ON MOTION OF AN ELASTIC WIRE
AND SINGULAR PERTURBATION
OF A 1-DIMENSIONAL PLATE EQUATION

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1. Introduction and preliminaries

Consider a springy circle wire in the euclidean space $\mathbb{R}^3$. We characterize such a
wire as a closed curve $\gamma = \gamma(x)$ with unit line element and fixed length. For such a
curve, its elastic energy is given by

$$E(\gamma) = \int_0^L |\gamma_{xx}|^2 \, dx.$$  

Solutions of the corresponding Euler-Lagrange equation are called elastic curves. Closed elastic curves in the euclidean space are classified in [7]. We discuss on motion of a circle wire governed by the elastic energy.

We will see that the equation becomes an initial value problem for $\gamma = \gamma(x, t)$:

$$\begin{cases}
\gamma_t + \partial_x^4 \gamma + \mu \gamma_t = \partial_x \{(w - 2|\gamma_{xx}|^2)\gamma_x\}, \\
-w_{xx} + |\gamma_{xx}|^2 w = 2|\gamma_{xx}|^4 - |\partial_x^3 \gamma|^2 + |\gamma_{tx}|^2, \\
\gamma(x, 0) = \gamma_0(x), \quad \gamma_t(x, 0) = \gamma_1(x), \quad (\gamma_{0x}, \gamma_{1x}) = 0.
\end{cases}$$

( EW )

Here, $\mu$ is a constant which represents the resistance, and the ODE for $w$ corresponds
to the constrained condition $(\gamma_x, \gamma_{tx}) \equiv 0$ (i.e., $|\gamma_x| \equiv 1$.) When the resistance $\mu$ is
very large, we can analyze the behavior of the solution replacing the time parameter $t$
to $\tau = \mu^{-1} t$. Then, (EW) becomes

$$\begin{cases}
\mu^{-2} \gamma_{\tau \tau} + \partial_x^4 \gamma + \gamma_{\tau} = \partial_x \{(w - 2|\gamma_{xx}|^2)\gamma_x\}, \\
-w_{xx} + |\gamma_{xx}|^2 w = 2|\gamma_{xx}|^4 - |\partial_x^3 \gamma|^2 + \mu^{-2}|\gamma_{tx}|^2, \\
\gamma(x, 0) = \gamma_0(x), \quad \gamma_{\tau}(x, 0) = \mu \gamma_1(x), \quad (\gamma_{0x}, \gamma_{1x}) = 0.
\end{cases}$$

( EW' )

And, when $\mu \to \infty$, we get, omitting initial data $\gamma_{\tau}(x, 0)$,
The equation (EP), treated in [4] and [5], has a unique all time solution for any initial data, and the solution converges to an elastic curve. In this paper, we will prove:

1) The equation (EW) has a unique short time solution for any initial data. (Corollary 3.13.)

2) If \( \mu \) is large, then the solution of (EW') exists for long time, and converges to a solution of (EP) when \( \mu \to \infty \). (Corollary 4.10.)

Note that in 2), the derivative \( \gamma_\tau(x, 0) = \mu \gamma_1(x) \) diverges when \( \mu \to \infty \).

If (EW) contained no 3rd derivatives \( \partial^3 \gamma \) and was not coupled with ODEs, i.e., if our equation was \( \gamma_{tt} + \partial^3 \gamma + \mu \gamma_t = F(\gamma, \gamma_x, \gamma_{xx}, \gamma_t) \), it is standard to show the short time existence of solutions. (See [9] Section 11.7.) Being coupled is not main difficulty to solve the equation. We can overcome it by careful estimation similar to [4]. However, the difficulty due to the presence of 3rd derivatives is essential. We will overcome the difficulty using the new unknown variable \( \xi := \gamma_x \in S^2 \). As we will see in Lemma 2.2, the equation for \( \xi \) does not contain 3rd derivatives \( \nabla_x^2 \xi \). Owing to the lack of the term, we will be able to solve (EW') by a usual method: perturb to a parabolic equation and show the solution of the parabolic equation converges to a solution of the original equation. This will be done in Section 3.

REMARK 1.1. In this paper, we only treat curves in the 3-dimensional euclidean space \( \mathbb{R}^3 \). But, the result holds also on the case of any dimensional euclidean space, with no modification of proofs.

By similarity, we may assume that the length of the initial curve \( \gamma_0 \) is 1. From now on, a closed curve means a map from \( S^1 \equiv \mathbb{R}/\mathbb{Z} \) into the euclidean space \( \mathbb{R}^3 \) or the unit sphere \( S^2 \). The inner product of vectors is denoted by \( (\ast, \ast) \), and the norm is denoted by \( |\ast| \). We also use the covariant derivation \( \nabla \) on \( S^2 \). For a tangential vector field \( X(x) \) along a curve \( \gamma(x) \) on \( S^2 \), the covariant derivative is defined by \( \nabla_x X := (X'(x))^T \). The covariant differentiation is non-commutative, because the curvature tensor \( R \) of \( S^2 \) is non-zero. For example, if \( X(x, t) \) is a tangential vector field along a family \( \gamma(x, t) \) of curves on \( S^2 \), we have

\[
\nabla_x \nabla_x X - \nabla_x \nabla_x X = R(\gamma_x, \gamma_t)X = (\gamma_t, X)\gamma_x - (\gamma_t, X)\gamma_x.
\]

For functions on \( S^1 \) and vector fields along a closed curve, we use \( L_2 \)-inner product \( (\ast, \ast) \) and \( L_2 \)-norm \( |\ast| \). Sobolev \( H^n \)-norm is denoted by \( |\ast|_n \). For a tensor field along the closed curve on \( S^2 \), \( |\ast|_n \) is defined using covariant derivation. That is, \( |\ast|_2^2 := \sum_{l=0}^n |\nabla_x^l \ast|^2 \). We also use \( C^n \) norm \( |\ast|_{(n)} \). In particular, \( |\ast|_{(0)} = \max|\ast| \).
2. The equations

To derive the equation of motion, we use Hamilton’s principle. For a moving curve \( \gamma = \gamma(t, x) \), the velocity energy is given by \( \| \gamma_t \|^2 \) and the elastic energy is given by \( \| \gamma_{xx} \|^2 \). (By rescaling, we omit coefficients.) Therefore, the real motion is a stationary point of the integral

\[
L(\gamma) := \int_{t_1}^{t_2} \| \gamma_t \|^2 - \| \gamma_{xx} \|^2 \, dt.
\]

That is, the integral

\[
L' := \int_{t_1}^{t_2} (\gamma_t, \delta_t) - (\gamma_{xx}, \delta_{xx}) \, dt
\]

should vanish for all \( \delta = \delta(t, x) \) satisfying \( \delta(t_1, x) = \delta(t_2, x) = 0 \) and the constrained condition \( (\gamma_x, \delta_x) = 0 \).

From integration by parts, we see

\[
L' = \int_{t_1}^{t_2} -(\gamma_{tt} + \partial_x^4 \gamma, \delta) \, dt.
\]

On the other hand, the orthogonal complement of the space \( V := \{ \delta \mid (\gamma_x, \delta_x) \equiv 0 \} \) at each time \( t \) is \( V^\perp = \{(u\gamma_x)_x \mid u = u(x)\} \). Therefore, \( \gamma \) is stationary if and only if \( \gamma_t \in V \) and \( \gamma_{tt} + \partial_x^4 \gamma = (u\gamma_x)_x \) for some function \( u = u(t, x) \).

**Remark 2.1.** Many papers (e.g., [2], [3]) apply Hamilton’s principle using \( |\gamma_{xt}|^2 + |\gamma_t|^2 \) as the kinetic energy, and gets a wave equation. The wave equation is completely different from (EW). A linear version of our equation can be found, for example, in [1] p. 246.

This difference can be explained as follows. We characterize a planer thick wire of length \( L \), of radius \( R \) and of unit weight per length as a map \( u = u(x, y) : [0, L] \times [-R, R] \to \mathbb{R}^2 \) such that \( u(x, y) = \gamma(x) + y J \gamma_x(x) \), where \( \gamma \) is a curve of unit line element and \( J \) is the \( \pi/2 \) rotation. When \( u \) moves, i.e. when we consider a family \( u = u(x, y, t) \) of such curves, the velocity energy becomes

\[
\frac{1}{2R} \int_0^L dx \int_{-R}^R |u_t(x, y)|^2 dy = \| \gamma_t \|^2 + \frac{1}{3} R^2 \| \gamma_{xt} \|^2.
\]

Hence, our wire is infinitely thin, while previous papers treat thick wires.

