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THE SECOND LOWER LOEWY TERM OF THE PRINCIPAL INDECOMPOSABLE OF A MODULAR GROUP ALGEBRA

JULIO P. LAFUENTE and CONCHITA MARTÍNEZ-PÉREZ[†‡]

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1. Introduction

Let G be a finite group and consider a field $\mathbb{K}$ of prime characteristic p. Let P be the projective cover of the trivial $\mathbb{K}G$-module, which we denote by $\mathbb{K}$, and J the Jacobson radical $\mathrm{J}(\mathbb{K}G)$ of the group algebra $\mathbb{K}G$. Let $e \in \mathbb{K}G$ be a primitive idempotent such that $P = e\mathbb{K}G$. We are concerned with the second term

$$eJ/eJ^2$$

of the lower Loewy series of P. It is a completely reducible $\mathbb{K}G$-module, whose composition factors are just the irreducible $\mathbb{K}G$-modules V such that there exists a nonsplit $\mathbb{K}G$-module extension $0 \to V \to E \to \mathbb{K} \to 0$ (see [7, VII 16.8]).

Gaschütz (see [7, VII §15]) gives a complete description of eJ/eJ^2 for $\mathbb{K} = \mathbb{F}_p$, the field of p elements, and G a p-soluble group: Its composition factors are precisely the abelian complemented p-chief factors of G, counting the multiplicities. Later Willems shows [12] that for any G each complemented p-chief factor of G appears as a component of eJ/eJ^2 with multiplicity not less than that as a (complemented) chief factor of G. Okuyama and Tsushima [10] define a filtration of eJ/eJ^2 from a chief series of G, which provides a new proof of these results and makes explicit the relationship between the chief factors of G and the composition factors of eJ/eJ^2.

In this paper we give a description of eJ/eJ^2 for any G and any field $\mathbb{K}$ of characteristic p, which only depends on the knowledge of what occurs for certain almost simple sections of G, by means of the development of a reduction theorem of Kovács [8]. As an application we obtain the terms of the filtration of Okuyama and Tsushima corresponding to any chief factor of any G.

2. Notations and basic facts

We denote by $\mathrm{Irr}(G, \mathbb{K})$ the set of irreducible $\mathbb{K}G$-modules. If $V \in \mathrm{Irr}(G, \mathbb{K})$, then, as P is the projective cover of $\mathbb{K}$,

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$$H^1(G, V) \cong \mathrm{Ext}_{\mathbb{K}G}(\mathbb{K}, V) \cong \mathrm{Hom}_{\mathbb{K}G}(eJ, V) \cong \mathrm{Hom}_{\mathbb{K}G}(eJ/eJ^2, V)$$

[5]. Therefore, if we denote by $\ell_2^G(V)$ the multiplicity of V as component of eJ/eJ^2, then

$$\ell_2^G(V) = \dim_{\mathrm{End}_{\mathbb{K}G}(V)} H^1(G, V).$$

($\mathrm{End}_{\mathbb{K}G}(V)$ is a division ring, because of Schur's lemma [6, 10.5].) We set

$$\mathcal{C}(G, \mathbb{K}) = \left\{V; V \in \mathrm{Irr}(G, \mathbb{K}),\ \ \ell_2^G(V) \neq 0\right\}$$

(here we identify isomorphic modules, that is $\mathcal{C}(G, \mathbb{K})$ consists actually of isomorphism classes of modules). On the other hand, $\mathrm{Ext}_{\mathbb{K}G}(\mathbb{K}, V) \cong \mathrm{E}(\mathbb{K}, V)$ [5], whence if $\xi \in H^1(G, V)$, then ξ represents an equivalence class of $\mathbb{K}G$-module extensions

$$0 \to V \to E \to \mathbb{K} \to 0.$$

We put then $\mathrm{C}_G(\xi) = \mathrm{C}_G(E)$ and

$$\mathcal{C}_1(G, \mathbb{K}) = \left\{V \in \mathcal{C}(G, \mathbb{K}); \exists \xi \in H^1(G, V) \text{ such that } \mathrm{C}_G(\xi) < \mathrm{C}_G(V)\right\}.$$

Recall that $\mathcal{C}_1(G, \mathbb{F}_p)$ is the set of the abelian complemented p-chief factors of G [11, 2.4(1)].

