (l_1(z), \dots, l_m(z)) \langle l_i, l_j \rangle_H \quad \text{if } u = f(l_1, \dots, l_m) \in \mathcal{FC}_b^\infty.$$

Assume that $B(z) = -z + v(z)$, $v : E \rightarrow H$. Because of (B.1), (B.3) it follows by [2, Lemma 5.1.] that $v \in L^2(E, H; \mu)$. In particular we have $\|z\|_E \in L^p(E, \mu)$ for all $p \geq 1$. Let γ be a Gaussian measure on E with covariance $\langle \cdot, \cdot \rangle_H$. By [2, Theorem 3.5.] $d\mu = \varphi^2 d\gamma$ where φ is in the Sobolev space $H^{1,2}(E; \gamma)$. Furthermore the logarithmic derivative β_H^μ of μ associated with H exists and admits the representation $\beta_H^\mu(z) = -z + (2\nabla\varphi/\varphi)(z)$. Note that possibly $\text{supp}(\mu) \neq E$. Nevertheless, since every $k \in E'$ is well- μ -admissible (thus in particular (A.1) holds)

$$\mathcal{E}^0(u, v) := \frac{1}{2} \int \langle \nabla u, \nabla v \rangle_H d\mu, \quad u, v \in \mathcal{FC}_b^\infty,$$

is well-defined and closable on $L^2(E; \mu)$ and the closure is a symmetric quasi-regular Dirichlet form. Let $(L^0, D(L^0))$ be the associated generator. It is easy to see that $\mathcal{FC}_b^\infty \subset D(L^0)$ and

$$L^0 u = \frac{1}{2} \Delta_H u + \frac{1}{2} {}_{E'} \langle \nabla u, \beta_H^\mu \rangle_E, \quad u \in \mathcal{FC}_b^\infty.$$

Set $\bar{\beta} := (1/2)(B - \beta_H^\mu)$. Clearly $\bar{\beta} \in L^2(E, H; \mu)$ and by (27) since $\int L^0 u d\mu = 0$, $u \in \mathcal{FC}_b^\infty$

$$(28) \quad \int \langle \bar{\beta}, \nabla u \rangle_H d\mu = 0 \quad \text{for all } u \in \mathcal{FC}_b^\infty.$$

As in 6.1 we then construct a conservative diffusion $\mathbf{M} = (\Omega, \mathcal{F}_\infty, (X_t)_{t \geq 0}, (P_z)_{z \in E_\Delta})$ associated to the part on $L^2(E; \mu)$ (which we denote by $(L, D(L))$) of the closure on $L^1(E, \mu)$ of $L^0u + \langle \bar{\beta}, \nabla u \rangle_H$, $u \in D(L^0)_b$. Note that $Lu = (1/2)\Delta_H u + (1/2) \times_{E'} \langle \nabla u, B \rangle_E$, $u \in \mathcal{FC}_b^\infty$. Surely (A.2) is satisfied and clearly $Lu_k = (1/2)_{E'} \langle k, B \rangle_E$ hence (A.4') holds. By Theorem 6.1 and Remark 6.2(ii) we then have P_z -a.s. for \mathcal{E}^α -q.e. $z \in E$ (thus in particular P_μ -a.s.)

$$(29) \quad X_t = z + W_t - \frac{1}{2} \int_0^t X_s ds + \frac{1}{2} \int_0^t v(X_s) ds$$

where $(W_t)_{t \geq 0}$ is an E -valued $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion starting at $0 \in E$ with covariance $\langle \cdot, \cdot \rangle_H$ and where $(1/2) \int_0^t X_s ds$, $(1/2) \int_0^t v(X_s) ds$ are in the sense of Bochner P_z -a.s. for \mathcal{E}^α -q.e. $z \in E$. Note that $\int (v - 2(\nabla\varphi/\varphi), \nabla u)_H d\mu = 0$ for all u in \mathcal{FC}_b^∞ . Let $(L', D(L'))$ denote the adjoint operator of $(L, D(L))$ on $L^2(E; \mu)$. Clearly, since $L'u_k = {}_{E'} \langle k, (1/2)id_E + (2\nabla\varphi/\varphi) - (1/2)v \rangle_E$ and $\|id_E\|_E, \|2\nabla\varphi/\varphi\|_E, \|v\|_E \in L^2(E; \mu)$ the coprocess $\widehat{\mathbf{M}} = (\widehat{\Omega}, \widehat{\mathcal{F}}_\infty, (\widehat{X}_t)_{t \geq 0}, (\widehat{P}_z)_{z \in E_\Delta})$ associated to $(L', D(L'))$ weakly solves

