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# UNIQUENESS OF THE MOST SYMMETRIC NON-SINGULAR PLANE SEXTICS

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## 0. Introduction

Let $C$ be a compact Riemann surface of genus $g \geq 2$. The order of the holomorphic automorphism group $\mathrm{Aut}(C)$ takes the value $84(g-1), 48(g-1), 40(g-1), 36(g-1), 30(g-1)$ or less by Hurwitz' theorem ([5, Chap. 6] or [1, Chap. 5]). A homogeneous polynomial $f \in \mathbf{C}[x, y, z]$ with $n = \deg f \geq 1$ defines an algebraic curve $C(f)$ in the projective plane $\mathbf{P}^2$ over the complex number field $\mathbf{C}$. As is well known $C(f)$ is a compact Riemann surface of genus $(n-1)(n-2)/2$ if $C(f)$ is non-singular. Particularly a non-singular plane quartic(resp. sextic) has genus $g = 3$(resp. $g = 10$). Let $\mathrm{Aut}(f)$ be the subgroup of the projectivities $PGL(3, \mathbf{C})$ of $\mathbf{P}^2$ consisting of all projectivities $(A)$ defined by $A \in GL(3, \mathbf{C})$ such that $f_A$ is proportional to $f$. Here $f_A(x, y, z) = f((x, y, z)({}^tA^{-1}))$ by definition. Clearly $\mathrm{Aut}(f)$ coincides with the projective automorphism group of $C(f)$, if $f$ is irreducible. It is also known that a holomorphic automorphism of a non-singular curve $C(f)$ of degree $n \geq 4$ is induced by a projectivity $(A) \in PGL(3, \mathbf{C})$ [9, Theorem 5.3.17(3)]. Therefore $\mathrm{Aut}(C(f))$=$\mathrm{Aut}(f)$ if $C(f)$ is non-singular of degree $n \geq 4$. By abuse of terminology we say that a homogeneous polynomial $f$ is non-singular or singular accoding as $C(f)$ is.

As is well known, the Klein quartic $f_4 = x^3y + y^3z + z^3x$ is the most symmetric in the sense that $|\mathrm{Aut}(f_4)| = 84 \times (3-1)$. It is also known that if $|\mathrm{Aut}(f)| = 168$ for a non-singular plane quartic $f$, then $f$ is projectively equivalent to $f_4$. A.Wiman has shown that for the following non-singular sextic

$$f_6 = 27z^6 - 135z^4xy - 45z^2x^2y^2 + 9z(x^5 + y^5) + 10x^3y^3,$$

$\mathrm{Aut}(f_6)$ is isomorphic to the simple group $A_6 \simeq PSL(2, 3^2)$[11], as a result $|\mathrm{Aut}(f_6)| = 40(g-1) = 360$. We call $f_6$ the Wiman sextic. He has also shown that the group $\mathrm{Aut}(f_6)$ acts transitively on the set of 72 flexes of $C(f_6)$. We can show even that no three flexes are collinear [6]. Our main results are

**Theorem.** *Let $f$ be a non-singular plane sextic defined over* $\mathbf{C}$. *Then*
(1) $|\mathrm{Aut}(f)| \leq 360$.
(2) $|\mathrm{Aut}(f)| = 360$ *if and only if $f$ is projectively equivalent to the Wiman sextic* $f_6$.

(1) will be proved in §1 according to [4], while (2) will be shown in §2. We can show that the most symmetric non-singular plane curve of degree 3, 5 or 7 is projectively equivalent to the Fermat curve [7].

We recall a well known fact: Let $R_A:\mathbf{C}[x,y,z] \longrightarrow \mathbf{C}[x,y,z]$ be a mapping defined by $R_A f = f_A$ for $A \in GL(3,\mathbf{C})$ and $f \in \mathbf{C}[x,y,z]$. Then $R_A$ is a ring-automorphim of the polynomial ring $\mathbf{C}[x,y,z]$. Since $(f_A)_B = f_{BA}$ for $A, B \in GL(3,\mathbf{C})$, the assignment $A \longrightarrow R_A$ is a group homomorphims of $GL(3,\mathbf{C})$ into $\mathrm{Aut}(\mathbf{C}[x,y,z])$.

We write $a \sim b$ when two quantities $a$ and $b$ such as polynomials or matrices are proportional. $E_3$ stands for the $3 \times 3$ unite matrix, and $e_i$ for the $i$-th column vector of $E_3 (1 \le i \le 3)$.

## 1. The maximum order of the automorphism group of non-singular plane sextics

Let $f$ be a non-singular plane sextic. In this section we will show that the order of the projective automorphism group $\mathrm{Aut}(f)$ can take the value neither $84 \times 9$ nor $48 \times 9$ (Theorem (1)). Otherwise, for some $f$ $\mathrm{Aut}(f)$ has a subgroup of order $3^3$ by Sylow's theorem. Thus it suffices to show the following theorem.

**Theorem 1.1.** *Let $f$ be a non-singular plane sextic. If $27 \,|\, |\mathrm{Aut}(f)|$, then $|\mathrm{Aut}(f)| < 360$.*

Our approach is elementary, but involves much computation. There exist eactly five groups of order 27 up to group isomorphism [3, 4.4]. They are three abelian groups and two non-abelian groups: (1) $\mathbf{Z}_{27}$ (2) $\mathbf{Z}_9 \times \mathbf{Z}_3$ (3) $\mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3$ (4) $a^9 = 1$, $b^3 = 1$, $b^{-1}ab = a^4$ (5) $a^3 = 1$, $b^3 = 1$, $c^3 = 1$, $ab = bac$, $ca = ac$, $cb = bc$. The group (5) is isomorphic to the matrix group

$$E(3^3) = \left\{ M(\alpha,\beta,\gamma) = \begin{bmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}; \quad \alpha,\ \beta,\ \gamma \in \mathbf{F}_3 \right\}.$$

We find projective representaions of these groups in the projective plane $\mathbf{P}^2$ defined over $\mathbf{C}$, and find a non-singular invariant sextic $f$, if any. We can manage to estimate the order of the projective automorphism group $\mathrm{Aut}(f)$.

**Lemma 1.2.** *Let $\varepsilon$ be a primitive 9-th root of $1 \in \mathbf{C}$. If $G_9$ is a subgroup of $PGL(3,\mathbf{C})$, isomorphic to $\mathbf{Z}_9$, then $G_9$ is conjugate to one of the following three groups in $PGL(3,\mathbf{C})$:*
(1) $\langle(\mathrm{diag}[1,\varepsilon,\varepsilon])\rangle$ (2) $\langle(\mathrm{diag}[1,\varepsilon,\varepsilon^2])\rangle$ (3) $\langle(\mathrm{diag}[1,\varepsilon,\varepsilon^3])\rangle$.

Proof. By our assumption $G_9$ is generated by a projective transformation $(A)$, where $A \in GL(3, \mathbf{C})$ satisfies $A^9 = E_3$ and $\mathrm{ord}((A)) = 9$, namely $G_9 = \langle (A) \rangle$. Therefore it is conjugate to $\langle (\mathrm{diag}[1, \varepsilon^i, \varepsilon^j]) \rangle$ for some $0 \le i \le j \le 8$ with $(i, j) \neq (0, 0), (0, 3), (0, 6), (3, 3), (3, 6), (6, 6)$. If $(i, j) = (0, j)$ with $j \not\equiv 0 \bmod 3$ or $i = j \not\equiv 0 \bmod 3$, then $G_9$ is conjugate to (1). If $1 \le i < j \le 8$ with $(i, j) \not\equiv (0, 0) \bmod 3$, then $G_9$ is conjugate to (2) or (3) according as $(i, j) \in \{(1, 2), (1, 5), (1, 8), (2, 4), (2, 7), (4, 5), (4, 8), (5, 7), (7, 8)\}$ or $(i, j) \in \{(1, 3), (1, 4), (1, 6), (1, 7), (2, 3), (2, 5), (2, 6), (2, 8), (3, 4), (3, 5), (3, 7), (3, 8), (4, 6), (4, 7), (5, 6), (5, 8), (6, 7), (6, 8)\}$. □

**Lemma 1.3.** *Let $\lambda_j \in \mathbf{C} (1 \le j \le n)$ be mutually distinct, and let $f_{j,A} = \lambda_j f_j$ for some $A \in GL(3, \mathbf{C})$ and $f_j \in \mathbf{C}[x, y, z]$. If $f = f_1 + \cdots + f_n \neq 0$ satisfies $f_A = \lambda f$ for some $\lambda \in \mathbf{C}$, then $\lambda = \lambda_i$ for some $i$, and $f_j = 0$ for $j \neq i$.*

Proof. We have $\lambda^k f = {\lambda_1}^k f_1 + \cdots + {\lambda_n}^k f_n$ for $0 \le k < n$. Multiplying the inverse of the Vandermonde matrix, we get $f_j = c_j f (1 \le j \le n)$ for some $c_j \in \mathbf{C}$. Thus $c_j(\lambda_j - \lambda) f = 0$. Since $f$ is assumed not to be the zero polynomial, the lemma follows. □

**Proposition 1.4.** *Let $f$ be a plane sextic. If* $\mathrm{Aut}(f)$ *has a subgroup $G_9$ isomorphic to $\mathbf{Z}_9$, then $C(f)$ has a singular point.*

Proof. Let $A_1 = \mathrm{diag}[1, \varepsilon, \varepsilon]$, $A_2 = \mathrm{diag}[1, \varepsilon, \varepsilon^2]$ and $A_3 = \mathrm{diag}[1, \varepsilon, \varepsilon^3]$. By Lemma 1.2 we may assume that $f_{A_j^{-1}} = \lambda_j f$ for some $\lambda_j \in \mathbf{C} (1 \le j \le 3)$. Since $A_j^9 = E_3$, it follows that ${\lambda_j}^9 = 1$. In addition any monomial $m$ satisfies $m_{A_j^{-1}} = \varepsilon^i m$ for some $i$. Suppose that a homogeneous polynomial $f'(x, y, z)$ of degree $d \ge 2$. Then $(1, 0, 0)$ is a singular point of $C(f')$ if and only if $f'$ contains none of three monomials $x^d$, $x^{d-1}y$ and $x^{d-1}z$. In the following table we summarize the values $i$ such that $m_{A_j^{-1}} = \varepsilon^i m$ for each $j = 1, 2, 3$ and for special 9 monomials. The proposition is immediate from the table.

| | $x^6$ | $x^5y$ | $x^5z$ | $y^6$ | $y^5x$ | $y^5z$ | $z^6$ | $z^5x$ | $z^5y$ |
|---|---|---|---|---|---|---|---|---|---|
| (1) | 0 | 1 | 1 | 6 | 5 | 6 | 6 | 5 | 6 |
| (2) | 0 | 1 | 2 | 6 | 5 | 7 | 3 | 1 | 2 |
| (3) | 0 | 1 | 3 | 6 | 5 | 8 | 0 | 6 | 7 |