In this paper, we treat slightly more general equation, equation with resistance \( \mu \). That is,

\[
\gamma_{tt} + \mu \gamma_t + \partial_x^4 \gamma = (u \gamma_x)_x,
\]
coupled with an ODE for $u$, which is derived from the constrained condition: $|\gamma_x| \equiv 1$. From

$$0 = \partial_t^2 |\gamma_x|^2 = 2(\gamma_{ttx}, \gamma_x) + 2|\gamma_{tx}|^2,$$

the unknown $u$ satisfies

$$(-\partial^5_x \gamma + \partial^2_x (u \gamma_x) - \nu \gamma_{tx}, \gamma_x) = -|\gamma_{tx}|^2.$$

Using $|\gamma_x|^2 \equiv 1$, we can rewrite this to

$$-u_{xx} + |\gamma_{xx}|^2 u = 2\partial_x^2 |\gamma_{xx}|^2 - |\partial^3_x \gamma|^2 + |\gamma_{tx}|^2,$$

and, putting $w := u + 2|\gamma_{xx}|^2$, we get (EW).

Since the principal part of (EW) is the operator of the plate equation:

$$u_{tt} + \partial_x^5 u,$$

we perturb it to a parabolic operator:

$$u_{tt} - 2\varepsilon u_{txx} + (1 + \varepsilon^2)\partial_x^4 u$$

$$= (\partial_t - (\varepsilon + \sqrt{-1})\partial_x^2)(\partial_t - (\varepsilon - \sqrt{-1})\partial_x^2)u$$

with $\varepsilon > 0$. It is possible to show that a perturbed equation of (EW)

$$\begin{cases}
\gamma_{tt} - 2\varepsilon \gamma_{txx} + (1 + \varepsilon^2)\partial_x^4 \gamma + \mu \gamma_t = \partial_x \{(w - 2|\gamma_{xx}|^2)\gamma_x\}, \\
-w_{xx} + |\gamma_{xx}|^2 w = 2|\gamma_{xx}|^4 - |\partial^3_x \gamma|^2 + |\gamma_{tx}|^2, \\
\gamma(x, 0) = \gamma_0(x), \quad \gamma_t(x, 0) = \gamma_1(x), \quad (\gamma_{0x}, \gamma_{1x}) = 0
\end{cases}$$

has a short-time solution. However, we cannot get uniform estimate when $\varepsilon \to 0$, because $\partial_x \{(w - 2|\gamma_{xx}|^2)\gamma_x\}$ contains the third derivative of $\gamma$. To overcome this difficulty, we convert (EW) to an equation on $S^2$, and “remove” the third derivative.

We introduce a new unknown function $\xi$ by $\xi = \gamma_x$. The function $\xi$ is a family of closed curves on $S^2$.

**Lemma 2.2.** The equation (EW) is equivalent to equation

$$\begin{cases}
\nabla_x \xi_t + \nabla_x^2 \xi + \mu \xi_t = (w - |\xi_x|^2)\nabla_x \xi_x + 2w \xi_x - \frac{3}{2} \partial_x |\xi_x|^2 \xi_x, \\
-w_{xx} + |\xi_x|^2 w = |\xi_x|^2 - |\nabla_x \xi|^2 + |\xi_x|^4, \\
\xi(x, 0) = \xi_0(x), \quad \xi_t(x, 0) = \xi_1(x), \quad \int_0^1 \xi_0 \, dx = \int_0^1 \xi_1 \, dx = 0,
\end{cases}$$

(EW$\xi$)
and (EP) is equivalent to equation

\[
\begin{aligned}
\xi_x + \nabla_x^2 \xi_x &= (w - |\xi_x|^2) \nabla_x \xi_x + 2w_t \xi_x - \frac{3}{2} \partial_3 |\xi_x|^2 \xi_x, \\
-w_{xx} + |\xi_x|^2 w &= -|\nabla_x \xi_x|^2 + |\xi_x|^4, \\
\xi(x, 0) &= \xi_0(0), \quad \int_0^1 \xi_0 \, dx = 0.
\end{aligned}
\]

(EP\^x)

Proof. It is straightforward to check the following decomposition:

\[
\begin{aligned}
\xi_{xx} &= \nabla_x \xi_x - |\xi_x|^2 \xi, \\
\xi_{tt} &= \nabla_t \xi_t - |\xi_t|^2 \xi, \\
\partial_x^3 \xi &= \nabla_x^2 \xi_x - |\xi_x|^2 \xi_x - \frac{3}{2} \partial_x |\xi_x|^2 \xi, \\
\partial_x^4 \xi &= \nabla_x^3 \xi_x - |\xi_x|^2 \nabla_x \xi_x - \frac{5}{2} \partial_x |\xi_x|^2 \xi_x + (|\nabla_x \xi_x|^2 + |\xi_x|^4 - 2 \partial_x^2 |\xi_x|^2) \xi.
\end{aligned}
\]

Using these formulas, we see that the \(x\)-derivatives of (EW) imply (EW\^x). Conversely, (EW\^x) implies the equation

\[
\xi_{tt} + \partial_t^4 \xi + \mu \xi_t = \partial_x^2 \{(w - 2|\xi_x|^2) \xi\}.
\]

Under the assumption: \(\int_0^1 \xi_0 \, dx = \int_0^1 \xi_1 \, dx = 0\), we see that the closedness condition: \(\int_0^1 \xi \, dx \equiv 0\) is satisfied. Let \(\gamma\) be the solution of an ODE:

\[
\begin{aligned}
\gamma_{tt} + \mu \gamma_t &= -\partial_x^3 \xi + \partial_x \{(w - 2|\xi_x|^2) \xi\}, \\
\gamma(x, 0) &= \gamma_0(x), \quad \gamma_t(x, 0) = \gamma_1(x).
\end{aligned}
\]

Then

\[
\gamma_{xtt} + \mu \gamma_{xt} = -\partial_x^4 \xi + \partial_x^2 \{(w - 2|\xi_x|^2) \xi\} = \xi_{tt} + \mu \xi_t
\]

and \((\gamma_x - \xi)_t + \mu (\gamma_x - \xi) = 0\). Hence \(\gamma_x \equiv \xi\) and \(\gamma\) is a solution of (EW).

A similar calculation gives the equivalence of (EP) and (EP\^x). \(\square\)

3. Short time existence

In this section, we fix \(\mu \in \mathbb{R}\).

To perturb (EW\^x), we introduce a function \(\rho(x, y)\). Since \(\xi_0\) is the derivative of a closed curve \(\gamma_0\) in the euclidean space, each component of \(\xi_0\) takes 0 at some \(x\).

Therefore, by Wirtinger’s inequality, we have \(\|\xi_0x\| \geq \pi^2 \|\xi_0\|^2 \geq \pi^2\). (It is known in fact that \(\|\xi_0x\|^2 \geq 4\pi^2\).) Let \(\delta(r)\) be a \(C^\infty\) function on \(\mathbb{R}\) such that \(\delta(r) = 1\) on \(|r| \leq \pi^2/8\), \(\delta(r) = 0\) on \(\pi^2/4 \leq |r|\) and \(0 \leq \delta(r) \leq 1\) on \(\pi^2/8 \leq |r| \leq \pi^2/4\). We put

\[
\rho(x, y) = \pi^2 + \delta(y^2 - |\xi_0x(x)|^2)(y^2 - \pi^2).
\]
Fix an interval $I$ such that $|\xi_0(x)|^2 \geq \pi^2/2$ for any $x \in I$. If $x \in I$ and $|y^2 - |\xi_0(x)|^2| \leq \pi^2/4$, then $\rho(x, y) \geq \min\{\pi^2, y^2\} \geq \pi^2/4$. And if $|y^2 - |\xi_0(x)|^2| \geq \pi^2/4$, then $\rho(x, y) = \pi^2$. Therefore, for any function $u(x)$,

$$\int_0^1 \rho(x, u(x)) \, dx \geq \frac{\pi^2}{4} \int_I \, dx.$$ 

**Remark 3.1.** Below, we use the function $\rho$ only to ensure $\rho \geq 0$ everywhere and $\int_0^1 \rho(x, u(x)) \, dx$ is bounded from below by a positive constant. Note that $\rho(x, y) := y$ satisfies this requirement if $\xi = \gamma_x$ for some closed curve $\gamma$ in the euclidean space.

**Proposition 3.2.** Let $\xi_0(x)$ be a $C^\infty$ closed curve on $S^2$ with $\|\xi_0\| \geq \pi$ and $\xi_1(x)$ a $C^\infty$ tangent vector field along $\xi_0$. Let $\rho$ be the function defined as above. Then, equation

$$(EW^{\xi}) \begin{cases} \nabla_x \xi_t - 2\varepsilon \nabla^2_x \xi_t + (1 + \varepsilon^2) \nabla^3_x \xi_t + \mu \xi_t \\ = (w - |\xi_1|^2) \nabla_x \xi_t + 2w_x \xi_t - \frac{3}{2} \partial_x |\xi_1|^2 \xi_t, \\ -w_{xx} + \rho(x, |\xi_1|^2)w = |\xi_t|^2 - |\nabla_x \xi_1|^2 + |\xi_1|^4, \\ \xi(x, 0) = \xi_0(0), \quad \xi_t(x, 0) = \xi_1(x) \end{cases}$$

has a $C^\infty$ solution on some interval $0 \leq t < T$.