$$\widehat{X}_t = z + \widehat{W}_t - \frac{1}{2} \int_0^t \widehat{X}_s ds + \int_0^t 2 \frac{\nabla\varphi}{\varphi}(\widehat{X}_s) ds - \frac{1}{2} \int_0^t v(\widehat{X}_s) ds$$

for \mathcal{E}^α -q.e. $z \in E$ where $(\widehat{W}_t)_{t \geq 0}$ is an E -valued $(\widehat{\mathcal{F}}_t)_{t \geq 0}$ -Brownian motion starting at $0 \in E$ with covariance $\langle \cdot, \cdot \rangle_H$.

6.3. An Itô-type formula Let $(\overline{G}_\alpha)_{\alpha > 0}$ be the resolvent associated to $(\overline{L}, D(\overline{L}))$. Since $\overline{G}_\alpha|_{L^2} = G_\alpha$, $\alpha > 0$, it follows that $D(L)_b \subset D(\overline{L})_b$ is dense w.r.t. the L^1 -graph norm. Note that since $1 \in D(L)$, the 1-reduced function e_f exists for all $f \in L^\infty(E; \mu)$. Let $u_n := n\overline{G}_n u$, $u \in D(\overline{L})_b$. By [16, Lemma 4.4.(iii)] we have that $e_{u-u_n} + e_{u_n-u} \xrightarrow[n \rightarrow \infty]{} 0$ in $D(\mathcal{E}^0)$, hence in $L^2(E; \mu)$. Furthermore $\bar{e}(A^{[u-u_n]}) = (-\overline{L}(u-u_n), u-u_n) \xrightarrow[n \rightarrow \infty]{} 0$. Now (cf. Remark 4.3) by Theorem 4.5(ii) the decomposition (17) extends to $A^{[u]}$, $u \in D(\overline{L})_b$. Similar to the finite dimensional case [16, Remark 1.7.(iii)] one can show that $D(\overline{L})_b$ is an algebra. Hence the decomposition (17) extends to $A^{[p(f)]}$, where $f = (f_1, \dots, f_n)$ is an n -tuple of elements in $D(\overline{L})_b$ and p is a real polynomial in n variables. Let $\Phi \in C^1(\mathbb{R}^n)$ with $\Phi(0) = 0$. Let p_j , $j \in \mathbb{N}$, χ be similar as in the proof of Theorem 5.5. We then have by Lemma 4.2 for any $\mu \in \widehat{\mathcal{S}}_{00}$, $T > 0$

$$P_\mu \left(\sup_{0 \leq t \leq T} |(\Phi - p_j)(\widehat{f})(X_t)| > \varepsilon \right) \leq \frac{e^T}{\varepsilon} n \|h\|_{\mathcal{H}} \sum_{i=1}^n \left\| \frac{\partial((\Phi - p_j)\chi)}{\partial x_i} \right\|_\infty \|e_{f_i} + e_{-f_i}\|_{\mathcal{H}}.$$

Furthermore

$$\bar{e}(A^{[(\Phi - p_j)(f)]}) \leq n \sum_{i=1}^n \left\| \frac{\partial((\Phi - p_j)\chi)}{\partial x_i} \right\|_\infty^2 (-\overline{L}f_i, f_i).$$

Hence, by Theorem 4.5(ii), the decomposition (17) extends to $A^{[\Phi(f)]}$. As a consequence Theorem 5.6 applies to the martingale part $M^{[\Phi(f)]}$ of the decomposition.

ACKNOWLEDGEMENT. I would like to thank M. Röckner for suggesting me to further develop the theory of generalized Dirichlet forms in the way implemented in this paper. I would also like to thank W. Stannat for valuable discussions.

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