□

**Proposition 1.5.** *No subgroup of $PGL(3, \mathbf{C})$ is isomorphic to $\mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3$.*

Proof. Assume that a subgroup $G$ of $PGL(3, \mathbf{C})$ is isomorphic to $\mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3$. Then there exist $A_1$, $A_2$, $A_3 \in GL(3, \mathbf{C})$ such that $A_1^3 = A_2^3 = A_3^3 = E_3$, $A_i A_j \sim A_j A_i$ for any $1 \le i < j \le 3$, and $G = \langle (A_1), (A_2), (A_3) \rangle$. Let $\omega$ be a primitive 3rd root of

1. We may assume that $G$ contains $(W)$ of the form $(\mathrm{diag}[1,1,\omega])$ or $(\mathrm{diag}[1,\omega,\omega^2])$. We will show that the first case implies the second case. Since $WA_j \sim A_jW$, the (3,1), (3,2), (1,3) and (2,3) components of $A_j (j=1,2,3)$ vanish. So we can assume that $A_1 = \mathrm{diag}[\omega^m,\omega^n,\omega]$ for some $0 \leq m,n < 3$, If $n = m$, then $n \neq 1$, and $A_2 = \mathrm{diag}[\omega^{m'},\omega^{n'},\omega]$ with $n' \neq m'$. Thus $(\mathrm{diag}[1,\omega,\omega^2]) \in G$. We will show that the assumption $(A) = (\mathrm{diag}[1,\omega,\omega^2]) \in G$ leads to a contradiction. Let $P_1 = (1,0,0)$, $P_2 = (0,1,0)$, and $P_3 = (0,0,1)$. Then $G$ fixes 3-point set $K = \{P_1, P_2, P_3\}$, because $(A)$ and $(A_j)$ commute. Since some $A_j$ is not diagonal, the homomorphism $\varphi$ from $G$ to the permutaion group of $K$ cannot be trivial. Since $|G| = 27$, it cannot be surjective. Thus $|\varphi(G)| = 3$, and $|\mathrm{Ker}\varphi| = 9$. In other words evry projectivety $(\mathrm{diag}[1,\omega^i,\omega^j])$ belongs to $G$. Since $G$ is commutative, any element of $G$ is induced by a diagonal matrix of order 3. This implies that $|G| = 9$, a desired contradiction. □

We turn to the group $E(3^3)$. See the paragraph just below Theorem 1.1 for the definition of the group and its element $M(\alpha,\beta,\gamma)$.

**Lemma 1.6.** (1) *Let*

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}.$$

*The map $\phi$ defined by $\phi(M(\alpha,\beta,\gamma)) = (B_1^\alpha B_2^\beta B_3^{\gamma-\alpha\beta})$ is an isomorphism of $E(3^3)$ into $PGL(3,\mathbf{C})$.*
(2) *If $G$ is a subgroup of $PGL(3,\mathbf{C})$ and isomorphic to $E(3^3)$, then $G$ is conjugate to $\phi(E(3^3))$.*

Proof. (1) Let $M_1 = M(1,0,0)$, $M_2 = M(0,1,0)$, $M_3 = M(0,0,1)$. Then $M(\alpha,\beta,\gamma) = M_1^\alpha M_2^\beta M_3^{\gamma-\alpha\beta}$. First we will prove that $\phi$ is a homomorphism by showing $\phi(M_jM(\alpha,\beta,\gamma)) = \phi(M_j)\phi(M(\alpha,\beta,\gamma))$. Clearly

$$\begin{aligned} M_1M(\alpha,\beta,\gamma) &= M(\alpha+1,\beta,\gamma+\beta) \\ M_2M(\alpha,\beta,\gamma) &= M(\alpha,\beta+1,\gamma) \\ M_3M(\alpha,\beta,\gamma) &= M(\alpha,\beta,\gamma+1). \end{aligned}$$

On the other hand , $B_j^3 = E_3$, $B_1B_2 = B_2B_1B_3$, $B_3B_1 = B_1B_3$, and $B_3B_2 = B_2B_3$. So $\phi$ is a homomorphism. Since $B_2$ and $B_3$ are diagonal, it is easy to see that $\phi$ is injective. Note that $\phi(E(3^3))$ does not depend on the choice of $\omega$, a primitive 3rd root of 1.
(2) Let $\phi'$ be an isomorphim of $E(3^3)$ into $PGL(3,\mathbf{C})$, and $\phi'(M_j) = (B_j')$. We may assume $B_3' = B_3$ or $B_3' = B_2$. The latter case is impossible. Since $B_3'B_1' \sim B_1'B_3'$ and $B_3'B_2' \sim B_2'B_3'$, we may assume $B_1' = \mathrm{diag}[\omega_1,\omega_2,1]$, and (1,3), (2,3), (3,1) and (3,2) components of $B_2'$ are equal to zero. It is not difficult to see that $B_1'B_2' \sim B_2'B_1'B_3'$ is

impossible. So let $B_3' = B_3$ and let $e_i$ denote the $i$-th unit column vector so that $E_3 = [e_1, e_2, e_3]$. A matrix $B \in GL(3, \mathbf{C})$ satisfies $BB_3 \sim B_3B$ if and only if either $B$ is diagonal or takes the form either $[e_2, e_3, e_1]\mathrm{diag}[a, b, c]$ or $[e_3, e_1, e_2]\mathrm{diag}[a, b, c]$. First assume that $B_2' = \mathrm{diag}[\omega_1, \omega_2, \omega_3]$. We may assume $1 = \omega_1 = \omega_2 \neq \omega_3$ (if necessary, we replace $\omega$ by $\omega^2$). Furthermore, we may assume $\omega_3 = \omega$(if necessary, we replace $\omega$ by $\omega^2$) so that $B_2' = B_2$. Since $(B_2')$ and $(B_1')$ do not commute, $B_1'$ cannot be diagonal. It turns out $B_1' = [e_3, e_1, e_2]\mathrm{diag}[a, b, c]$. By use of a diagonal matrix, we may assume that $a = b = c$, namely $B_1' = B_1$. Secondly assume that $B_1' = \mathrm{diag}[\omega_1, \omega_2, \omega_3]$. We note that the map sending $M(\alpha, \beta, \gamma)$ to $M(\beta, \alpha, \gamma)$ is an anti-isomorphism. Therefore $\phi'$ gives an isomorphism $\phi''(M(\alpha, \beta, \gamma)) = \phi'(M(\beta, \alpha, \gamma)^{-1})$. $\phi''$ is an isomorphism whose type we have discussed. Namely, $\phi'(E(3^3))$= $\phi''(E(3^3))$ is conjugate to $\phi(E(3^3))$. Thirdly and finally assume that neither $B_1'$ nor $B_2'$ is diagonal. Let $B_2' = [e_2, e_3, e_1]$(without loss of generality we may take $a = b = c = 1$). Then we can show that if $B_1'$ takes the form either $[e_2, e_3, e_1]\mathrm{diag}[a, b, c]$ or $[e_3, e_1, e_2]\mathrm{diag}[a, b, c]$ with $|\{a, b, c\}| = 2$, $ac = b^2\omega$ and $a^2 = bc\omega^2$, then $\phi'(M(\alpha, \beta, \gamma)) = (B_1'^{\alpha} B_2'^{\beta} B_3'^{\gamma-\alpha\beta})$ is an isomorphism(if $|\{a, b, c\}| = 1$ *or* 3, this $\phi'$ cannot be an isomorphism). Clearly $\phi'(E(3^3)) = \phi(E(3^3))$. The case $B_2' = [e_3, e_1, e_2]$ can be reduced to the case $B_2' = [e_2, e_3, e_1]$ by use of the matrix $[e_1, e_3, e_2]$. □

Let $f \in \mathbf{C}[x_1, x_2, x_3]$ be a homogeneous polynomial and let $h$ be its Hessian $\mathrm{Hess}(f) = \det[f_{jk}]$, where $f_{jk} = (\partial^2/\partial x_j \partial x_k)f$.

**Lemma 1.7.** *Let $A = [a_{jk}] \in GL(3, \mathbf{C})$, and let $f$ be a homogeneous polynomial in $\mathbf{C}[x_1, x_2, x_3]$ such that $f_{A^{-1}} = \lambda f$. Then $h_{A^{-1}} = \lambda^3(\det A^{-1})^2 h$, where $h = \mathrm{Hess}(f)$.*

Proof. Let $y_j = \sum_{k=1}^3 a_{jk}x_k$. By our assumption $\lambda f(x_1, x_2, x_3) = f(y_1, y_2, y_3)$. Hence

$$\begin{aligned} \lambda f_j(x_1, x_2, x_3) &= \sum_{\ell} f_\ell(y_1, y_2, y_3) a_{\ell j} \\ \lambda f_{jk}(x_1, x_2, x_3) &= \sum_{\ell} \sum_{\ell'} f_{\ell\ell'}(y_1, y_2, y_3) a_{\ell' k} a_{\ell j}. \end{aligned}$$

The second equality yields $\lambda^3 h(x_1, x_2, x_3) = h_{A^{-1}}(x_1, x_2, x_3)(\det A)^2$. □

**Lemma 1.8.** *Let the marices $B_j$ be as in* Lemma 1.6. *A non-singular sextic $f$ is invariant under all $(B_j)$ if and only if*

$$f \sim x^6 + y^6\alpha^2 + z^6\alpha + \kappa(x^3y^3 + y^3z^3\alpha^2 + z^3x^3\alpha),$$

*where $\alpha^3 = 1$ with $(\kappa^2 - 4\alpha^2)(\kappa^3 - 3\alpha\kappa^2 + 4) \neq 0$.*

Proof. First we will show that a non-singular sextic $f$ invariant under all $(B_j)$ takes the form as in the lemma. Note that $f_{B_3^{-1}} = \omega^j f$ and $f_{B_2^{-1}} = \omega^k f$ for some $j, k \in \{0, 1, 2\}$. One can easily see that unless $(j, k) = (0, 0)$, $f$ is singular. So $f$ takes the form $f = a_1x^6 + a_2y^6 + a_3z^6 + a_4x^3y^3 + a_5y^3z^3 + a_6z^3x^3$. Since $f_{B_1^{-1}} = a_3x^6 + a_1y^6 + a_2z^6 + a_6x^3y^3 + a_4y^3z^3 + a_5z^3x^3$ must be equal to $\lambda f$, where $\lambda^3 = 1$(note that $B_1^3 = E_3$), we get $(a_1, a_2, a_3) = \lambda(a_3, a_1, a_2)$, and $(a_4, a_5, a_6) = \lambda(a_6, a_4, a_5)$. Therefore $a_2 = \lambda a_1$, $a_3 = \lambda^2 a_1$, $a_5 = \lambda a_4$, $a_6 = \lambda^2 a_4$. We note that $a_1 \neq 0$, because, otherwise, $f$ is singular.