**Proof.** We can prove unique short-time existence of $(EW^{\xi})$ by a similar method with that used in [4]. Here, we mention only two steps. One is an estimation of the ODE for $w$. Lemma 3.3 with the function $\rho$ ensures estimation of $w$ by $\xi$. Another, Lemma 3.4, is a crucial point to use the contraction principle. □

**Lemma 3.3 ([4] Lemma 4.1, Lemma 4.2).** Let $a$ and $b$ be $L_1$-functions on $S^1$ such that $a \geq 0$ and $\|a\|_{L_1} > 0$. Then, the ODE for a function $w$ on $S^1$

$$-w'' + aw = b$$

has a unique solution $w$, and the solution $w$ is estimated as

$$\max|w| \leq 2\{1 + \|a\|_{L_1}^{-1}\} \cdot \|b\|_{L_1},$$

$$\max|w'| \leq 2\{1 + \|a\|_{L_1}^{-1}\} \cdot \|b\|_{L_1}.$$ 

Moreover, there exists universal constants $C > 0$ and $N > 0$ depending on $n$ such that

$$\|w\|_{n+2} \leq C(1 + \|a\|_N^n) \|b\|_n,$$

$$\|w\|_{(n+2)} \leq C(1 + \|a\|_{(n)}^N) \|b\|_{(n)}.$$
Lemma 3.4. We consider a linear PDE for \( u \)

\[
\begin{align*}
  u_{tt} - 2\varepsilon u_{xx} + (1 + \varepsilon^2)\partial_x^4 u &= f, \\
  u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x).
\end{align*}
\]

If \( f \in C^{2\alpha} \), \( u_0 \in C^{4+2\alpha}_x \) and \( u_1 \in C^{2+2\alpha}_x \), then there is a unique solution \( u \in C^{4+2\alpha} \).

Moreover, we have an estimation:

\[
\|u\|_{C^{4+2\alpha}} \leq C\left(\|f\|_{C^{2\alpha}} + \|u_0\|_{C^{4+2\alpha}} + \|u_1\|_{C^{2+2\alpha}}\right),
\]

where \( \| \cdot \|_{C^{2\alpha}} \) means the Hölder norm for x-direction, and \( \| \cdot \|_{C^{4+2\alpha}} \) means the weighted Hölder norm (t-derivatives are counted twice of x-derivatives.)

Proof. We decompose the equation to a parabolic equation as

\[
\begin{align*}
  u_t - (\varepsilon + \sqrt{-1})u_{xx} &= v, \\
  v_t - (\varepsilon - \sqrt{-1})v_{xx} &= f.
\end{align*}
\]

Using the fundamental solution

\[
\Gamma(x, t) = \frac{1}{2\sqrt{\pi\varepsilon}} \exp\left(-\frac{x^2}{4(\varepsilon \pm \sqrt{-1})t}\right)
\]

of the parabolic operator \( \partial_t - (\varepsilon \pm \sqrt{-1})\partial_x^2 \), we can estimate as

\[
\begin{align*}
  \|u\|_{C^{4+2\alpha}} &\leq C\left(\|v\|_{C^{2\alpha}} + \|u_0\|_{C^{4+2\alpha}}\right) \\
  &\leq C\left(\|f\|_{C^{2\alpha}} + \|v_0\|_{C^{4+2\alpha}} + \|u_0\|_{C^{4+2\alpha}}\right) \\
  &\leq C\left(\|f\|_{C^{2\alpha}} + \|u_1\|_{C^{2+2\alpha}} + \|u_0\|_{C^{4+2\alpha}}\right).
\end{align*}
\]

When we take the limit \( \varepsilon \to 0 \) in (EW^{\varepsilon}), we should note that the term \( \nabla_x^3 \xi_x \) is quasi-linear, and contains the third derivative of \( \xi \). In fact, in local coordinate system,

\[
\nabla_x^3 \xi_x = \left\{ \partial_x^4 \xi_x^p + 4\Gamma_q^p \cdot (\xi_q \partial_x^2 \xi_x^q) \right\} \frac{\partial}{\partial x^p} + \text{[lower order terms]}.
\]

However, when we integrate it by parts, we can treat it as though it contained no third derivatives.

Lemma 3.5. For any \( K > 0 \), there are \( T > 0 \) and \( M > 0 \) with the following property:

Let \( \xi \) be a solution of (EW^{\varepsilon}) with \( \varepsilon \in [0, 1] \) on an interval \( [0, t_1) \subset [0, T) \). If its initial value satisfies \( \|\xi_1\|_{L^2}^2 + \|\xi_0\|_{L^2}^2 \leq K \), then \( \|\xi_t\|_{L^2}^2 + \|\xi_x\|_{L^2}^2 \leq M \) holds on \( 0 \leq t < t_1 \).
Proof. Put

\[ f = (w - \rho(x, |\xi|^2)) \nabla_x \xi_x + 2w_x \xi_x - \frac{3}{2} \partial_x |\xi|^2 \xi_x. \]

We can estimate

\[
\frac{1}{2} \frac{d}{dt} \left\{ \|\xi_t\|^2 + (1 + \varepsilon^2) \|\nabla_x \xi_t\|^2 \right\}
\]

\[ = \langle \xi_t, \nabla \xi_t \rangle + (1 + \varepsilon^2) \langle \nabla_x \xi_t, \nabla_x \nabla_x \xi_t \rangle \]

\[ = \langle \xi_t, \nabla \xi_t \rangle + (1 + \varepsilon^2) \|\nabla_x \xi_t\|^2 + (1 + \varepsilon^2) \langle R(\xi_t, \xi_t) \xi_t, \nabla_x \xi_t \rangle \]

\[ \leq \langle \xi_t, 2\varepsilon \nabla_x^2 \xi_t + f \rangle - \mu \|\xi_t\|^2 + C \max|\xi_t|^2 \|\xi_t\| \|\nabla_x \xi_t\| \]

\[ \leq -2\varepsilon \|\nabla_x \xi_t\|^2 + \langle \xi_t, f \rangle - \mu \|\xi_t\|^2 + C \|\xi_t\|^2 \|\xi_t\| \|\nabla_x \xi_t\| \]

\[ \leq (1 - \mu) \|\xi_t\|^2 + \|f\|^2 + C \|\xi_t\|^2 (\|\xi_t\|^2 + \|\nabla_x \xi_t\|^2), \]

and,

\[
\frac{1}{2} \frac{d}{dt} \|\xi_t\|^2 = \langle \xi_t, \nabla \xi_t \rangle = -\langle \nabla_x \xi_t, \xi_t \rangle \leq \|\nabla_x \xi_t\|^2 + \|\xi_t\|^2. \]

Here, by Lemma 3.3, \( \|f\| \leq C(1 + \|\xi_t\|^2 + \|\xi_t\|^2)^{\lambda_1} \). Therefore, putting \( X(t) := 1 + \|\xi_t\|^2 + (1 + \varepsilon^2) \|\xi_t\|^2 \), we get

\[ X'(t) \leq C_1 X(t)^{\lambda_2}, \]

and, \( X(t) \) is bounded from above by a solution of the ODE: \( y'(t) = C_1 y(t)^{\lambda_2} \).

Remark 3.6. If we use original equation of \( \gamma \), which contains \( \partial_x^3 \gamma \) in the right hand side, the term \( \langle \gamma_t, \partial^2_x \gamma \rangle \) appears in the estimation. Since we need the term \(-2\varepsilon \|\gamma_x\|^2 \) to cancel \( \langle \gamma_t, \partial^3_x \gamma \rangle \), we cannot get uniform estimate with respect to \( \varepsilon \), and the following proof will fail.

Lemma 3.7. For any \( K > 0 \) and \( n \geq 0 \), there is \( M > 0 \) with the following property:

Let \( \xi \) be a solution of (\( EW^\varepsilon \)) with \( \varepsilon \in [0, 1] \) on \([0, T)\). If its initial value satisfies \( \|\xi_0\|, \|\xi_0\|_{n+1} \leq K \), and if it satisfies \( \|\xi_t\|, \|\xi_t\|_1^2 \leq K \) on \( 0 \leq t < T \), then \( \|\xi_t\|_n, \|\xi_t\|_{n+1} \leq M \) holds on \( 0 \leq t < T \).