A $\mathbb{K}G$-module V can be considered as a (faithful) $\mathbb{K}G/\mathrm{C}_G(V)$-module. We put

$$\mathcal{C}_0(G, \mathbb{K}) = \left\{V \in \mathcal{C}(G, \mathbb{K}); \ell_2^{G/\mathrm{C}_G(V)}(V) \neq 0\right\}.$$

If $\mathbb{F} \subseteq \mathbb{K}$ is a field extension and M is an $\mathbb{F}G$-module, then we set $M_{\mathbb{K}} = M \otimes_{\mathbb{F}} \mathbb{K}$ for the scalar extension.

If $V \in \mathrm{Irr}(G, \mathbb{K})$, then a unique (up to isomorphisms) $\hat{V} \in \mathrm{Irr}(G, \mathbb{F}_p)$ is determined such that V is a component of $\hat{V}_{\mathbb{K}}$. In this case $H^1(G, V) \neq 0$ if and only if $H^1(G, \hat{V}) \neq 0$, $\mathrm{C}_G(V) = \mathrm{C}_G(\hat{V})$ and $V \in \mathcal{C}_1(G, \mathbb{K})$ if and only if $\hat{V}$ is isomorphic to a complemented chief factor of G [9, §1]. Therefore $V \in \mathcal{C}_\varepsilon(G, \mathbb{K})$ if and only if $\hat{V} \in \mathcal{C}_\varepsilon(G, \mathbb{F}_p)$, $\varepsilon = \emptyset, 0, 1$.

Proposition 2.1. *If $\mathbb{F} \subseteq \mathbb{K}$ is a field extension, let $V \in \mathrm{Irr}(G, \mathbb{K})$ and $U \in \mathrm{Irr}(G, \mathbb{F})$ be such that V is a component of $U_{\mathbb{K}}$. Then*

$$\dim_{\mathrm{End}_{\mathbb{K}G}(V)} H^n(G, V) = \dim_{\mathrm{End}_{\mathbb{F}G}(U)} H^n(G, U), \quad n = 1, 2, \ldots$$

Proof. Let

$$\mathcal{P} : \cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to \mathbb{F} \to 0$$

be a minimal projective resolution of $\mathbb{F}$. Then $\dim_{\mathrm{End}_{\mathbb{F}G}(U)} H^n(G,U)$ is the multiplicity of U as component of $P_{n+1}/P_n\,\mathrm{J}(\mathbb{F}G)$.

On the other hand, $\mathrm{J}(\mathbb{F}G)_{\mathbb{K}} \cong \mathrm{J}(\mathbb{K}G)$ [7, VII 1.5]. As $\mathbb{K}$ is of prime characteristic, $U_{\mathbb{K}}$ is a direct sum of pairwise non-isomorphic irreducible $\mathbb{K}G$-modules [7, VII 1.15]. Then we have that the multiplicity of U as component of $P_{n+1}/P_n\,\mathrm{J}(\mathbb{F}G)$ is equal to the multiplicity of V as component of $(P_{n+1}/P_n\,\mathrm{J}(\mathbb{F}G))_{\mathbb{K}}$. And also we have that $\mathcal{P}_{\mathbb{K}}$ is a minimal projective resolution of $\mathbb{K}$.