Let $f = x^6 + y^6\alpha^2 + z^6\alpha + \kappa(x^3y^3 + y^3z^3\alpha^2 + z^3x^3\alpha)$, where $\alpha^3 = 1$. Obviously $f$ is invariant under all $(B_j)$. We will discuss when $C(f)$ has a singular point. Simple computation yields

$$\begin{aligned} f_x &= 3x^2(2x^3 + \kappa y^3 + \kappa\alpha z^3) \\ f_y &= 3y^2(\kappa x^3 + 2\alpha^2 y^3 + \alpha^2\kappa z^3) \\ f_z &= 3z^2(\alpha\kappa x^3 + \alpha^2\kappa y^3 + 2\alpha z^3). \end{aligned}$$

If $(a, b, c)$ is a common zero of the three linear forms in $x^3, y^3, z^3$ above, then the determinant of the coefficient matrix vanishes, namely $\kappa^3 - 3\alpha\kappa^2 + 4 = 0$. Conversely, if this determinant vanishes, then $C(f)$ has clearly a singular point. If the determinant does not vanish and $C(f)$ has a singular point $(a, b, c)$, then one of $a, b, c$ is equal to zero and $4\alpha^2 - \kappa^2 = 0$. It is clear that $C(f)$ has a singular point if $4\alpha^2 - \kappa^2 = 0$. Thus $C(f)$ has a singular point if and only if $(\kappa^3 - 3\alpha\kappa^2 + 4)(4\alpha^2 - \kappa^2) = 0$. □

**Lemma 1.9.** $|\mathrm{Aut}(f)| < 360$, *where $f$ is a non-singular sextic given in* Lemma 1.8.

Proof. The Hessian $h = \mathrm{Hess}(f)$ takes the form $54h_1h_2$, where $h_1 = xyz$ and

$$\begin{aligned} h_2 = {} & 20\alpha\kappa^2(x^9 + y^9 + z^9) + (-5\alpha\kappa^3 + 20\alpha^2\kappa^2 + 100\kappa)(x^6y^3 + y^6z^3 + z^6x^3) \\ & +(-5\alpha^2\kappa^3 + 20\kappa^2 + 100\alpha\kappa)(x^3y^6 + y^3z^6 + z^3x^6) + (35\kappa^3 - 75\alpha\kappa^2 + 500)x^3y^3z^3. \end{aligned}$$

We consider a set of lines $L = \{\ell;\ \ell \textit{ is a line such that } \ell | h\}$. By Lemma 1.7 $\mathrm{Aut}(f)$ acts on $L$ as $(A)\ell = \{(A)P; P \in \ell\}$. Denoting the line $x = 0$ by $\ell_x$, let $G_x = \{(A) \in \mathrm{Aut}(f); (A)\ell_x = \ell_x\}$. Obviously $|\mathrm{Aut}(f)\ell_x| \leq |L| \leq 12$. By the way we remark that $|L| = 12$ for $f' = x^6 + y^6 + z^6 - 10(x^3y^3 + y^3z^3 + z^3x^3)$(Indeed, the $3 \times 3$ matrix $B$ whose row vectors are $[1, 1, 1]$, $[1, \omega, \omega^2]$ and $[1, \omega^2, \omega]$, $\omega$ being a primitive the third root of 1, satisfies $f'_{B^{-1}} = -27f'$). Assume $(A) \in G_x$. Without loss of generality $A$ takes the form

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a & b & c \\ a' & b' & c' \end{bmatrix} \in GL(3, \mathbf{C}).$$

Putting $Y = by + cz$ and $Z = b'y + c'z$, we get $f_{A^{-1}} = p_0x^6 + x^5p_1(Y, Z) + x^4p_2(Y, Z) + x^3p_3(Y, Z) + x^2p_4(Y, Z) + xp_5(Y, Z) + p_6(Y, Z)$. Since this polynomial is proportional to $f$, $p_5(Y, Z) = 6a\alpha^2Y^5 + 3\kappa\alpha^2(a'Y^3Z^2 + aY^2Z^3) + 6a'\alpha Z^5$ must vanish, namely $a = a' = 0$. Now $f_{A^{-1}} = x^6 + \kappa x^3(Y^3 + Z^3\alpha) + Y^6\alpha^2 + \kappa Y^3Z^3\alpha^2 + Z^6\alpha$. Assuming first $\kappa \neq 0$, we will show that $|G_x| = 18$ to the effect that $|\mathrm{Aut}(f)| \leq 18 \times 12 = 216$. By simple computaion $Y^3 + Z^3\alpha = y^3(b^3 + b'^3\alpha) + 3y^2z(b^2c + b'^2c'\alpha) + 3yz^2(bc^2 + b'c'^2\alpha) + z^3(c^3 + c'^3\alpha)$.

Since this must be equal to the polynomial $y^3 + z^3\alpha$, it follows that $b^2c + b'^2c'\alpha = 0$, and $bc^2 + b'c'^2\alpha = 0$. Multiplying $c$ and $b$ to each equality and then by subtraction, we get $b'c'(cb' - bc') = 0$, namely $b'c' = 0$, because $A$ is non-singular. If $b' = 0$, then $c = 0$, $b^3 = 1$, $c'^3 = 1$. It can be immediately seen that with these values $(A)$ really belongs to $G_x$. If $c' = 0$, then $b = 0$, $b'^3 = \alpha^2$, $c^3 = \alpha$. It can be also verified that with these values $(A)$ belongs to $G_x$. Thus, if $\kappa \neq 0$, then $|G_x| = 2 \times 9$. If $\kappa = 0$, then $h = \mathrm{const}x^4y^4z^4$, in particular, $L = \{x, y, z\}$. One can see easily that $G_x$ consisits of $2 \times 6^2$ points. Since $\mathrm{Aut}(f)$ acts transitively on $L$, we have $|\mathrm{Aut}(f)| = |L| \times |G_x| = 216$ (see [10, p. 171] or [8] for the automorphism group of the Fermat curves). □

## 2. Uniqueness of sextics with $|\mathrm{Aut}(f)|$=360

In the previous section we have shown that $|\mathrm{Aut}(f)| \leq 360$ for a non-singular plane sextic $f$. It is, therefore, reasonable to call a non-singular plane sextic $f$ satisfying $|\mathrm{Aut}(f)| = 360$, the most symmetric. The Wiman sextic

$$f_6 = 27z^6 - 135z^4xy - 45z^2x^2y^2 + 9z(x^5 + y^5) + 10x^3y^3$$

is known to be the most symmetric [11]. The aim of this section is to prove the

**Theorem 2.1.** *The most symmetric sextics are projectively equivalent to the Wiman sextic.*

As a byproduct another proof of $|\mathrm{Aut}(f_6)| = 360$ will be given (see Proposition 2.22).

There are five groups of order 8 up to isomorhism ([3, chap. 4]):

1) $\mathbf{Z}_8$
2) $\mathbf{Z}_2 \times \mathbf{Z}_4$
3) $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$
4) $Q_8$, which is generated by $a$ and $b$ satisfying $a^4 = 1$, $b^2 = a^2$, and $ba = a^{-1}b$
5) $D_8$, which is generated by $a$ and $b$ satisfying $a^4 = 1$, $b^2 = 1$, and $ba = a^{-1}b$.

In a series of lemmas we will show that if $f$ is the most symmetric sextic, then the Sylow 2-subgroup of $\mathrm{Aut}(f)$ is isomorphic to $D_8$.

**Lemma 2.2.** *A subgroup $G_8$ of $PGL(3, \mathbf{C})$ is isomorphic to $\mathbf{Z}_8$, if and only if $G_8$ is conjugate to one of the following groups*:

$$(1)\ \langle(\mathrm{diag}[1,1,\varepsilon])\rangle \quad (2)\ \langle(\mathrm{diag}[1,\varepsilon,\varepsilon^2])\rangle$$
$$(3)\ \langle(\mathrm{diag}[1,\varepsilon,\varepsilon^3])\rangle \quad (4)\ \langle(\mathrm{diag}[1,\varepsilon,\varepsilon^4])\rangle.$$

Proof. Suppose that $G_8$ and $\mathbf{Z}_8$ are isomorphic. Then there exists an $A \in GL(3,\mathbf{C})$ such that $G_8 = \langle(A)\rangle$. Since $(A)$ is of finite order, $A$ is diagonalizable; $T^{-1}AT \sim \mathrm{diag}[1,\varepsilon^i,\varepsilon^j](0 \le i < j \le 7)$, where $\varepsilon$ is a primitive 8-th root of 1. Clearly $(i,j) \notin \{(0,2),(0,4),(0,6),(2,4),(2,6),(4,6)\}$. If $i = 0$, then $G_8$ is conjugate to (1). If $(i,j) \in \{(1,2),(1,7),(2,5),(3,5),(3,6),(6,7)\}$, then $G_8$ is conjugate to (2). If $(i,j) \in \{(1,3),(1,6),(2,3),(2,7),(5,6),(5,7)\}$, then $G_8$ is conjugate to (3). Finally if $(i,j) \in \{(1,4),(1,5),(3,4),(3,7),(4,5),(4,7)\}$, then $G_8$ is conjugate to (4). □

**Lemma 2.3.** *The projective automorphism group* Aut$(f)$ *of a non-singular sextic $f$ has a subgroup isomorphic to $\mathbf{Z}_8$, if and only if $f$ is projectively equivalent to a sextic of the form $f' = x^6 + Bx^2y^2z^2 + y^5z + yz^5$ with $B^3 + 27 \neq 0$.*

Proof. Assume that Aut$(f)$ has a subgroup isomorphic to $\mathbf{Z}_8$. Let $A$ denote one of the follwoing four matrices; $\mathrm{diag}[1,1,\varepsilon]$, $\mathrm{diag}[1,\varepsilon,\varepsilon^2]$, $\mathrm{diag}[1,\varepsilon,\varepsilon^3]$, $\mathrm{diag}[1,\varepsilon,\varepsilon^4]$, where $\varepsilon$ is a primitive 8-th root of 1. By Lemma 2.2 $f$ is projectively equivalent to a sextic $f'$ such that $f'_{A^{-1}} = \varepsilon^j f'$ for some $0 \le j < 8$. One can easily see that such an $f'$ is singular except for the case $(A,j) = (\mathrm{diag}[1,\varepsilon,\varepsilon^3],0)$(see the proof of Proposition 1.4). In this exceptional case $f'$ is a linear combination of monomials $x^6, x^2y^2z^2, y^5z, yz^5$. Since $f'$ is assumed to be non-singular, it takes the form $x^6 + Bx^2y^2z^2 + (y^5z + yz^5)$ up to projective equivalence. Suppose that $C(f')$ has a singular point $(a,b,c)$. It is immediate that $abc \neq 0$. It is a common zero of $f_1 = 3x^4 + By^2z^2$, $f_2 = 2Bx^2yz + 5y^4 + z^4$ and $f_3 = 2Bx^2yz + y^4 + 5z^4$. Being on $C(f_2)$ and $C(f_3)$, $(a,b,c)$ satisfies $Ba^2c + 3b^3 = 0$ and $Ba^2b + 3c^3 = 0$, hence $B^2a^4 = 9b^2c^2$. Since $B^2f_1(a,b,c) = 0$, we get $(27 + B^3)b^2c^2 = 0$, namely $B^3 + 27 = 0$. Conversely, if $B^3 + 27 = 0$, then $(\sqrt{-3/B},1,1)$ is a singular point of $C(f')$. □

We cite two theorems concerning a flex of a plane curve.