Proof. The claim holds for \( n = 0 \) by taking \( M = K \). We prove the claim by induction. Suppose that the claim holds for \( n \). In particular, we know bounds of \( \|\xi_t\|_n \),
Therefore, we have

\[
\| \nabla \nabla_{x}^{n+1} \xi_{i} - \nabla_{x}^{n+1} \nabla_{x} \xi_{i} \| = \left\| \sum_{i=0}^{n} \nabla_{x}^{i} \left( R(\xi_{i}, \xi_{i}) \nabla_{x}^{n-i} \xi_{i} \right) \right\|
\]

\[
\leq C \sum_{i+j \leq n} \left\| \nabla_{x}^{i} \xi_{i} \nabla_{x}^{j} \xi_{j} \right\| \leq C \sum_{i+j \leq n} \| \xi_{i} \| \| \xi_{j} \| j+1 \leq C \| \xi_{i} \|_{n+1},
\]

\[
\| \nabla \nabla_{x}^{n+2} \xi_{i} - \nabla_{x}^{n+3} \xi_{i} \| = \left\| \sum_{i=0}^{n+1} \nabla_{x}^{i} \left( R(\xi_{i}, \xi_{i}) \nabla_{x}^{n+1-i} \xi_{i} \right) \right\|
\]

\[
\leq C \left( \| \xi_{i} \| \| \nabla_{x}^{n+1} \xi_{i} \| + \sum_{i=0}^{n+1} \| \nabla_{x} \xi_{i} \| \right) \leq C \left( \| \xi_{i} \|_{1} \| \xi_{x} \|_{n+1} + \| \xi_{i} \|_{n+1} \right)
\]

\[
\leq C \| \xi_{i} \|_{n+1},
\]

\[
\| w \|_{n+2} \leq C(1 + \| \rho(x, |x|^2) |_{n} \| \xi_{i} \|^2 - |\nabla_{x} \xi_{i} |^2 + |\xi_{i} |^4 \|_{n}
\]

\[
\leq C \left( \sum_{i+j \leq n} \| \xi_{i} \| \| \xi_{j} \| j+1 \right) + \sum_{i+j \leq n, i \leq j} \| \nabla_{x} \xi_{i} \| \| \nabla_{x} \xi_{j} \| j+1 + 1 \right)
\]

\[
\leq C(\| \xi_{i} \|_{n+1} + \| \xi_{x} \|_{1} \| \xi_{x} \|_{n+2} + 1) \leq C(\| \xi_{i} \|_{n+1} + \| \xi_{x} \|_{n+2} + 1).
\]

Put

\[
f := (w - |x|^2) \nabla_{x} \xi_{x} + 2w_{x} \xi_{x} - \frac{3}{2} \partial_{x} |x|^2 \xi_{x}.
\]

Then,

\[
\| f \|_{n+1} \leq C(1 + \| \xi_{x} \|_{n+2} + \| \xi_{x} \|_{n+2} \| \xi_{x} \|_{1} + \| w \|_{n+2} \| \xi_{x} \|_{1} + \| w \|_{2} \| \xi_{x} \|_{n+1})
\]

\[
\leq C(1 + \| \xi_{x} \|_{n+2} + \| w \|_{n+2}) \leq C(1 + \| \xi_{x} \|_{n+2} + \| \xi_{x} \|_{n+1}).
\]

Using these, we have

\[
\frac{1}{2} \frac{d}{dt} \left\{ \| \nabla_{x}^{n+1} \xi_{i} \|^2 + (1 + \varepsilon^2) \| \nabla_{x}^{n+2} \xi_{x} \|^2 \right\}
\]

\[
= (\nabla_{x}^{n+1} \xi_{i}, \nabla_{x}^{n+1} \nabla_{x} \xi_{i}) + (1 + \varepsilon^2)(\nabla_{x}^{n+2} \xi_{x}, \nabla_{x}^{n+2} \xi_{x})
\]

\[
\leq (\nabla_{x}^{n+1} \xi_{i}, \nabla_{x}^{n+1} \nabla_{x} \xi_{i}) + (1 + \varepsilon^2)(\nabla_{x}^{n+2} \xi_{x}, \nabla_{x}^{n+3} \xi_{x})
\]

\[
+ C(\| \xi_{i} \|_{n+1} + \| \xi_{x} \|_{n+2})(1 + \| \xi_{i} \|_{n+1})
\]

\[
\leq (\nabla_{x}^{n+1} \xi_{i}, \nabla_{x}^{n+1} (f + 2\varepsilon \nabla_{x}^{2} \xi_{i} - \mu \xi_{i})) + C(1 + \| \xi_{i} \|_{n+1} + \| \xi_{x} \|_{n+2})
\]

\[
\leq (\nabla_{x}^{n+1} \xi_{i}, 2\varepsilon \nabla_{x}^{n+3} \xi_{i}) + C(1 + \| \xi_{i} \|_{n+1} + \| \xi_{x} \|_{n+2})
\]

\[
\leq C(1 + \| \nabla_{x}^{n+1} \xi_{i} \|^2 + (1 + \varepsilon^2) \| \nabla_{x}^{n+2} \xi_{x} \|^2).
\]
**Lemma 3.8.** For any smooth initial data \( \{\xi_0, \xi_1\} \), \( K > 0 \), \( T > 0 \) and \( m, n \geq 0 \), there is \( M > 0 \) with the following property:

Let \( \xi \) be a solution of \((\text{EW}^\varepsilon)\) with \( \varepsilon \in [0, 1] \) on \([0, T)\). If \( \|\xi_t\|, \|\xi_t\|_1 \leq K \) on \( 0 \leq t < T \), then \( \xi \) is smooth on \( S^1 \times [0, T) \), and the derivatives are bounded as \( \|\nabla_t^m \xi\|_{(t)} \leq M \).

Proof. By Lemma 3.7, the claim holds for \( m \leq 1 \). Suppose that the claim holds up to \( m \). In particular, we have \( C_\infty^\infty \) bounds of \( \xi \) and \( \nabla_t^{m-1} \xi_t \). Therefore, using

\[-(\partial_t^j w)_{xx} + \partial_t^j w = \partial_t^j f - \sum_{0 < i < j} \binom{i}{j} \partial_t^i \rho \partial_t^{j-i} w\]

for \( 0 \leq j \leq m - 1 \), we have \( C_\infty^\infty \) bounds of \( \partial_t^{m-1} w \). Since \( \nabla_t^{m+1} \xi_t \) is expressed as a polynomial of these lower derivatives, we get the result.

**Proposition 3.9.** The equation \((\text{EW}^\varepsilon)\) has a short time solution for any smooth initial data.

Proof. We put \( K := \|\xi_1\|^2 + \|\xi_0x\|^2 \) and take \( T > 0 \) in Lemma 3.5. Then, by Lemma 3.8, any solution has a priori estimate on \( 0 \leq t < T \).

Let \([0, T_\varepsilon)\) be the maximal interval such that a solution exists for \( \varepsilon \). If \( T_\varepsilon < T \), then \( \xi \) is smoothly and uniformly bounded on \([0, T_\varepsilon)\), hence can be continued beyond \( T_\varepsilon \). This contradicts to the definition of \( T_\varepsilon \), therefore we see that \( T_\varepsilon \geq T \). We conclude that a solution \( \xi \) exists on the interval \([0, T)\) for each \( \varepsilon > 0 \), and these \( \xi \)'s have smooth uniform bounds on \( S^1 \times [0, T) \).

Therefore, taking a sequence \( \varepsilon_i \to 0 \), we get a solution of

\[
\begin{align*}
\nabla_t \xi_t + \nabla_x^2 \xi_t + \mu \xi_t &= (w - |\xi_t|^2)\nabla_x \xi_t + 2w_x \xi_t - \frac{3}{2} \partial_x |\xi_t|^2 \xi_t, \\
-w_{xx} + \rho(x, |\xi_t|^2)w &= |\xi_t|^2 - |\nabla_x \xi_t|^2 + |\xi_t|^4, \\
\xi(x, 0) &= \xi_0(0), \quad \xi_t(x, 0) = \xi_1(x).
\end{align*}
\]

Since \( \rho(x, |\xi_t|^2) = |\xi_t|^2 \) when \( \xi_t \) is sufficiently close to \( \xi_{0x} \), we have a solution \( \xi \) of \((\text{EW}^\varepsilon)\) on some time interval. Once we have a short time solution \( \xi \) of \((\text{EW}^\varepsilon)\), we can estimate the solution as Lemma 3.8, and the solution \( \xi \) can be continued to the interval \([0, T)\).

**Proposition 3.10.** Let \( \xi \) and \( \tilde{\xi} \) be solutions of \((\text{EW}^\varepsilon)\) on \([0, T)\). If \( \xi \) and \( \tilde{\xi} \) have same smooth initial data, then they identically coincide.