We consider again dimensions in $\mathcal{P}_{\mathbb{K}}$ and have the claim. □

Corollary 2.2. *Let* $V \in \mathrm{Irr}(G,\mathbb{K})$. *Then*

$$\ell_2^G(V) = \ell_2^G(\hat{V}), \quad \ell_2^{G/\mathrm{C}_G(V)}(V) = \ell_2^{G/\mathrm{C}_G(\hat{V})}(\hat{V}).$$

Denote by $\mathrm{cm}^G(V)$ the multiplicity of $\hat{V}$ as complemented chief factor in a chief series of G. As another immediate consequence we have the validity of the following equality, which appears in [1, 2.10(b)] for the case $\mathbb{K} = \mathbb{F}_p$:

Corollary 2.3. *Let* $V \in \mathrm{Irr}(G,\mathbb{K})$. *Then we have*

$$\ell_2^G(V) = \mathrm{cm}^G(V) + \ell_2^{G/\mathrm{C}_G(V)}(V).$$

3. The second Loewy term

Recall that a *primitive* group is a group G with a maximal subgroup H such that $\mathrm{core}_G(H) = 1$, $\mathrm{core}_G(H)$ being the intersection of all conjugate of H in G. Then G has exactly either one minimal normal subgroup or two nonabelian minimal normal subgroups. If G has a single nonabelian minimal normal subgroup, then we say $G \in \mathcal{P}_2$.

A particular consequence of Kovács reduction theorem [8] is that, if $U \in \mathrm{Irr}(G,\mathbb{F}_p)$ is faithful and $H^1(G,U) \neq 0$, then $G \in \mathcal{P}_2$ and $p \mid |\mathrm{S}(G)|$ (where $\mathrm{S}(G)$, the *socle* of G, is the product of the minimal normal subgroups of G). From the above proposition we have that this is also true for any faithful irreducible module in $\mathrm{Irr}(G,\mathbb{K})$.

Proposition 3.1. *The following two assertions are equivalent*:
(a) *There exists a faithful irreducible* $\mathbb{K}G$*-module* V *such that* $H^1(G,V) \neq 0$.
(b) $G \in \mathcal{P}_2$ *and* $p \mid |\mathrm{S}(G)|$.

Proof. It suffices to show that (b) $\Longrightarrow$ (a). This follows from the fact that $\mathrm{F}_p(G) = \cap\{\mathrm{C}_G(V); V \in \mathcal{C}(G,\mathbb{K})\}$ [2, Theorem 1], as $\mathrm{F}_p(G) = 1$ and $\mathrm{S}(G)$ is contained in each nontrivial normal subgroup of G. □

Corollary 3.2. *Set* $\mathrm{n}_0(G) = \left\{C; C \lhd G,\ \ G/C \in \mathcal{P}_2,\ \ p \,\middle|\, |\,\mathrm{S}(G/C)|\right\}$. *Then*

$$\mathrm{n}_0(G) = \{\mathrm{C}_G(V); V \in \mathcal{C}_0(G, \mathbb{K})\}\,.$$

Proof. By the definition of $\mathcal{C}_0(G, \mathbb{K})$ and Proposition 3.1, it is clear that if $V \in \mathcal{C}_0(G, \mathbb{K})$, then $\mathrm{C}_G(V) \in \mathrm{n}_0(G)$. Assume now that $C \in \mathrm{n}_0(G)$. By Proposition 3.1 there exists a faithful irreducible $\mathbb{K}G/C$-module V such that $H^1(G/C, V) \neq 0$, that is such that $\ell_2^{G/C}(V) \neq 0$. As the inflation map $H^1(G/C, V) \to H^1(G, V)$ is a monomorphism [5, VI 8.1], $V \in \mathcal{C}(G, \mathbb{K})$. As $C = \mathrm{C}_G(V)$ we conclude that $V \in \mathcal{C}_0(G, \mathbb{K})$. □

Let $C \in \mathrm{n}_0(G)$. Then $\mathrm{S}(G/C)$ is the only minimal normal subgroup of G/C and is nonabelian. Therefore it is the product of isomorphic nonabelian simple groups. Let S/C be a simple component of $\mathrm{S}(G/C)$, $A = \mathrm{N}_G(S/C)$ and $B = \mathrm{C}_G(S/C)$. In these conditions we say $A/B \in \mathrm{a}(C)$. Observe that A/B is an *almost simple* group, that is a group in $\mathcal{P}_2$ with simple socle (isomorphic to S/C).