**Theorem 2.4** ([2, p. 70]). *A point $P$ on an irreducible plane curve $C(f)$ is a simple point if and only if the local ring $\mathcal{O}_P(f)$ is a discrete valuation ring. In this case, if $L = ax + by + cz$ is a line through $P$ different from the tangent to $C(f)$ at $P$, then the image $\ell$ of $L$ in $\mathcal{O}_P(f)$ is a uniformizing parameter for $\mathcal{O}_P(f)$.*

**Theorem 2.5** ([2, p. 116]). *Let $h$ be the Hessian of an irreducible $f$.*

(1) *$P$ lies both on $C(h)$ and $C(f)$, if and only if $P$ is a flex or a multiple point of $f$.*

(2) *The intersection number $I(P, h \cap f)$ is equal to* 1 *if and only if $P$ is an ordinary*

*flex.* (*Note that if $P$ is a simple point of $C(f)$ and $C(\ell)$ is the tangent at $P$ to $C(f)$, then $I(P, h \cap f) = \operatorname{ord}_P^f(h)$* [2, p. 81], *which is equal to $I(P, \ell \cap f) - 2 =$ $\operatorname{ord}_P^f(\ell) - 2$* [2, Proof on p. 116].)

The following lemma shows that a Sylow 2-subgroup of $\operatorname{Aut}(f)$ of the most symmetric sextic $f$ cannot be isomorphic to $\mathbf{Z}_8$.

**Lemma 2.6.** *If $f' = x^6 + Bx^2y^2z^2 + y^5z + yz^5$ with $B^3 + 27 \neq 0$, then $|\operatorname{Aut}(f')| <$ 360.*

Proof. Since $f'(x, 1, z) = x^6 + Bx^2z^2 + z + z^5$, $P = (0, 1, 0)$ is a flex of $C(f')$. The tangent to $C(f')$ at $P$ is $C(z)$. Since $\operatorname{ord}_P^{f'}$ is a discrete valuation of the local ring $\mathcal{O}_P(f')$, and $x$ is a uniformizing parameter of the ring, namely $\operatorname{ord}_P^{f'}(x) = 1$, we get $\operatorname{ord}_P^{f'}(z) = 6$. Simple calculation yields the Hessian $h' = \operatorname{Hess}(f')$, which takes the form $-360B^2x^8y^2z^2 - 750x^4\{y^8 + z^8 + (10500 + 40B^3)y^4z^4\} - 160b^2x^2(y^7z^3 + y^3z^7) - 50B(y^{10}z^2 + y^2z^{10}) + 700By^6z^6$. So $I(P, h' \cap f') = \operatorname{ord}_P^{f'}(h') = 4$. This value can be obtained as $\operatorname{ord}_P^{f'}(z) - 2$ by Theorem 2.5 (2). Let $G_P = \{(A) \in \operatorname{Aut}(f'); (A)P = P\}$. Since $(A) \in G_P$ fixes $P$ as well as the tangent $C(z)$, we may assume that

$$A = \begin{bmatrix} a & 0 & c \\ a' & b' & c' \\ 0 & 0 & 1 \end{bmatrix}.$$

The condition $f'_{A^{-1}} \sim f'$ implies that $a' = c' = 0$, because $5(b'y)^4(a' + c'z)z$ must vanish in $f'_{A^{-1}}$. Such an $(A)$ belongs to $G_P$ if and only if $b'^4 = 1$, $a^6 = b'$, and $Ba^2b' = B$. Thus $|G_P|$ is equal to 8 or 24 according as $B \neq 0$ or $B = 0$. In the case $B \neq 0$, we evaluate the order of the group $\operatorname{Aut}(f')$ as follows:

$$4\left(\frac{|\operatorname{Aut}(f')|}{|G_P|}\right) = I(P, h' \cap f')\left(\frac{|\operatorname{Aut}(f')|}{|G_P|}\right) \leq \sum_Q I(Q, h' \cap f') = 12 \times 6.$$

Thus $|\operatorname{Aut}(f')| \leq 144$.

Suppose $B = 0$. In this case $h' = -750x^4(y^8 - 14y^4z^4 + z^8)$, and $h'$ contains 9 linear factors; $x$ with multiplicity four, and $\sqrt{-1}^j(7 \pm 4\sqrt{3})y - z (0 \leq j \leq 3)$ with multiplicity one. Let $G_x = \{(A) \in \operatorname{Aut}(f'); (A)\ell_x = \ell_x\}$, where $\ell_x$ stands for the line $C(x)$. By Lemma 1.7 $G_x = \operatorname{Aut}(f')$. We shall show that $|G_x| = 144$. Assume that $(A) \in G_x$. $(A)$ fixes both $C(f)$ and $C(x)$. Note that each tangent to $C(f)$ at the intersection $\in C(f) \cap C(x)$ passes through $(1, 0, 0)$. So $(A)$ fixes $(1, 0, 0)$ as well. Thus $A$ takes the form

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 0 & b' & c' \end{bmatrix}$$

up to constant multiplication. Putting $Y = by + cz$, $Z = b'y + c'z$, we write $f'_{A^{-1}}$ as $Y^5Z+YZ^5+x^6$. Now $(A)$ belongs to $G_x$ if and only if $y^5z+yz^5 = Y^5Z+YZ^5$. The right-hand side takes the form $y^6(b^5b'+bb'^5)+\cdots+z^6(c^5c'+cc'^5)$. Therefore $bb'(b^4+b'^4) = 0$, and $cc'(c^4 + c'^4) = 0$. If $b = 0$, then it follows immediately that $c' = 0$, and $c^5b' = cb'^5 = 1$. The number of such an $(A)$ is equal to 24. Similarly the case $b' = 0$ gives another 24 elements of $G_x$. The case $cc' = 0$ does not give new $(A) \in G_x$. We turn to the case $bb'cc' \neq 0$. In this case $b^4 + b'^4 = c^4 + c'^4 = 0$. Since the coefficient of $y^4z^2$ vanishes, $b^2c^2 + b'^2c'^2 = 0$. Under these conditions the coefficients of $y^2z^4$, $y^3z^3$ vanish. The coefficients of $y^5z$ and $yz^5$ yield the condition $1 = -4b^4(bc' - b'c)$ and $1 = 4c^4(bc' - b'c)$ respectively. In particular $c^4 = -b^4$. Therefore if $bb'cc' \neq 0$, then $(A) \in G_x$ if and only if $b^4 + b'^4 = 0$, $c^4 + c'^4 = 0$, $b^4 + c^4 = 0$, $b^2c^2 + b'^2c'^2 = 0$, and $4b^4(-bc' + b'c) = 1$. Thus $b' = \sqrt{-1}^j(1 + \sqrt{-1})b/\sqrt{2}$, $c' = \sqrt{-1}^k(1 + \sqrt{-1})c/\sqrt{2}$ with $0 \leq j, k \leq 3$ and $j + k \equiv 0 \bmod 2$, $c = \sqrt{-1}^\ell(1 - \sqrt{-1})b/\sqrt{2}$ with $0 \leq \ell \leq 3$ such that $4b^6(\sqrt{-1}^j - \sqrt{-1}^k)\sqrt{-1}^\ell = 1$. It is easy to see that each $j$ gives one admissible value of $k$, that $\ell$ can be arbitrary, and that $b$ can take six values for an addmissible $(j, k, \ell)$. Consequently there exist $4 \times 4 \times 6$ $(A) \in G_x$ such that $bb'cc' \neq 0$. Hence $|G_x| = 24 + 24 + 96 = 144$. This completes the proof of Lemma 2.6. □

**Lemma 2.7.** *A subgroup $G_8$ of $PGL(3, \mathbf{C})$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_4$, if and only if $G_8$ is conjugate to one of the following two groups*:
(1) $\langle(\mathrm{diag}[-1, 1, 1]),\ (\mathrm{diag}[1, \sqrt{-1}, \sqrt{-1}])\rangle$
(2) $\langle(\mathrm{diag}[-1, 1, 1]),\ (\mathrm{diag}[1, \sqrt{-1}, \sqrt{-1}^2])\rangle$.

Proof. Assume that $G_8$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_4$. Then there exist commuting $(A)$, and $(B)$ in $PGL(3, \mathbf{C})$ of order 2 and 4 respectively. We may assume that $A^2 = E_3$ and $B$ takes the form either $\mathrm{diag}[1, 1, \sqrt{-1}]$ or $\mathrm{diag}[1, \sqrt{-1}, \sqrt{-1}^2]$. First suppose that $B = \mathrm{diag}[1, 1, \sqrt{-1}]$. Since $AB \sim BA$, (1,3),(2,3),(3,1) and (3,2) components of $A$ vanish. We may assume that (3,3) component of $A$ is equal to 1. Since $A$ is diagonalizable, we may assume that $A = \mathrm{diag}[-1, 1, 1]$. Secondly assume that $B = \mathrm{diag}[1, \sqrt{-1}, \sqrt{-1}^2]$. Since $AB \sim BA$, and $A$ is involutive, it follows that $A$ is diagonal; $A = \mathrm{diag}[a, b, 1]$. If $a = b$, then $a = -1$. There exists a $T \in GL(3, \mathbf{C})$ such that $T^{-1}AT \sim \mathrm{diag}[-1, 1, 1]$ and $T^{-1}BT \sim \mathrm{diag}[1, \sqrt{-1}^3, \sqrt{-1}^2]$, hence $T^{-1}B^3T \sim \mathrm{diag}[1, \sqrt{-1}, \sqrt{-1}^2]$. The case $a \neq b$ can be dealt with similarly. □

**Lemma 2.8.** *If a plane sextic is invariant under the group* (1) *or* (2) *in* Lemma 2.7, *then it is singular.*

Proof. Let $A = \mathrm{diag}[-1, 1, 1]$, $B_1 = \mathrm{diag}[1, 1, \sqrt{-1}]$, $B_2 = \mathrm{diag}[1, \sqrt{-1}, \sqrt{-1}^2]$, and let $B$ denote either $B_1$ or $B_2$. As in the proof of Proposition 1.4 we can show easily that a sextic $f$ satisfying $f_{B^{-1}} \sim f$ and $f_{A^{-1}} \sim f$ is singular. Indeed, if $f$ contains $x^6$, then $f_{B^{-1}} = f$, hence three monomials $z^6$, $z^5x$, $z^5y$ or three monomials $y^6$,