Proof. To express the difference of two solutions, we use local coordinates. We fix the initial value \( \{\xi_0, \xi_1\} \), and take a local coordinate \( U \) which contains the initial
value $\xi_0$. In $U$, $(\text{EW}^\xi)$ is expressed as:

$$
\begin{cases}
\xi_t^p + \theta_\xi^p + 4f_q^p(\xi)\xi_q^p \partial_\xi^p = F_p[\xi_{xx}, w_x, \xi_t], \\
-w_{xx} + g_q(\xi)\xi_q^q \partial_\xi^q w = G[\xi_{xx}, \xi_t],
\end{cases}
$$

where $F_p[\xi_{xx}, w_x, \xi_t]$ is a polynomial of $\xi_t^q$, $\xi_{xx}^q$, $w_x$, $w_x$, $\xi_t^q$, functions of $\xi_q^q$, and $G[\xi_{xx}, \xi_t]$ is a polynomial of $\xi_t^q$, $\xi_{xx}^q$, $\xi_t^q$, functions of $\xi_q^q$. (We only note highest derivatives.)

Let $\{\xi, \tilde{w}\}$ be another solution of $(\text{EW}^\xi)$ on $[0, t_1)$ $(t_1 \leq T)$. Applying Lemma 3.5 and Lemma 3.8 with $\varepsilon = 0$, we have smooth bounds of $\xi$ and $\xi_t$. We put $\zeta := \tilde{\xi} - \xi$, $u := \tilde{w} - w$. Then, we see that

$$
\xi_t^p + \theta_\xi^p + 4f_q^p(\xi)\xi_q^p \partial_\xi^p
$$
equals to a sum of terms containing at least one of $\zeta_x^q$, $\zeta_{xx}^q$, $u$, $u_x$, $\zeta_t$ or the difference of the values of a function at $\tilde{\xi}$ and $\xi$. Similarly,

$$
-w_{xx} + g_q(\xi)\xi_q^q \partial_\xi^q u
$$
equals to a sum of terms containing at least one of $\zeta_x^q$, $\zeta_{xx}^q$, $\zeta$, or the difference of the values of a function at $\tilde{\xi}$ and $\xi$.

Therefore, we can estimate $\zeta$ and $u$ linearly:

$$
\begin{align*}
|\xi_t^p + \theta_\xi^p + 4f_q^p(\xi)\xi_q^p \partial_\xi^p| &\leq C(|\zeta| + |\zeta_x^q| + |\zeta_{xx}^q| + |u| + |u_x| + |\zeta_t|), \\
|-w_{xx} + g_q(\xi)\xi_q^q \partial_\xi^q u| &\leq C(|\zeta| + |\zeta_x^q| + |\zeta_{xx}^q| + |\zeta_t|).
\end{align*}
$$

Regarding $\zeta$ as a vector field along $\xi$, these inequalities can be written using covariant derivation along $\xi$:

$$
\begin{align*}
\|\nabla_x^2 \zeta + \nabla_x^2 \zeta\| &\leq C(\|\zeta\|_2 + \|u\|_1 + \|\nabla_x \zeta\|), \\
\|u\|_1 + \|\xi_x^q\|_1^2 &\leq C(\|\zeta\|_2 + \|\nabla_x \zeta\|).
\end{align*}
$$

Thus we have $\|u\|_1 \leq C(\|\zeta\|_2 + \|\nabla_x \zeta\|)$, and

$$
\frac{d}{dt} [\|\nabla_x \zeta\|_2^2 + \|\xi_t^q\|_2^2]
$$

$$
= 2(\nabla_x \zeta, \nabla_x^2 \zeta) + 2(\zeta, \nabla_x^2 \zeta) + 2(\nabla_x \zeta, \nabla_x \nabla_x \zeta) + 2(\nabla_x^2 \zeta, \nabla_x \nabla_x \zeta) + 2(\nabla_x^2 \zeta, \nabla_x^2 \zeta) + 2(\nabla_x \zeta, \nabla_x \nabla_x \zeta) + C(\|\zeta\|_2^2 + \|\nabla_x \zeta\|_2^2)
$$

$$
\leq C(\|\nabla_x \zeta\|_2^2 + \|\xi_t^q\|_2^2),
$$

from which we see that $(\|\nabla_x \zeta\|_2^2 + \|\xi_t^q\|_2^2)e^{-C_1t}$ is non-increasing, hence identically vanishes.
This proof applies at any time $t_0$ such that $\tilde{\xi}(t_0) = \xi(t_0)$. Therefore, the set $\{t \mid \tilde{\xi}(t) = \xi(t)\}$ is open and closed in $[0, T)$, hence agrees to $[0, T)$.

Combining Proposition 3.9 and Proposition 3.10, we get the following

**Theorem 3.11.** The equation $(EW^\xi)$ has a unique short time solution for any smooth initial data.

**Remark 3.12.** To show this theorem, we did not assume that $\mu \geq 0$. Hence the result is time-invertible. That is, a unique solution exists on some open time interval $(-T, T)$ containing $t = 0$.

**Corollary 3.13.** The equation $(EW)$ has a unique short time solution for any smooth initial data.

### 4. Singular perturbation

In this section, we assume that $\mu > 0$ and change the time variable $t$ of $(EW^\xi)$ to $\mu^{-1}t$.

\[
\begin{align*}
\mu^{-2}\nabla_t \xi_t + \nabla_x^2 \xi_x + \xi_t &= (w - |\xi_x|^2)\nabla_x \xi_x + 2w_x \xi_x - \frac{3}{2} \partial_x |\xi_x|^2 \xi_x, \\
-w_{xx} + |\xi_x|^2 w &= \mu^{-2} |\xi_x|^2 - |\nabla_x \xi_x|^2 + |\xi_x|^4, \\
\xi(x, 0) &= \xi_0(0), \quad \xi_t(x, 0) = \mu \xi_1(x), \quad \int_0^1 \xi_0 dx = \int_0^1 \xi_1 dx = 0.
\end{align*}
\]

First, we show uniform existence and boundedness of solutions with respect to large $\mu$. Constants $T$, $M$ below are independent of $\mu$.

**Lemma 4.1.** For any $K > 0$, there are $T > 0$ and $M > 0$ with the following property:

If $\xi$ is a solution of $(EW^\xi\mu)$ on an interval $[0, t_1) \subset [0, T)$ and if its initial value satisfies $\|\xi_0\|, \|\xi_1\| \leq K$, then $\|\xi\|_1, \mu^{-1} \|\xi\| \leq M$ holds on $0 \leq t < t_1$.

**Proof.** It is similar to the proof of Lemma 3.5. We put

\[
f = (w - |\xi_x|^2)\nabla_x \xi_x + 2w_x \xi_x - \frac{3}{2} \partial_x |\xi_x|^2 \xi_x,
\]

and we have

\[
\frac{1}{2} \frac{d}{dt} \left\{ \mu^{-2} \|\xi\|^2 + \|\nabla_x \xi_x\|^2 \right\} + \|\xi_t\|^2 = \langle \xi_t, f \rangle + \langle \nabla_x \xi_x, R(\xi_t, \xi_x) \rangle 
\leq \left( \frac{1}{4} + \frac{1}{4} \right) \|\xi_t\|^2 + \|f\|^2 + C(\|\xi_x\|^2 \|\nabla_x \xi_x\|)^2.
\]
Here, \( \|f\|^2 \) is bounded by a polynomial of \( X := \mu^{-2} \|\xi_t\|^2 + \|\nabla_x \xi_t\|^2 + \|\xi_t\|^2 \). Combining it with \( d\|\xi_t\|^2/dt \leq \|\xi_t\|^2 + \|\nabla_x \xi_t\|^2 \), we have a \( \mu \)-independent estimate of time derivative of \( X \) by a polynomial of \( X \). Therefore, there is a \( \mu \)-independent time \( T > 0 \) such that \( \|\xi_t\| \leq C\mu \) and \( \|\xi_t\|_1 \leq C \) on \([0, T)\).

**Lemma 4.2.** For any \( K > 0 \) and \( n > 0 \), there are \( M > 0 \) and \( \mu_0 > 0 \) with the following property:

Let \( \xi \) be a solution of \((\text{EW}^\xi)\) on \([0, T)\) with \( \mu \geq \mu_0 \). If its initial value satisfies \( \|\xi_0\|_{n+1}, \|\xi_1\|_n \leq K \) and if it satisfies \( \|\xi_n\|_1, \mu^{-1} \|\xi_n\| \leq K \) on \([0, T)\), then it holds that \( \|\xi_n\|_{n+1}, \|w\|_{n+1}, \mu^{-1} \|\xi_n\|_1 \leq M \) on \([0, T)\).