If $H \leq G$ and V is a $\mathbb{K}G$-module, then we set

$$V^H = \{v \in V; vh = v\ \ \forall h \in H\}$$

and write $V \downarrow_H$ for the $\mathbb{K}H$-module obtained from V by restricting the action to $\mathbb{K}H$.

If W is a $\mathbb{K}H$-module, then we set $W \uparrow^G = W \otimes_{\mathbb{K}H} \mathbb{K}G$.

Lemma 3.3. *Consider* $C \in \mathrm{n}_0(G)$, $A/B \in \mathrm{a}(C)$ *and assume that* W *is a faithful irreducible* $\mathbb{K}A/B$*-module. Then*

(a) $W \uparrow^G \in \mathrm{Irr}(G, \mathbb{K})$, $W \cong (W \uparrow^G)^B$ *and* $\mathrm{C}_G(W \uparrow^G) = C$.

(b) $\ell_2^{A/B}(W) = \ell_2^{G/C}(W \uparrow^G)$.

(c) $\ell_2^A(W) = \ell_2^G(W \uparrow^G)$ *and* $\mathrm{cm}^A(W) = \mathrm{cm}^G(W \uparrow^G)$.

Proof. (a) We may assume that $C = 1$. Then $G \in \mathcal{P}_2$. Set $N = \mathrm{S}(G)$. Let $V \in \mathrm{Irr}(G, \mathbb{K})$ be a component of the head $\mathrm{H}(W \uparrow^G) := W \uparrow^G /(W \uparrow^G)J$ of $W \uparrow^G$. By Nakayama's theorem [6, V 16.6], W is a submodule of $\mathrm{S}(V \downarrow_A)$, and so $W \downarrow_N$ is a submodule of $V \downarrow_N$.

Let $\{g_1, \ldots, g_n\}$ be a transversal of A in G, with $g_1 = 1$. Then, by putting $S_i = S^{g_i}$, $B_i = B^{g_i}$, we have $N = S_1 \times \cdots \times S_n$, $B_i = \mathrm{C}_G(S_i)$. Set moreover for $1 \leq i \leq n$

$$V_i = V^{B_i}, \quad U_i = V_1 + \cdots + V_{i-1} + V_{i+1} + \cdots + V_n, \quad M_i = V_i \cap U_i.$$

We have

$$S_i \leq \bigcap_{j \neq i} B_j \leq \bigcap_{j \neq i} \mathrm{C}_G(V_j) = \mathrm{C}_G(U_i), \quad B_i \leq \mathrm{C}_G(V_i)$$

and hence $N \leq S_i B_i \leq \mathrm{C}_G(M_i)$. Therefore $M_i \subseteq V^N$. As V is an irreducible $\mathbb{K}G$-module and $N \trianglelefteq G$, either $V^N = V$ or $V^N = 0$. Assume that $V^N = V$. As $W \downarrow_N$ is a submodule of $V\downarrow_N$, $N \le \mathrm{C}_A(W)$. Then $B < BN \le \mathrm{C}_A(W)$, contradicting the fact that W is a faithful A/B-module.

So we have that $V^N = 0$. In particular $M = 0$, that is $V_1 + \cdots + V_n$ is a direct sum. As $Wg_i \subseteq V_i$, we have that also $Wg_1 + \cdots + Wg_n$ is a direct sum, and hence

$$W\uparrow^G \cong Wg_1 \oplus \cdots \oplus Wg_n \le V.$$

As $\dim_{\mathbb{K}} V \le \dim_{\mathbb{K}} W\uparrow^G$, we have that $V \cong W\uparrow^G$. Clearly $W \cong (W\uparrow^G)^B$. And $\mathrm{C}_G(V) = \mathrm{core}_G(\mathrm{C}_A(W)) = \mathrm{core}_G(B) = 1$.
(b) By Shapiro's lemma [3, 6.3], $H^1(A/C, W) \cong H^1(G/C, W\uparrow^G)$. By [8, 3.5], $\mathrm{End}_{\mathbb{K}A/C}(W) \cong \mathrm{End}_{\mathbb{K}G/C}(W\uparrow^G)$. Therefore $\ell_2^{A/C}(W) = \ell_2^{G/C}(V)$.