$y^5x$, $y^5z$ do not appear in $f$ according as $B = B_1$ or $B = B_2$. Suppose the monomial $x^6$ does not appear in $f$. If $f$ contains $x^5y$, then $f_{A^{-1}} = -f$ and $f_{B^{-1}} \sim f$ so that three monomials $z^6$, $z^5x$, $z^5y$ do not appear in $f$, namely $(0, 0, 1)$ is a singular point of $C(f)$. If $f$ contains $x^5z$, then $f_{A^{-1}} = -f$ and $f_{B^{-1}} \sim f$ so that three monomials $z^6$, $z^5x$, $z^5y$ do not appear in $f$. Finally if $f$ contains none of three monomials $x^6$, $x^5y$, and $x^5z$, then $(1, 0, 0)$ is a singualr point of $C(f)$. □

**Lemma 2.9.** *No subgroup of $PGL(3, \mathbf{C})$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.*

Proof. Let $(A)$ and $(B)$ be mutually distinct commuting involutions. We may assume that $A = \mathrm{diag}[-1, 1, 1]$, and $B = \mathrm{diag}[1, -1, 1]$. Assume that an involution $(C)$ commutes with both of them. Then $C$ is diagonal, hence $(C) \in \langle (A),\ (B)\rangle$. Namely, mutually distinct three commuting involutions in $PGL(3, \mathbf{C})$ generate a subgroup of order 4. □

**Lemma 2.10.** *A subgroup $G_8$ of $PGL(3, \mathbf{C})$ is isomorphic to $Q_8$, if and only if $G_8$ is conjugate to $\langle (\mathrm{diag}[1, \sqrt{-1}, \sqrt{-1}^3]),\ ([e_1, e_3, e_2]\mathrm{diag}[1, \sqrt{-1}, \sqrt{-1}])\rangle$, where $e_i$ is the $i$-th column vector of the unit matrix $E_3$.*

Proof. $G_8$ is isomorphic to $Q_8$, if and only if it is generated by some $(A)$ of order 4 and $(B)$ such that $(B)^2 = (A)^2$ and $(B)(A) = (A)^{-1}(B)$. Suppose that $G_8$ is isomorphic to $Q_8$. Since $(A)$ has order 4, we may assume that $A$ takes the form either $\mathrm{diag}[1, 1, \sqrt{-1}]$ or $\mathrm{diag}[1, \sqrt{-1}, \sqrt{-1}^3]$, for subgroups $\langle (\mathrm{diag}[1, \sqrt{-1}^j, \sqrt{-1}^k])\rangle (0 < j < k < 4)$ are mutually conjugate. If $A = \mathrm{diag}[1, 1, \sqrt{-1}]$, we can show easily that no $B \in GL(3, \mathbf{C})$ satisfies $ABA \sim B$. If $A = \mathrm{diag}[1, \sqrt{-1}, \sqrt{-1}^3]$, then, up to constant multiplication, $B = [e_1, e_3, e_2]\mathrm{diag}[1, b, c]$ with $bc = -1$ alone satisfies $B^2 \sim A^2$ and $ABA \sim B$. Transforming $B$ by a diagonal matrix we get the lemma. □

**Lemma 2.11.** *Any $Q_8$-invariant sextic is singular.*

Proof. Let $A = \mathrm{diag}[1, \sqrt{-1}, \sqrt{-1}^3]$ and $B = [e_1, e_3, e_2]\mathrm{diag}[1, \sqrt{-1}, \sqrt{-1}]$, and $f$ is a sextic. Suppose $f_{A^{-1}} = \sqrt{-1}^j f$ for some $0 \le j \le 3$. $f$ is a linear combination of monomials $m$ in $x, y, z$ satisfying $m_{A^{-1}} = \sqrt{-1}^j m$. If $j = 2$, then $f$ contains none of $x^6$, $x^5y$, and $x^5z$ so that $(1, 0, 0)$ is a singular point of $C(f)$. If $j \in \{1, 3\}$, then $x$ divides $f$. Finally if $j = 0$, then $f$ is a linear combination of eight monomials: $x^6, x^2y^4, x^2y^2z^2, x^2z^4, x^4yz, y^5z, y^3z^3, yz^5$. Since we also require that $f_{B^{-1}} \sim f$, $f$ is either a linear combination of the leading four monomials or a linear combination of the remaining four monomials. In either case $f$ is reducible. □

We have so far shown that a Sylow 2-subgroup of $\mathrm{Aut}(f)$ of the most symmetric sextic $f$ is isomorphic to $D_8$. We turn to the study of a Sylow 5-subgroup of $\mathrm{Aut}(f)$

of the most symmetric sextic $f$.

**Lemma 2.12.** *A subgroup $G_5$ of $PGL(3, \mathbf{C})$ is isomorphic to $\mathbf{Z}_5$ if and only if $G_5$ is conjugate to either $G_{5,1} = \langle(\mathrm{diag}[1, 1, \varepsilon])\rangle$ or $G_{5,2} = \langle(\mathrm{diag}[1, \varepsilon, \varepsilon^2])\rangle$, where $\varepsilon$ is a primitive 5-th root of 1.*

Proof. We can argue as in the proof of Lemma 2.2. □

**Proposition 2.13.** *Let $f$ be a non-singular sextic. If* $\mathrm{Aut}(f)$ *contains a subgroup conjugate to $G_{5,1}$ in* Lemma 2.12, *then* $|\mathrm{Aut}(f)| < 360$.

Proof. Let a sextic $f$ satisfy $f_{A^{-1}} = \varepsilon^j f$, where $A = \mathrm{diag}[1, 1, \varepsilon]$. It turns out that unless $j = 0$, $f$ is singular. In the case $j = 0$, $f$ is a linear combination of monomials $x^{6-k}y^k$ $(0 \le k \le 6)$, $xz^5$ and $yz^5$. By change of variables $x' = ax+by$ and $y' = cx+dy$, we may assume that

$$f = C_0x^6 + C_1x^5y + C_2x^4y^2 + C_3x^3y^3 + C_4x^2y^4 + C_5xy^5 + C_6y^6 + xz^5,$$

where $C_6 = 1$, because if $C_6 = 0$, then $f$ is reducible. So $P = (0, 0, 1)$ is a flex of $C(f)$, $C(x)$ is the tangent there to $C(f)$, $y$ is a uniformizing parameter of $\mathcal{O}_P(f)$, and $\mathrm{ord}_P^f(x) = 6$. Let $h = \mathrm{Hess}(f)$. By Theorem 2.5 (2) $I(P, h \cap f) = \mathrm{ord}_P^f(x) - 2 = 4$. Using Bezout's theorem we get $4|\mathrm{Aut}(f)P| \le \sum_Q I(Q, h \cap f) = 72$. Let $G_P = \{(B) \in \mathrm{Aut}(f); (B)P = P\}$. If $|G_P| < 20$, then $|\mathrm{Aut}(f)| = |\mathrm{Aut}(f)P||G_P| < 360$. We will try to show that $|G_P| < 20$. Let $(B) \in G_P$. Then the first, the second and the third row of $B$ takes the form $[a, 0, 0]$, $[b, 1, 0]$, and $[a', b', c]$. Since $f_{B^{-1}} \sim f$, $a' = b' = 0$. Now $f_{B^{-1}}$ is of the following form:

$$\begin{aligned}
f_{B^{-1}} = {} & x^6(C_0a^6 + C_1a^5b + C_2a^4b^2 + C_3a^3b^3 + C_4a^2b^4 + C_5ab^5 + C_6b^6) \\
& + x^5y(C_1a^5 + 2C_2a^4b + 3C_3a^3b^2 + 4C_4a^2b^3 + 5C_5ab^4 + 6b^5) \\
& + x^4y^2(C_2a^4 + 3C_3a^3b + 6C_4a^2b^2 + 10C_5ab^3 + 15b^4) \\
& + x^3y^3(C_3a^3 + 4C_4a^2b + 10C_5ab^2 + 20b^3) \\
& + x^2y^4(C_4a^2 + 5C_5ab + 15b^2) \\
& + xy^5(C_5a + 6b) + y^6 + xz^5ac^5.
\end{aligned}$$

This polynomial is proportional to $f$, hence, equal to $f$. Therefore $ac^5 = 1$, and $b = C_5(1 - a)/6$. Substituting $b$ in the coefficients of $x^2y^4$, we get $(a^2 - 1)(C_4 - 5{C_5}^2/12) = 0$. If $C_4 \ne 5{C_5}^2/12$, then $a^2 = 1$, hence $|G_P| \le 10$. Suppose $C_4 = 5{C_5}^2/12$. Comparing the coefficients of $x^3y^3$, we get $(a^3 - 1)(C_3 - 5{C_5}^3/54) = 0$. Suppose $C_3 = 5{C_5}^3/54$(otherwise, $|G_P| \le 15$). Now

$$f = \left(x\frac{C_5}{6} + y\right)^6 + x^6\left(1 - \frac{{C_5}^6}{6^6}\right) + x^5y\left(C_1 - \frac{{C_5}^5}{6^4}\right) + x^4y^2\left(C_2 - \frac{{C_5}^4}{2 \cdot 6^3}\right) + xz^5.$$

By change of variables $x' = x$, $y' = xC_5/6 + y$, and $z' = z$, we get a projectively equivalent sextic, which will be denoted by $f$ again: $f = D_0x^6 + D_1x^5y + D_2x^4y^2 + y^6 + xz^5$. If $(B) \in G_P$, then $B = \mathrm{diag}[a, 1, c]$, where

$$D_0a^6 = D_0, \quad D_1a^5 = D_1, \quad D_2a^4 = D_2, \quad \text{and} \quad ac^5 = 1.$$

If $D_1D_2 \neq 0$, then $a = 1$, hence $|G_P| = 5$. If $D_1 = 0$ and $D_2 \neq 0$, then $D_0 \neq 0$, hence $a^2 = 1$ so that $|G_P| = 10$. Finally suppose that $D_1 \neq 0$, $D_2 = 0$ and that $f$ is non-singular, namely $6^6D_0^5 \neq 5^5D_1^6$. Then the line $C(z)$ intersects $C(f)$ at distinct six points. Besides $h = \mathrm{Hess}(f) = 250z^3h'$, where $h' = -3y^4z^5 + 24(3D_0x + 2D_1y)x^4y^4 - 2D_1{}^2x^9$. Note that $h'$ has no linear factors. Indeed, none of linear factors $z - \alpha x - \beta y$, $x - \alpha y$, and $y - \beta x$ divides $h'$. Let $G_z = \{(B) \in \mathrm{Aut}(f);\ (B) \text{ fixes the line } C(z)\}$. Since $\mathrm{Aut}(f) \subset \mathrm{Aut}(h)$ by Lemma 1.7, $(B) \in \mathrm{Aut}(f)$ fixes a line $C(z)$ and hence the point $P$ (see the proof of Lemma 2.6). In particular $G_z = \mathrm{Aut}(f) = G_P$, and $B$ takes the form $\mathrm{diag}[a, 1, c]$, where $a^5 = 1$ and $ac^5 = 1$. In particular $|\mathrm{Aut}(f)| = |G_P| \leq 5 \times 5$. □

**Lemma 2.14.** *Let $f$ be a non-singular sextic. The automorphism group of $f$ contains a subgroup conjugate to $G_{5,2}$, if and only if $f$ is projectively equivalent to one of the following forms*:
(1) $x^6 + C_1x^3yz^2 + C_2y^2z^4 + C_3x^2y^3z + x(y^5 + z^5)$
(2) $z^6 + Bz^4xy + Cz^2x^2y^2 + Dz(x^5 + y^5) + Ex^3y^3$
*If $f$ is the sextic* (1), *then* $|\mathrm{Aut}(f)| < 360$.