**Proof.** It is similar to the proof of Lemma 3.7. Suppose that we have bounds:

\[ \|\xi_{n+1}\|, \mu^{-1} \|\xi_n\|_1 \leq M. \]

They imply that \( \|\xi_{n+1}\|_1, \mu^{-1} \|\xi_n\|_{n+1} \leq C \), and,

\[ \|w\|_{n+2}, \|f\|_{n+1} \leq C(1 + \mu^{-1} \|\xi_{n+1}\| + \|\xi_n\|_{n+2}) \leq C(1 + \mu^{-1} \|\nabla_x^{n+1} \xi_t\| + \|\nabla_x^{n+2} \xi_t\|). \]

Using this, we have

\[
\frac{1}{2} \frac{d}{dt} \left\{ \mu^{-2} \|\nabla_x^{n+1} \xi_t\|^2 + \|\nabla_x^{n+2} \xi_t\|^2 \right\} + \|\nabla_x^{n+1} \xi_t\|^2
\]

\[= \langle \nabla_x^{n+1} \xi_t, \mu^{-2} \nabla_x \nabla_x^{n+1} \xi_t \rangle + \langle \nabla_x^{n+2} \xi_t, \nabla_x \nabla_x^{n+2} \xi_t \rangle + \|\nabla_x^{n+1} \xi_t\|^2 \]

\[\leq \langle \nabla_x^{n+1} \xi_t, \mu^{-2} \nabla_x^{n+1} \xi_t \rangle + \langle \nabla_x^{n+2} \xi_t, \nabla_x^{n+2} \xi_t \rangle + \|\nabla_x^{n+1} \xi_t\|^2 \]

\[\quad + C\mu^{-2} \|\nabla_x^{n+1} \xi_t\| \cdot \mu \|\xi_t\|_{n+1} + C \|\xi_{n+2}\| \|\xi_{n+1}\| \]

\[\leq \langle \nabla_x^{n+1} \xi_t, \nabla_x^{n+1} f \rangle + (C\mu^{-1} + \frac{1}{8})(\|\nabla_x^{n+1} \xi_t\|^2 + \|\xi_t\|^2) + C \|\xi_{n+2}\|^2 \]

\[\leq \left(C, mu^{-1} + \frac{1}{4}\right)(\|\nabla_x^{n+1} \xi_t\|^2 + \|\xi_t\|^2) + C(1 + \|\nabla_x^{n+2} \xi_t\|^2). \]

Assuming that \( \mu \geq 4C_1 \) and combining it with the first estimation:

\[ \frac{1}{2} \frac{d}{dt} \left\{ \mu^{-2} \|\xi_t\|^2 + \|\nabla_x \xi_t\|^2 \right\} \leq -\frac{1}{2} \|\xi_t\|^2 + C, \]

we can estimate

\[ X(t) := \mu^{-2}(\|\nabla_x^{n+1} \xi_t\|^2 + \|\xi_t\|^2) + (\|\nabla_x^{n+2} \xi_t\|^2 + \|\nabla_x \xi_t\|^2) \]

by \( X(t) \leq C(1 + X(t)) \). Hence we have \( \|\xi_t\|_{n+2} \leq C, \|\xi_t\|_{n+1} \leq C \mu \). Substituting it to the estimate of \( \|w\|_{n+2} \), we get \( \|w\|_{n+2} \leq C \).

**Proposition 4.3.** For any initial data \( \xi_0 \) and \( \xi_1 \), there is \( T > 0 \) such that \((\text{EW}^\xi)\) has a solution on \([0, T)\) for each \( \mu > 0 \). Moreover, for any \( n \geq 0 \), there are \( \mu_0 > 0 \)
and $M > 0$ such that the solution with $\mu \geq \mu_0$ satisfies $\|\xi_t\|_n, \|w\|_n \leq M$ and $\|\xi_t\|_n \leq M\mu$ on $[0, T)$.

Proof. Using Lemma 4.1 and Lemma 4.2, the proof is similar to that of Proposition 3.9.

Let $\{\eta, v\}$ be a solution of the limiting equation $(\mu \to \infty)$ of (EW$^{\mu}$) omitting initial data $\xi_\xi(x, 0)$.

\[
\begin{align*}
\eta_t + \nabla_2^2 \eta_t &= (v - |\eta_t|^2)\nabla_2 \eta_t + 2v \eta_t - \frac{3}{2} \partial_x |\eta_t|^2 \eta_t, \\
-v_{xx} + |\eta_t|^2 v &= -|\nabla_2 \eta_t|^2 + |\eta_t|^4, \\
\eta(x, 0) &= \xi_0(0).
\end{align*}
\]

In [4] (Theorem 7.5), we know that the corresponding equation for closed curves in the euclidean space has a unique all time solution. Therefore, (EP$^\mu$) has a unique all time solution, via Lemma 2.2.

We regard function $\eta$ as the 0-th approximation of $\xi$ for $\mu \to \infty$. To compare $\xi$ and $\eta$, we divide the interval $[0, \infty)$ so that the image $\xi(S^1 \times I)$ of each subinterval $I$ is contained in a local coordinate $U$ of $S^2$. For a solution $\xi$ and an interval $[t_0, t_1) \subset I$ such that $\xi(S^1 \times [t_0, t_1))$ is contained in $U$, we denote by $\{\zeta, u\}$ the difference between $\xi$ and $\eta$ in the local coordinate, i.e., $\zeta^p := \xi^p - \eta^p$, $u := w - v$. We use the local expression of (EW$^{\mu}$):

\[
\begin{align*}
\mu^{-2}(\zeta^p_{tt} + \Gamma_q^p(\xi)\zeta^q_{x} \zeta_t^r) + \partial^p_q \xi^p + 4\Gamma_q^p(\xi)\zeta^q_{x} \partial^p_q \zeta_t^r + \zeta_t^p &= F^p[\xi_{xx}, w_x], \\
-w_{xx} + g_{qr}(\xi)\zeta^q_{x} \zeta_t^r w &= \mu^{-2}g_{qr}(\xi)\zeta^q_{x} \zeta_t^r + G[\xi_{xx}], \\
\zeta(x, 0) &= \xi_0(0), \quad \zeta_t(x, 0) = \mu \xi_1(x), \quad \int_0^1 \xi_0 dx = \int_0^1 \xi_1 dx = 0,
\end{align*}
\]

where $F^p[\xi_{xx}, w_x]$ are polynomials of $\xi_x$, $\xi_{xx}$, $w$, $w_x$, functions of $\xi$, and $G[\xi_{xx}]$ is a polynomial of $\xi_x$, $\xi_{xx}$, functions of $\xi$. (We only note highest derivatives.) Since the local expression of (EP$^\mu$) is given by the above equations substituting $\mu^{-1} = 0$, $\{\zeta, u\}$ satisfies

\[
\begin{align*}
\mu^{-2}(\zeta^p_{tt} + 2\Gamma_q^p(\eta)\eta^q_{x} \zeta_t^r) + \partial^p_q \xi^p + 4\Gamma_q^p(\eta)\eta^q_{x} \partial^p_q \zeta_t^r + \zeta_t^p &= F^p[\eta_{xx}, w_x] - F^p[\eta_{xx}, v_x] - 4\Gamma_q^p(\xi)\zeta^q_{x} \partial^p_q \zeta_t^r - 4(\Gamma_q^p(\xi) - \Gamma_q^p(\eta))\eta^q_{x} \partial^p_q \zeta_t^r, \\
-w_{xx} + g_{qr}(\xi)\zeta^q_{x} \zeta_t^r u &= \mu^{-2}g_{qr}(\xi)\zeta^q_{x} \zeta_t^r + 2(\Gamma_q^p(\xi) - \Gamma_q^p(\eta))\eta^q_{x} \zeta_t^r,
\end{align*}
\]

$\zeta(x, 0) = 0, \quad \zeta_t(x, 0) = \mu \xi_1(x).$
We regard \( \zeta \) as a vector field along \( \eta \). Then, we can rewrite the above expression as

\[
\begin{align*}
(\text{EW}^\zeta) & \quad \mu^{-2} \nabla_2^2 \zeta + \nabla^4 \zeta + \nabla \zeta \\
& = L_1[\nabla^2 \zeta, u_x] + Q_1[\nabla \zeta, u_x; \nabla^3 \zeta, u_x] - \mu^{-2}[\nabla \eta_x + L_2[\zeta] + Q_2[\nabla \zeta; \nabla \zeta]], \\
& - u_{xx} + |\xi|^2 u \\
& = \mu^{-2}[|\eta_x|^2 + L_3[\nabla \zeta] + Q_3[\nabla \zeta; \nabla \zeta]] + L_4[\nabla^2 \zeta] + Q_4[\nabla^2 \zeta; \nabla^2 \zeta], \\
(|\xi|^2 &= |\eta_x|^2 + L_5[\nabla \zeta] + Q_5[\nabla \zeta; \nabla \zeta]), \\
\zeta(x, 0) &= 0, \quad \eta_x(x, 0) = \mu \xi_1(x),
\end{align*}
\]

where \( L_i \) are linear, \(|Q_i(\alpha; \beta)| \leq C|\alpha| |\beta|\). (We only note highest derivatives.) To get estimate of \( \{\zeta, u\} \), we need following

**Lemma 4.4** ([5] Lemma 1.5). For any \( K_1, K_2 > 0 \) and any \( T > 0 \), there are \( M > 0 \) and \( \mu_0 > 0 \) with the following property:

If \( \mu \geq \mu_0 \) and \( X(t), Y(t) \) and \( Z(t) \) are non-negative functions on \([0, T)\) such that

\[
X(0) \leq K_1 \mu^{-2}, \quad |X'(0)| \leq K_1, \quad Y(0) \leq K_1, \quad Z(0) \leq K_1 \mu^2,
\]

and that

\[
\begin{align*}
\mu^{-2}X''(t) + X'(t) & \leq K_1 \left(X(t) + \mu^{-2}Z(t) + \mu^{-2}\right) - K_2 Y(t), \\
Y'(t) + \mu^{-2}Z'(t) & \leq K_1 \left(Y(t) + 1\right) - K_2 Z(t),
\end{align*}
\]

on \([0, T)\), then they satisfy

\[
X(t) < M \mu^{-2}, \quad Y(t) < M \quad \text{and} \quad Z(t) < M \mu^2
\]

on \([0, T)\).