Proof. Let $A = \mathrm{diag}[1, \varepsilon, \varepsilon^2]$. Then each of the two sextics (1) and (2), say $f$, satisfies $f_{A^{-1}} \sim f$. Assume that $(A) \in \mathrm{Aut}(f)$ for a sextic $f$, namely $f_{A^{-1}} = \varepsilon^j f (j = 0, 1, 2, 3, 4)$. If $j = 3$ or $j = 4$, $f$ is singular. According as $j \in \{0, 2\}$ or $j = 1$, $f$ takes the form (1) or (2) up to projective equivalence. Assuming that $f$ takes the form (1), we shall show that $|\mathrm{Aut}(f)| < 360$. $P = (0, 1, 0)$ is a flex of $f$, and $C(x)$ is the tangent there. So $z$ is a uniformizing parameter of $\mathcal{O}_P(f)$. Since $\mathrm{ord}_P^f(x) \geq 4$, we can estimate the intersection number: $I(P, h \cap f) = \mathrm{ord}_P^f(x) - 2 \geq 2$, where $h$ is the Hessian of $f$. Let $G_P = \{(B) \in \mathrm{Aut}(f); (B)P = P\}$. If $(B) \in G_P$, then the first, the second and the third row of $B$ takes the form $[1, 0, 0]$, $[a, b, c]$, and $[a', 0, c']$ repectively, because $(B)$ fixes the line $C(x)$(i.e. $[1, 0, 0]B \sim [1, 0, 0]$) and $(B)P = P$. Since $f_{B^{-1}} \sim f$ and $C_2 \neq 0$, we get $c = 0$, $a = 0$, $a' = 0$, $b^5 = 1$ and $c' = b^2$. Thus $|G_P| = 5$. By Bezout's theorem $2|\mathrm{Aut}(f)|/|G_P| = 2|\mathrm{Aut}(f)P| \leq \sum_Q I(Q, h \cap f) \leq 72$, that is, $|\mathrm{Aut}(f)| \leq 180$. □

By Lemma 2.14 the most symmetric sextic is projectively equivalent to the following sextic :

$$f = z^6 + Bz^4xy + Cz^2x^2y^2 + Dz(x^5 + y^5) + Ex^3y^3.$$

Let $I = [e_2, e_1, e_3]$, where $E_3 = [e_1, e_2, e_3]$ is the unit matrix. Clearly $f_I = f$. If $f$ is the most symmetric sextic, then any Sylow 2-subgroup of $\mathrm{Aut}(f)$ is isomorphic to the group $D_8$. By Sylow's theorem the involution $(I)$ belongs to a Sylow 2-subgroup of $\mathrm{Aut}(f)$.

**Lemma 2.15.** (1) *If $g$ is an involution of $D_8$, then there exists an involution $g' \in D_8 \setminus \{g\}$ such $gg' = g'g$.*
(2) *Let $g$ and $g'$ be mutually distinct commuting involutions of $D_8$. Then one of the following cases takes place.*
1) *There exists an element $c \in D_8$ of order 4 such that $c^2 = g,\ g'cg' = c^{-1}$.*
2) *There exists an element $c \in D_8$ of order 4 such that $c^2 = g',\ gcg = c^{-1}$.*
3) *There exists an element $c \in D_8$ of order 4 such that $c^2 = gg',\ gcg = c^{-1}$.*

Proof. Let $a,\ b$ be generators of $D_8$ such that $a^4 = 1,\ b^2 = 1$ and $ba = a^{-1}b$. So $a$ generates a cyclic group $H$ of order 4, and $D_8 = H + bH$. An element $g \in D_8$ is an involution if and only if $g \in \{a^2\} \cup bH$. (1) If $g = a^2$, then we can take $g' = ba^2$. If $g = ba^j$, we can take $g' = ba^{j+2}$. (2) If $g = a^2$, then $g' \in bH$. So we can take $c = a$. If $g' = a^2$, then we can take $c = a$. Finally if $g, g' \in bH$, then $gg' = a^2$. So we can take $c = a$. □

**Lemma 2.16.** *Assume that $f = z^6 + Bz^4xy + Cz^2x^2y^2 + Dz(x^5+y^5) + Ex^3y^3$ is non-singular. If there exists an involution $(A) \in \mathrm{Aut}(f) \setminus \{(I)\}$ such that $(A)(I) = (I)(A)$, then $A$ takes the form*

$$\begin{bmatrix} \alpha & \beta & \gamma \\ \beta & \alpha & \gamma \\ \lambda & \lambda & 1 \end{bmatrix}, \quad \text{where} \quad \alpha + \beta + 1 = 0,\ \ \alpha\beta + 1 = 0,\ \ \gamma\lambda = 2,$$

*and*

$$(\star) \qquad \gamma^2 B = 12 - \gamma^5 D,\ \ \gamma^4 C = 48 + \gamma^5 D,\ \ \gamma^6 E = 64 - 2\gamma^5 D.$$

*Conversely, if $(\star)$ holds for some $\gamma \neq 0$, then the above matrix $A$ gives an involution $(A) \in \mathrm{Aut}(f) \setminus \{(I)\}$ such that $(A)(I) = (I)(A)$.*

Proof. Suppose that $\mathrm{Aut}(f)$ contains an involution $(A) \neq (I)$ commuting with $(I)$. Let $A = [a, b, c]$, where $a = [a_j]$, $b = [b_j]$ and $c = [c_j]$ are column vectors. We claim that $c_3 \neq 0$. Otherwise the condition $AI \sim IA$ yields $b_1 = \delta a_2,\ b_2 = \delta a_1$, $b_3 = \delta a_3$, and $c_2 = \delta c_1$. Since $A^2 \sim E_3$, we get $\delta = 1,\ a_1 + a_2 = 0$, and $c_1 a_3 = 2{a_1}^2$. However, $(A) \notin \mathrm{Aut}(f)$, because $f_{A^{-1}} = \sum z^j C_j$ with $C_1 = 10{a_1}^7 a_3 D(x+y)(x-y)^4 \not\sim D(x^5 + y^5)$. Note that $D \neq 0$ because of non-singularity of $f$. Thus we may assume that $c_3 = 1$. The condition $AI \sim IA$ implies that $a_2 = b_1,\ a_1 = b_2,\ a_3 = b_3$ and $c_2 = c_1$. We claim that $c_1 \neq 0$. If $c_1 = 0$, then the condition $(A) \in \mathrm{Aut}(f)$ yields

$a_3 = 0$ and $a_1 b_1 = 0$. Besides, by the condition $A^2 \sim E_3$, we get $A \sim E_3$ or $A \sim I$. Similarly $a_3 \neq 0$. For the sake of simplicity of notation we put $\alpha = a_1$, $\beta = b_1$, $\gamma = c_1$, and $\lambda = a_3$. Since $A^2 \sim E_3$, $\alpha + \beta + 1 = 0$, $2\alpha\beta + \gamma\lambda = 0$, and $\gamma\lambda \notin \{0, -1/2\}$. Under these conditions $A^2 = (2\gamma\lambda + 1)E_3$. Let $W = \mathrm{diag}[1, 1, 1/\gamma]$, $A' = W^{-1}AW$, and $f_{W^{-1}} = \gamma^{-6} f'$. $(A') \in \mathrm{Aut}(f')$, because $f'_{A'} = (f_{W^{-1}})_{A'} = f_{A'W^{-1}} = f_{W^{-1}A} = (f_A)_{W^{-1}} = (\mathrm{const} f)_{W^{-1}} = \mathrm{const} f_{W^{-1}} = \mathrm{const} f'$. By the next lemma $(A') \in \mathrm{Aut}(f')$ implies ($\star$). Conversely suppose ($\star$) holds. Let $f_{W^{-1}} = \gamma^6 f'$. By the next lemma there exists an involution $(A') \in \mathrm{Aut}(f') \setminus \{(I)\}$ such that $(A')(I) = (I)(A')$. Since $f'_W \sim f$, $A = WA'W^{-1}$ gives an involution $(A) \in \mathrm{Aut}(f) \setminus \{(I)\}$. □

**Lemma 2.17.** *Let $f$ be as in* Lemma 2.16, *and let*

$$A = \begin{bmatrix} a & b & 1 \\ b & a & 1 \\ d & d & 1 \end{bmatrix}, \quad \text{where} \quad a + b + 1 = 0, \quad 2ab + d = 0, \quad d \notin \left\{0, -\frac{1}{2}\right\}.$$

*Then $f_{A^{-1}} \sim f$ if and only if*

$$d = 2, \quad B = 12 - D, \quad C = 48 + D, \quad E = 64 - 2D.$$

Proof. We note that coefficents of $f_{A^{-1}}$ can be written without using $a$ and $b$. In fact we get the following formula.