**Lemma 4.5.** For any \( n \geq 0 \) and any \( K > 0 \), there are \( M > 0 \) and \( \mu_0 > 0 \) with the following property:

Let \( \{\zeta, u\} \) be the solution of (EW\(^\zeta\)) with \( \mu \geq \mu_0 \), defined on \([t_0, t_1) \subset (0, T)\). If \(|\zeta|_n \leq K \mu^{-1} \) at \( t = t_0 \), then \(|\zeta|_n \leq M \mu^{-1} \) holds on \([t_0, t_1)\).

Proof. Note that we have bounds of \( \{\xi, w\} \) and \( \{\eta, v\} \) by Proposition 4.3. Therefore, we know \(|\xi|_n \leq C, \quad |\nabla \zeta|_n \leq C \mu \) and \(|u|_n \leq C\). We may assume that \( \mu \geq \mu_0 \geq 1 \). For

\[
h := \mu^{-2}[|\eta_x|^2 + L_3[\nabla \zeta] + Q_3[\nabla \zeta; \nabla \zeta]] + L_4[\nabla^2 \zeta] + Q_4[\nabla^2 \zeta; \nabla^2 \zeta],
\]
we have
\[ \|h\|_n \leq C(\mu^{-2}(1 + \|\nabla_1 \zeta\|_n + \|\nabla_1 \zeta\|_1 \|\nabla_1 \zeta\|_n) + \|\zeta\|_{n+2} + \|\zeta\|_3 \|\zeta\|_{n+2}) \]
\[ \leq C(\mu^{-2} + \mu^{-1}\|\nabla_1 \zeta\|_n + \|\zeta\|_{n+2}), \]
and, \[ \|u\|_{n+2} \leq C\|h\|_n \leq C(\mu^{-2} + \mu^{-1}\|\nabla_1 \zeta\|_n + \|\zeta\|_{n+2}). \] And, for
\[ f := L_1[\nabla_2^2 \zeta, u_1] + Q_1[\nabla_2^2 \zeta, u_1; \nabla_1^2 \zeta, u_1] - \mu^{-2}(\nabla_i \eta_i + L_2[\zeta] + Q_2[\nabla_1 \zeta; \nabla_1 \zeta]), \]
we have
\[ \|f\|_n \leq C(\|\zeta\|_{n+2} + \|u\|_{n+1} + \mu^{-2}(1 + \|\nabla_1 \zeta\|_1 \|\nabla_1 \zeta\|_n)) \]
\[ \leq C(\|\zeta\|_{n+2} + \mu^{-2} + \mu^{-1}\|\nabla_1 \zeta\|_n). \]
Put \[ X_n(t) := \|\nabla_1^2 \xi\| \] and \[ Z_n(t) := \|\nabla_1^2 \nabla_1 \zeta\|. \] Then, we see that
\[ (X_0^2)' = 2(\zeta, \nabla_1 \xi) \leq 2X_0Z_0, \]
\[ (X^2)' = 2(\nabla_1 \zeta, \nabla_1 \nabla_1 \zeta) \leq -2(\nabla_1 \zeta, \nabla_1 \nabla_1 \zeta) + C\|\zeta\|_1 \|\zeta\| \]
\[ \leq 2X_0Z_0 + C(X_0^2 + X_1^2), \]
\[ \mu^{-2}(Z_i^2)' + 2Z_i^2 + (X_{i+2}^2)' \]
\[ = 2(\nabla_1^2 \xi, \mu^{-2}\nabla_1^2 \nabla_1^2 \zeta + \nabla_1^2 \nabla_1^2 \zeta) + 2(\nabla_1^2 \xi, \nabla_1^2 \nabla_1^2 \zeta) \]
\[ \leq 2(\nabla_1^2 \xi, \nabla_1^2 f) + C\|\nabla_1^2 \nabla_1^2 \zeta\| (\mu^{-2}\|\nabla_1^2 \zeta\|_1 + \|\zeta\|_{i-1}) + C\|\nabla_1^2 \zeta\| \|\zeta\|_{i+1} \]
\[ \leq C(Z_i(X_{i+2} + X_0) + \mu^{-2} + \mu^{-1}(Z_i + Z_0)) + C(X_{i+2}^2 + X_0^2). \]
Therefore,
\[ \mu^{-2}(\|\nabla_1^2 \zeta\|_n^2)' + (\|\zeta\|_{n+2}^2)' + 2\|\nabla_1^2 \zeta\|_n^2 \]
\[ \leq C\|\zeta\|_{n+2}^2 + C\|\nabla_1^2 \zeta\|_n^2 + C\mu^{-2} + C \sum_{i=0}^n Z_i(X_{i+2} + X_0) \]
\[ \leq \frac{1}{2}\|\nabla_1^2 \zeta\|_n^2 + C\|\zeta\|_{n+2}^2 + C_1\mu^{-1}\|\nabla_1^2 \zeta\|_n^2 + C\mu^{-2}, \]
\[ \mu^{-2}(\|\nabla_1^2 \zeta\|_n^2)' + (\|\zeta\|_{n+2}^2)' \leq C(\|\zeta\|_{n+2}^2 + \mu^{-2}) - \|\nabla_1^2 \zeta\|_n^2 \]
if \[ \mu \geq 2C_1. \]
We also have,
\[ \mu^{-2}(X_i^2)'' + (X_i^2)' + 2X_{i+2}^2 \]
\[ = 2\mu^{-2}\|\nabla_1^2 \zeta\|_n^2 + 2(\nabla_1^2 \zeta, \mu^{-2}\nabla_1^2 \nabla_1^2 \zeta + \nabla_1^2 \nabla_1^2 \zeta + \nabla_1^4 \zeta) \]
\[-3\mu^{-2}\|\nabla_i \nabla'i \nabla'i\|^2 + 2\langle \nabla'_i \nabla_i, \nabla'_i f \rangle
\]
\[+ C \mu^{-2} \|\nabla_i \nabla'\| \{\mu^{-2}(\|\nabla_i \nabla_i \|_{i=1} + \|\nabla_i \nabla_i \|_{i=2}) + \|\nabla_i \nabla_i \|_{i=1}\}
\leq 3\mu^{-2} \|v_i^2 + C \sum \{X_{i+2} + X_{i+1} + \mu^{-2} + \mu^{-1}(Z_i + Z_0)\}
\]
\[+ C \mu^{-2} \{X_{i+2} + X_{i+1} + \mu^{-2}(Z_i + Z_0) + X_i\}
\leq X_{i+2}^2 + C \{X_i^2 + X_0^2 + \mu^{-2}(Z_i^2 + Z_0^2) + \mu^{-4}\},
\]
\[\mu^{-2}(\|\nabla_i \nabla'i\|)^2 + (\|\nabla_i \nabla'i\|)^2 \leq C \{\|\nabla_i \nabla'i\|^2 + \mu^{-2}\|\nabla_i \nabla'i\|^2 + \mu^{-4}\} - \|\nabla_i \nabla'i\|^2.
\]

Setting \(X := \|\nabla_i \nabla'i\|^2, Y := \|\nabla'_i \nabla'i\|^2\) and \(Z := \|\nabla_i \nabla'i\|^2\) in Lemma 4.4, we have \(\|\nabla_i \nabla'i\| \leq C \mu^{-1}\). \(\square\)

**Lemma 4.6.** For any \(n, m \geq 0\) and \(K > 0\), there are \(M > 0\) and \(\mu_0 > 0\) with the following property:

Let \(\{\nabla_i, u\}^{\infty}\) be the solution of (EW\(^{\infty}\)) with \(\mu \geq \mu_0\), defined on \([t_0, t_1] \subset [0, T]\). If \(\|\nabla_i \nabla'i\| \leq K \mu_0^{m-1}\) at \(t = t_0\), then

\[
\|\nabla_i \nabla'i\| \leq M(\mu^{-1} + \mu_0^{m-1} e^{-\mu_0 t/2}),
\]
\[
\|\nabla'_i u\| \leq M(\mu^{-1} + \mu_0^{m-1} e^{-\mu_0 t/2})
\]

hold on \([t_0, t_1]\).