$$\begin{aligned} f_{A^{-1}} = {} & z^5(x+y)\{6d + B(4d-1) + C(2d-2) + D(2d-5) + E(-3)\} \\ & + z^4(x^2+y^2)\{15d^2 + B(-9/2+6d)d + C(1-5d+d^2) + D(10+5d) \\ & \quad + E(3-(3/2)d)\} \\ & + z^3(x^3+y^3)\{20d^3 + B(-8d^2+4d^3) + C(3d-4d^2) + D(-10-5d+10d^2) \\ & \quad + E(-1+3d)\} \\ & + z^3(x^2y+xy^2)\{60d^3 + B(4d-16d^2+12d^3) + C(-2+9d-4d^2) \\ & \quad + D(25d-10d^2) + E(-9-3d)\} \\ & + z^2(x^4+y^4)\{15d^4 + B(-7d^3+d^4) + C((3+(1/4))d^2-d^3) \\ & \quad + D(5-(25/2)d^2) + E((-3/2)d+(3/4)d^2)\} \\ & + z^2(x^3y+xy^3)\{60d^4 + B(6d^2-16d^3+4d^4) + C(-5d+5d^2) \\ & \quad + D(-20d-10d^2) + E(3-3d-3d^2)\} \\ & + z(x^4y+xy^4)\{30d^5 + B(4d^3-7d^4) + C(-4d^2-(1/2)d^3) \\ & \quad + D((15/2)d+(15/4)d^2-(15/2)d^3) + E(3d+(9/4)d^2)\} \\ & + z(x^3y^2+x^2y^3)\{60d^5 + B(12d^3-6d^4) + C(2d-4d^2-d^3) \\ & \quad + D(-25/2)d^2+5d^3) + E(-3-3d-(3/2)d^2)\} \\ & + (x^6+y^6)\{d^6 + B(-(1/2)d^5) + C((1/4)d^4) \end{aligned}$$

$$
\begin{aligned}
&+ D(-1-(5/2)d-(5/4)d^2)d + E(-(1/8)d^3)\} \\
&+(x^5y+xy^5)\{6d^6+B(d^4-d^5)+C(-d^3-(1/2)d^4) \\
&\quad + D(-d+(5/2)d^3)+E((3/4)d^2+(3/4)d^3\} \\
&+(x^4y^2+x^2y^4)\{15d^6+B(4d^4+(1/2)d^5)+C(d^2-(1/4)d^4) \\
&\quad + D((5/2)d^2+(5/4)d^3)+E(-(3/2)d-3d^2-(15/8)d^3)\} \\
&+z^6\{1+B+C+2D+E\} \\
&+z^4xy\{30d^2+B(1-7d+12d^2)+C(4-6d+2d^2)+D(-30d)+E(9+3d)\} \\
&+z^2x^2y^2\{90d^4+B(12d^2-18d^3+6d^4)+C(1-6d+(15/2)d^2+2d^3) \\
&\quad + D(45d^2)+E(9+9d+(9/2)d^2)\} \\
&+z(x^5+y^5)\{6d^5+B(-3d^4)+C(3/2)d^3 \\
&\quad + D(-1+(5/2)d+(35/4)d^2+(5/2)d^3)+E(-3/4)d^2\} \\
&+x^3y^3\{20d^6+B(6d^4+2d^5)+C(2d^2+2d^3+d^4) \\
&\quad + D(-5d^3)+E(1+3d+(9/2)d^2+(5/2)d^3\}.
\end{aligned}
$$

Since $z^5x$ does not appear in $f$, we have $3E = 6d + B(4d-1) + C(2d-2) + D(2d-5)$. Since the coefficients of $z^4x^2$, $z^3x^3$, $z^3x^2y$ vanish, and $d \neq -1/2$, we get a system of linear equations on $B$, $C$, and $D$ as follows:

$$
\begin{aligned}
&B(-2d+1)+C(1)+D\left(\frac{1}{2}d-5\right)=6d, \\
&B\left(4d^2-6d+\frac{2}{3}\right)+C\left(-2d+\frac{4}{3}\right)+D\left(12d-\frac{50}{3}\right)=-20d^2+4d, \\
&B(12d^2-26d+6)+C(-6d+8)+D(-12d+30)=-60d^2+36d.
\end{aligned}
$$

The determinant of the coefficient matirx is equal to $50(4d+2)(-d+2)/3$. We claim that $d=2$. Assume the contrary. Cramer's formula yields $B=6d$, $C=12d^2$, and $D=0$. On the other hand $D \neq 0$, because $f$ is assumed to be non-singular. Thus $d = 2$. The above system of linear equations on $B$, $C$, and $D$, together with the equality $3E = 6d + B(4d-1) + C(2d-2) + D(2d-5)$ yields equalities $B = 12 - D$, $C = 48 + D$, and $E = 64 - 2D$. By easy computaion we get $f_{A^{-1}} = 125f$. □

Suppose $f$ is the most symmetric sextic. By Lemma 2.14 we may assume that $f$ takes the form given in Lemma 2.16. By Lemma 2.16, we may further assume that $B = 12 - D$, $C = 48 + D$, $E = 64 - 2D$.

**Lemma 2.18.** *Let $f$ be a sextic of the form $z^6 + Bz^4xy + Cz^2x^2y^2 + Dz(x^5+y^5) + Ex^3y^3$ with $B = 12 - D$, $C = 48 + D$, $E = 64 - 2D$. Let $M = \mathrm{diag}[1, 1, m] (m \neq 0)$. Then $f_{M^{-1}}$ is the Wiman sextic*

$$f_6 = 27z^6 - 135z^4xy - 45z^2x^2y^2 + 9z(x^5+y^5) + 10x^3y^3,$$

*if and only if* $[D, 1/m] = [(9 \pm 15\sqrt{15}\sqrt{-1})/2, (-3 \pm \sqrt{15}\sqrt{-1})/12]$. *In particular if* $D^2 - 9D + 864 = 0$, *then* $f$ *is projectively equivalent to the Wiman sextic.*

Proof. It is evident that $f$ satisfies the condition if and only if the following 4 equalities hold:
(1) $(12 - D)/m^2 = -135/27$
(2) $(48 + D)/m^4 = -45/27$
(3) $D/m^5 = 9/27$
(4) $(64 - 2D)/m^6 = 10/27$.
The equalities (2) and (3) imply $(48 + D)m/D = -5$, while (3) and (4) yield $(64 - 2D)/Dm = 10/9$. Thus $(48 + D)(64 - 2D) + 50D^2/9 = 0$, namely $D^2 - 9D + 864 = 0$. $m^{-1} = -(48 + D)/(5D)$ gives the value of $m^{-1}$. Conversely, since
$m^{-2} = -(1 \pm \sqrt{15}\sqrt{-1})/24$, $m^{-4} = (-7 \pm \sqrt{15}\sqrt{-1})/288$,
$12 - D = 15(1 \mp \sqrt{15}\sqrt{-1})/2$, and $48 + D = 15(7 \pm \sqrt{15}\sqrt{-1})/2$,
(1) and (2) hold, hence (3) and (4) as well. □

**Lemma 2.19.** *Let* $f$ *be as in* Lemma 2.18, *and let*

$$A = \begin{bmatrix} a & b & 1 \\ b & a & 1 \\ 2 & 2 & 1 \end{bmatrix}, \quad \textit{where } a + b + 1 = 0, \textit{ and } ab + 1 = 0,$$

$$B = \mathrm{diag}[\delta, \delta^4, 1], \quad \textit{where } \delta \textit{ is a primitive 5-th root of } 1.$$

*Then* $(AB^2) \in \mathrm{Aut}(f)$ *and* $\mathrm{ord}((AB^2)) = 3$.

Proof. Let $G$ be the subgroup of $\mathrm{Aut}(f)$ generated by $(A)$, $(I)$ and $(B)$. Let $P_1 = (1, 0, 0)$. It is a flex of $C(f)$. We can show that the orbit $GP_1$ consists of 2+5+5 points, hence $|G| = 12 \times 5$. So it is no wonder that there is an $(M) \in G$ of oder 3. By Lemma 2.17 $(A) \in \mathrm{Aut}(f)$. Clearly $(B) \in \mathrm{Aut}(f)$. We will show that $c$, $c\omega$, $c\omega^2$ are the characteristic roots of $AB^2$ for some constant $c$. Let $\sqrt{5}$ be a solution to $x^2 = 5$(we do not assume $\sqrt{5} > 0$). To get a solution to $x^4 + x^3 + x^2 + x + 1 = 0$, put $y = x + x^{-1}$. Then $y^2 + y - 1 = 0$. So $y = (-1 \pm \sqrt{5})/2$, and $x^2 - yx + 1 = 0$. Let $a = (-1 + \sqrt{5})/2$, and $b = (-1 - \sqrt{5})/2$. Let $\delta$ be a solution of $x^2 - ax + 1 = 0$. Then $\delta^2 = a\delta - 1$, $\delta^3 = -a\delta - a$, $\delta^4 = a - \delta$, and $\delta^5 = 1$. $AB^2$ now takes the form

$$AB^2 = \begin{bmatrix} a^2\delta - a & \delta + 1 & 1 \\ -\delta - b & -a^2(\delta + 1) & 1 \\ 2(a\delta - 1) & -2a(\delta + 1) & 1 \end{bmatrix}.$$

By careful computation we get $\det(AB^2 + \sqrt{5}\mu) = 5\sqrt{5}(\mu^3 - 1)$. As is well known, if $AB^2 v_j = -\sqrt{5}\omega^j v_j$ and $v_j \neq 0$, then $V = [v_0, v_1, v_2]$ diagonalizes $AB^2$; $V^{-1}AB^2V =$

$-\sqrt{5}\mathrm{diag}[1, \omega, \omega^2]$. For example we may take

$$v_0 = \begin{bmatrix} (3+\sqrt{5})\delta - 1 - \sqrt{5} \\ -(3+\sqrt{5})\delta \\ 2 \end{bmatrix}, \quad v_1 = \begin{bmatrix} (3-\sqrt{5})\omega\delta + 2\omega + \sqrt{5} - 1 \\ (-3+\sqrt{5})\omega\delta + 2(\sqrt{5}-1)\omega + \sqrt{5} - 1 \\ 4 \end{bmatrix}.$$

Substituting $\omega^2$ for $\omega$ in $v_1$, we get $v_2$. □

**Lemma 2.20.** *Let $f$ be the sextic in* Lemma 2.18, *and let $V = [v_0, v_1, v_2] \in GL(3, \mathbf{C})$ be as in the proof of* Lemma 2.19. *Set $U = 2V$. Then*

$$\begin{aligned} f_{U^{-1}} = {} & 10240[x^6(-170 - 76\sqrt{5})(-27 + D) + (y^6 + z^6)(100 - 40\sqrt{5})D \\ & + x^3(y^3 + z^3)(-200 - 100\sqrt{5})D + y^3z^3(20 - 8\sqrt{5})(864 - 17D) \\ & + x(y^4z + yz^4)(-75 + 75\sqrt{5})D + x^4yz(75 + 33\sqrt{5})(108 + D) \\ & + x^2y^2z^2(5 + \sqrt{5})(1296 - 63D)]. \end{aligned}$$

Proof. Let $\lambda = 2\delta$. Then $\lambda^2 - (-1+\sqrt{5})\lambda + 4 = 0$. So the coefficients of $f_{U^{-1}}$ are $\mathbf{Z}$-linear combinations of $\sqrt{5}^j\omega^k\lambda^\ell$. Using computer, we get the reslut. □

REMARK. Let $f' = f_{U^{-1}}$. The involution $(B^{-1}IB) \in \mathrm{Aut}(f)$ gives rise to an involution $(J) = (U^{-1}B^{-1}IBU) \in \mathrm{Aut}(f')$, where $E_3 = [e_1, e_2, e_3]$, $I = [e_2, e_1, e_3]$ and $J = [e_1, e_3, e_2]$.

The next lemma completes the proof of Theorem 2.1.