Proof. We put \(V_i := \mu^{-1} + \mu e^{-\mu t/2}\). Note the log-convexity:

\[
V_i^2 \leq V_{i-1} V_{i+1} \quad \text{and} \quad V_i V_{i+k} \leq V_0 V_{i+k} \leq (1 + \mu_0^{-1}) V_{i+k} \quad \text{for} \quad j, k \geq 0.
\]

We know that \(\|\nabla_i \nabla'i\| \leq C \mu_0 \|u\| \leq C\) by Proposition 4.3, and \(\|\nabla_i \nabla'i\| \leq C \mu_0^{-1}\) by Lemma 4.5. In particular, \(\|\nabla_i \nabla'i\| \leq C V_{i-1}\) holds. We prove the estimate of \(\nabla_i \nabla'i\) and the estimate of \(\nabla_i \nabla'i\) assuming the estimate of \(\nabla_i \nabla'i\) for \(j < m\).

First, we estimate \(\nabla_i \nabla'i\). Put

\[
h := \mu^{-2}(\|u\|^2 + L_3[\nabla_i \nabla'i] + Q_3[\nabla_i \nabla'i] + L_4[\nabla_i^2 \nabla'i] + Q_4[\nabla_i^2 \nabla'i] + \nabla_i \nabla'i\).
\]

It is estimated as

\[
\|\nabla_i \nabla'i \| \leq C \mu^{-2}(1 + \|\nabla_i \nabla'i\| + V_{2m-1})
\]
\[
+ \|\nabla_i \nabla'i\| \|\nabla_i \nabla'i\| \|\nabla_i \nabla'i\| + V_{3} V_{2m-1} + V_{2m-1}
\]
\[
\leq C \mu^{-1} \|\nabla_i \nabla'i\| + V_{2m}.
\]
where $V_3^*$ appears only if $m \geq 2$. Therefore, we have
\[
\| \partial_t^m u\|_{(n+2)} \leq \| \partial_t^m u\|_{(n)} + C \sum_{j=1}^{m} \| \partial_t^j |\xi x|^2 \|_{(n)} \| \partial_t^{m-j} u\|_{(n)} \\
\leq C \{ \mu^{-1} \| \nabla t^{m+1} \zeta\|_{(n)} + V_{2m} \} + C \sum_{j=1}^{m} (1 + V_{2j-1}) V_{2(m-j)} \\
\leq C \{ \mu^{-1} \| \nabla t^{m+1} \zeta\|_{(n)} + V_{2m} \}.
\]

Now, we estimate $\nabla t^{m+1} \zeta$. Put
\[
f := L_1[\nabla t^2 \zeta, u_x] + Q_1[\nabla t^2 \zeta, u_x; \nabla t^2 \zeta, u_x] - \mu^{-2}(\nabla t \eta_t + L_2[\zeta] + \Omega_2[\nabla t \zeta; \nabla t \zeta]).
\]

Then,
\[
\| \nabla t^m f\|_{(n)} \leq C \{ V_{2m-1} + \| \partial_t^m u\|_{(n+1)} + \| u\|_{(n+1)} \| \partial_t^m u\|_{(n+1)} \\
+ \mu^{-2}(1 + V_{2m-1} + \| \nabla t \zeta\|_{(n)} \| \nabla t^{m+1} \zeta\|_{(n)} + V_3^* V_{2m-1}) \} \\
\leq C \{ \mu^{-1} \| \nabla t^{m+1} \zeta\|_{(n)} + V_{2m} \},
\]

where $V_3^*$ appears only if $m \geq 2$. Therefore,
\[
\| \nabla t^m (\mu^{-2} \nabla t^2 \zeta + \nabla t \zeta)\|_{(n)} \leq \| \nabla t^m \zeta\|_{(n+4)} + \| \nabla t^m f\|_{(n)} \\
\leq C \{ \mu^{-1} \| \nabla t^{m+1} \zeta\|_{(n)} + V_{2m} \}.
\]

Thus,
\[
\mu^{-2} \frac{\partial}{\partial t} |\nabla t^{m+1} \zeta|^2 + 2|\nabla t^{m+1} \zeta|^2 \\
= 2(\nabla t^m \nabla t^{m+1} \zeta, \mu^{-2} \nabla t^{m+1} \zeta + \nabla t^{m+1} \zeta) \\
\leq 2(\nabla t^m \nabla t^{m+1} \zeta, \nabla t^m (\mu^{-2} \nabla t^{m+2} \zeta + \nabla t^{m+1} \zeta) \} \\
+ C \mu^{-2} |\nabla t^m \nabla t^{m+1} \zeta|| \nabla t^{m+1} \zeta\|_{(n-1)} \} \\
\leq C |\nabla t^m \nabla t^{m+1} \zeta|| \mu^{-1} \| \nabla t^{m+1} \zeta\|_{(n)} + V_{2m} \}.
\]

From this, for $X(t) := \| \nabla t^{m+1} \zeta\|_{(n)}^2$, we have
\[
\mu^{-2} X'(t) + 2X(t) \leq C_1 \mu^{-1} X(t)^2 + C V_{2m} X(t) \leq \left( \frac{1}{2} + C_1 \mu^{-1} \right) X(t)^2 + C V_{2m}^2,
\]

where $X(t) = \lim_{\delta \to 0} \sup (X(t + \delta) - X(t))/\delta$.

We set $\mu_0 \leq 2C_1$. Then,
\[
\mu^{-2} X'(t) + X(t) \leq C_2 (\mu^{-2} + \mu^{4m} e^{-\mu t}),
\]
\[ X(t) \leq X(t_0)e^{-\mu^2 t} + C_2(\mu^{-2} + \mu^{2m+2}e^{-\mu^2 t}) \]
\[ \leq C(\mu^{-2} + \mu^{2m+2}e^{-\mu^2 t}), \]

that is, \( \|\partial^n u\|_{(\alpha+2)} \leq CV_{2m+1} \).

Substituting it to the estimate of \( \|\partial^n u\|_{(\alpha+2)} \), we get the estimation of \( \partial^n u \).

**Proposition 4.7.** For any initial data \( \{\xi_0, \xi_1\} \), any interval \( [t_0, t_1) \subset [0, T) \) and any local coordinate \( U \) of \( S^2 \) such that the image \( \eta(S^1 \times [t_0, t_1)) \) is contained in \( U \), there exists \( \mu_0 > 0 \) with the following property:

If \( \xi \) is a solution of \( (EW^\mu) \) on \( [0, T) \), then the image \( \xi(S^1 \times [t_0, t_1)) \) is contained in \( U \). Moreover, \( \xi \) uniformly converges to \( \eta \) on \( [0, T) \) when \( \mu \to \infty \).

**Proof.** We divide the interval \( [0, T) \) so that the image \( \eta(S^1 \times I) \) of each subinterval \( I \) is included to a local coordinate \( U_I \).

Note that \( \zeta \) is defined only on each short time interval.

Starting from \( t = 0 \) and applying this Lemma on each time interval where \( \{\zeta, u\} \) is defined, we see that \( \|\zeta\|_n \) is small for large \( \mu \).

We sum up these results, and get the following

**Theorem 4.8.** For any non-negative integers \( m, n \) and any positive number \( T \), there are positive numbers \( \mu_0 \) and \( M \) with the following properties:

For each \( \mu \geq \mu_0 \), there exists a solution \( \xi \) of \( (EW^\mu) \) on \( [0, T) \), and \( \xi \) uniformly converges to \( \eta \) when \( \mu \to \infty \). More precisely,

\[ |\partial^n \xi^p(\xi^p - \eta^p)| \leq M(\mu^{-1} + \mu^{2m-1}e^{-\mu^2 t/2}) \]

holds on each local coordinate.

**Remark 4.9.** In general, we cannot expect uniform estimation on the whole time \( [0, \infty) \). The limit \( \eta(\infty) \) can be an unstable elastic curve, and in that case, \( \xi(\infty) \) and \( \eta(\infty) \) discontinuously depend on the initial data.

**Corollary 4.10.** For any positive number \( T \), there exists a unique solution \( \gamma \) of \( (EW^\gamma) \) on \( [0, T) \) for sufficiently large \( \mu > 0 \). Moreover, the solution converges to a solution \( \eta \) of \( (EP) \) when \( \mu \to \infty \).
References


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