**Lemma 2.21.** *Let $f$ be the most symmetric sextic of the form in* Lemma 2.18. *Then $D^2 - 9D + 864 = 0$.*

Proof. A Sylow 3-subgroup of $\mathrm{Aut}(f)$ cannot be isomorphic to $\mathbf{Z}_9$ by Proposition 1.4. Therefore any Sylow 3-subgroup of $\mathrm{Aut}(f)$ is isomorphic to $\mathbf{Z}_3 \times \mathbf{Z}_3$ [3]. By Sylow's theorem there exists a Sylow 3-subgroup which contains $(X) = (AB^2)$ in Lemma 2.19. So there exists a $(Y) \in \mathrm{Aut}(f)\setminus\{\langle(X)\rangle\}$ of order 3 such $(X)(Y) = (Y)(X)$. Let $f_{U^{-1}} = 10240f'$, $(X') = (U^{-1}XU)$ (see Lemma 2.20 for the definition of $U$). We may assume that $X' = \mathrm{diag}[1, \omega, \omega^2]$. Then there exists a $(Y') \in \mathrm{Aut}(f')\setminus\{\langle(X')\rangle\}$ such that $X'Y' \sim Y'X'$, and $Y'^3 \sim E_3$. So without loss of generality $T = Y'$ takes the form either $\mathrm{diag}[1, 1, \omega]$ or $[e_2, e_3, e_1]\mathrm{diag}[a, b, 1]$. The former case is impossible, because $f'_{T^{-1}} \sim f'$ implies $f'_{T^{-1}} = f'$ despite the fact that $f'_{T^{-1}} \neq f'$(note that $D \neq 0$, for $f$ must be non-singular). Assume the second case for $T$. According as the monomial $x^2y^2z^2$ appears in $f'$ or not, we proceed as follows. $[x^jy^zk^\ell]$ denotes the coefficient of $x^iy^jz^\ell$ in $f'$. If $[x^2y^2z^2] = 0$, i.e. $D = 144/7$, then $f'$ does not have an automorphism of the form $(T)$. Indeed, the assumption $f'_{T^{-1}} = \mathrm{const}f'$ leads to a contradiciton as follows. Since $([x^6]x^6)_{T^{-1}} = \mathrm{const}[z^6]z^6$, $\mathrm{const} = [x^6]/[z^6] = (161 + 72\sqrt{5})/32$. By the two equalties $a^4b[xy^4z] = \mathrm{const}[x^4yz]$, and $ab[x^4yz] = \mathrm{const}[xyz^4]$, we get $a^3 = [x^4yz][x^4yz]/([xyz^4][xy^4z]) = 5(161+72\sqrt{5})/4^2$. On the other hand $a^6[y^6] = \mathrm{const}[x^6]$ gives $a^6 = \mathrm{const}[x^6]/[y^6] = (161+72\sqrt{5})^2/32^2$. Hence $a^6 \neq (a^3)^2$.

Suppose that $[x^2y^2z^2] \neq 0$. Then $f'_{T^{-1}} = a^2b^2f'$. Equivalently following nine equalities hold:

$$\begin{array}{lll} a^2b^2[x^6] = [y^6]a^6, & a^2b^2[x^3y^3] = [y^3z^3]a^3b^3, & a^2b^2[x^4yz] = [y^4zx]a^4b \\ a^2b^2[y^6] = [z^6]b^6, & a^2b^2[y^3z^3] = [z^3x^3]b^3, & a^2b^2[xy^4z] = [yz^4x]ab^4 \\ a^2b^2[z^6] = [x^6], & a^2b^2[z^3x^3] = [x^3y^3]a^3, & a^2b^2[xyz^4] = [yzx^4]ab. \end{array}$$

The second and the ninth equalities imply

$$0 = [x^3y^3][xyz^4] - [y^3z^3][x^4yz] = -6480(3+\sqrt{5})(D^2 - 9D + 864).$$

For the sake of completeness we will determine the values of $a$ and $b$ in the case $D^2 - 9D + 864 = 0$. By the second equality above we get $ab = [x^3y^3]/[y^3z^3]$. The eighth equality above yields $a = b^2$. So $b^3 = [x^3y^3]/[y^3z^3] = \{-100(2+\sqrt{5})D\}/\{(20 - 8\sqrt{5})(864 - 17D)\}$. Conversely if $a = b^2$ and $b^3 = \{-100(2+\sqrt{5})D\}/\{(20-8\sqrt{5})(864 - 17D)\}$ with $D^2-9D+864 = 0$, then above nine equalties hold. Clearly the sencond and the ninth equalities hold. Because $a^3 = b^6 = ([x^3y^3]/[y^3z^3])^2 = [x^6]/[y^6] = [x^6]/[z^6]$, the first and the seventh equalities hold. The third and the fifth ones hold too, because $ab = b^3 = [x^3y^3]/[y^3z^3] = [x^4yz]/[y^4zx] = [z^3x^3]/[y^3z^3]$. Since $[y^6] = [z^6]$, $[xy^4z] = [yz^4x]$, and $[x^3y^3] = [z^3x^3]$, the fourth, the sixth and the eighth ones hold. □

For the sake of completeness we will show the following proposition, which, together with Lemma 2.18, assures us that $|\mathrm{Aut}(f_6)| = 360$.

**Proposition 2.22.** *Let $f$ be a sextic of the form $z^6 + Bz^4xy + Cz^2x^2y^2 + Dz(x^5 + y^5) + Ex^3y^3$ with $B = 12 - D$, $C = 48 + D$, $E = 64 - 2D$, where $D^2 - 9D + 864 = 0$. Then $|\mathrm{Aut}(f)| = 360$.*

Proof. By Lemma 2.14 $|\mathrm{Aut}(f)|$ is a multiple of 5. By the proof of Lemma 2.21 $|\mathrm{Aut}(f)|$ is a multiple of 9. In view of Theorem (1) in the introduction it suffices to show that $\mathrm{Aut}(f)$ contains a subgroup isomorphic to $D_8$. Let $I = [e_2, e_1, e_3]$ and $A$ be as in Lemma 2.19. Clearly $(I) \in \mathrm{Aut}(f)$, and $(A) \in \mathrm{Aut}(f)$ by Lemma 2.17. We will show that there exists an $(M) \in \mathrm{Aut}(f)$ such that $(M)^2 = (I)$, and $(AM)^2 = (E_3)$(see Lemma 2.15 (2)). It is natural to diagonalize $A$ and $I$. Taking $a = (-1+\sqrt{5})/2$, and $b = (-1-\sqrt{5})/2$, we define

$$U = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & \sqrt{5}+1 & \sqrt{5}-1 \end{bmatrix},$$

and $W = UV$. Then $A'' = W^{-1}AW = \sqrt{5}\mathrm{diag}[1, 1, -1]$, and $I'' = W^{-1}IW = \mathrm{diag}[-1, 1, 1]$. Put $f' = f_{U^{-1}}$, and $f'' = f'_{V^{-1}}$. We look for an $M'' \in GL(3, \mathbf{C})$ such that $M''^2 \sim I''$, $A''M''^2 \sim E_3$ and $(M'') \in \mathrm{Aut}(f'')$(see Lemma 2.15(2)). Since $M''$ and $I''$ commute due to the first condition, we may assume that the first, the second and the third rows of $M''$ take the form $[\sqrt{-1}, 0, 0]$, $[0, a, b]$, and $[0, c, d]$ respectively. Either $a + d = 0$ or $a + d \neq 0$, $c = d = 0$ due to the condition $M''^2 \sim I''$. The second case is impossible, because $M''$ cannot be diagonal. Now the condition $A''M''^2 \sim E_3$ yields $a = d = 0$ and $bc = 1$. By careful computaion we get the explict form of $f''$:

$$
\begin{aligned}
f'' = {} & x^6(-E) \\
& +x^4[y^2\{3E + 10(1+\sqrt{5})D + (6+2\sqrt{5})C\} + yz\{-6E - 20D + 8D\} \\
& \quad + z^2\{3E + 10(1-\sqrt{5})D + (6-2\sqrt{5})C\}] \\
& +x^2[y^4\{-3E + 20(1+\sqrt{5})D - 2(6+2\sqrt{5})C - (56+24\sqrt{5})B\} \\
& \quad + y^3z\{12E + 20(-4-2\sqrt{5})D + 8(1+\sqrt{5})C - 16(6+2\sqrt{5})B\} \\
& \quad + y^2z^2\{-18E + 120D + 0C - 96B\} \\
& \quad + y^3z\{12E + 20(-4+2\sqrt{5})D + 8(1-\sqrt{5})C - 16(6-2\sqrt{5})B\} \\
& \quad + z^4\{-3E + 20(1-\sqrt{5})D - 2(6-2\sqrt{5})C - (56-24\sqrt{5})B\}] \\
& +x^0[y^6\{E + 2(1+\sqrt{5})D + (6+2\sqrt{5})C + (56+24\sqrt{5})B + 16(36+16\sqrt{5})\} \\
& \quad + y^5z\{-6E - 2(6+4\sqrt{5})D - 8(2+\sqrt{5})C - 16(1+\sqrt{5})B + 192(7+3\sqrt{5})\} \\
& \quad + y^4z^2\{15E + 10(3+\sqrt{5})D + 10(1+\sqrt{5})C - 40(1+\sqrt{5})B + 480(3+\sqrt{5})\} \\
& \quad + y^3z^3\{-20E - 40D + 0C + 0B + 1280\} \\
& \quad + y^2z^4\{15E + 10(3-\sqrt{5})D + 10(1-\sqrt{5})C - 40(1-\sqrt{5})B + 480(3-\sqrt{5})\} \\
& \quad + yz^5\{-6E - 2(6-4\sqrt{5})D - 8(2-\sqrt{5})C - 16(1-\sqrt{5})B + 192(7-3\sqrt{5})\} \\
& \quad + z^6\{E + 2(1-\sqrt{5})D + (6-2\sqrt{5})C + (56-24\sqrt{5})B + 16(36-16\sqrt{5})\}].
\end{aligned}
$$

We will show that $(M'') \in \mathrm{Aut}(f'')$ for some $b$ and $c$. The coeffients of $x^4yz$, $x^2y^3z$, $x^2yz^3$, $y^5z$, $yz^5$ and $y^3z^3$ in $f''$ vanish. Note that $E = 64 - D \neq 0$, for $D^2 - 9D + 864 = 0$. So such $b$ and $c$ exist if and only if $f''_{M''^{-1}} = -f''$. Let us denote by $[x^jy^kz^\ell]$ the coefficient of the monomial $x^jy^kz^\ell$ in $f''$. Then the following equalities hold:
(1) $b^2[x^4y^2] = -[x^4z^2]$ (2) $b^4[x^2y^4] = [x^2z^4]$ (3) $b^6[y^6] = -[z^6]$ (4) $b^2[y^4z^2] = -[y^2z^4]$.
We can show that the equality (1) implies (2) through (4). To be more precise, assume that $b$ is a solution to (1) for given $D$. (1) gives $b^4[x^4y^2]^2 = [x^4z^2]^2$, which implies (2), because $[x^4y^2]^2[x^2z^4] - [x^4z^2]^2[x^2y^4] = 0$. (1) and (2) give $b^6[x^4y^2][x^2y^4] = -[x^4z^2][x^2z^4]$, which implies (3). (4) is exactly the same condition as (1). This completes the proof of Proposition 2.22. □

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