Assume that $\hat{W}$ appears as a chief factor of A between C and B. Then $S \le \mathrm{C}_A(\hat{W}) = \mathrm{C}_A(W) = B = \mathrm{C}_A(S/C)$, a contradiction. In particular $\mathrm{cm}^{A/C}(W) = 0$. Therefore $\ell_2^{A/B}(W) = \ell_2^{A/C}(W)$, and hence $\ell_2^{A/B}(W) = \ell_2^{G/C}(V)$.
(c) Again by Shapiro's lemma, $\ell_2^A(W) = \ell_2^G(W\uparrow^G)$. From (b) and [1, 2.10(b)] we have that $\mathrm{cm}^A(W) = \mathrm{cm}^G(W\uparrow^G)$. □

We now deduce the validity of [8] for any field $\mathbb{K}$ of prime characteristic p:

Theorem 3.4 (Kovács Reduction.). *Consider $V \in \mathcal{C}_0(G,\mathbb{K})$, $A/B \in \mathrm{a}(C)$ and set $N/C = \mathrm{S}(G/C)$. Let $W = V^{B\cap N}$. Then $W \in \mathcal{C}_0(A,\mathbb{K})$, $\mathrm{C}_A(W) = B$, $\ell_2^{G/C}(V) = \ell_2^{A/B}(W)$ and $V \cong W\uparrow^G$.*

Proof. As $V \in \mathcal{C}_0(G,\mathbb{K})$, $\hat{V} \in \mathcal{C}_0(G,\mathbb{F}_p)$. Moreover $C := \mathrm{C}_G(V) = \mathrm{C}_G(\hat{V}) \in \mathrm{n}_0(G)$. By [8], $U := \hat{V}^{B\cap N} \in \mathcal{C}_0(A,\mathbb{F}_p)$, $\mathrm{C}_A(U) = B$ and $\ell_2^{G/C}(\hat{V}) = \ell_2^{A/B}(U)$.

Let $U_{\mathbb{K}} \cong W_1 \oplus \cdots \oplus W_r$, where each W_i is irreducible. Then $W_i \in \mathcal{C}_0(A,\mathbb{K})$ and $\mathrm{C}_A(W_i) = B$. Let now $\hat{V}_{\mathbb{K}} \cong V_1 \oplus \cdots \oplus V_s$, with each V_i irreducible and $V_1 \cong V$. Then we have

$$V_1 \oplus \cdots \oplus V_s \cong \hat{V}_{\mathbb{K}} \cong (U\uparrow^G)_{\mathbb{K}} \cong U_{\mathbb{K}}\uparrow^G \cong W_1\uparrow^G \oplus \cdots \oplus W_r\uparrow^G .$$

By Lemma 3.3 (a) each $W_i\uparrow^G$ is irreducible. Therefore, by the Krull-Remak-Schmidt theorem [6, I 12.3], we have that $r = s$ and, after rearranging the indices if necessary, $V_i \cong W_i\uparrow^G$, $1 \le i \le r$. Moreover, as $U = \hat{V}^{B\cap N}$, $U_{\mathbb{K}} \cong (\hat{V}_{\mathbb{K}})^{B\cap N}$, and therefore $W_i \cong V_i^{B\cap N}$. Finally, by Corollary 2.2, $\ell_2^{G/C}(V) = \ell_2^{G/C}(\hat{V}) = \ell_2^{A/B}(U) = \ell_2^{A/B}(W_1)$. □

This reduction theorem allows us to reduce also the study of $\mathcal{C}(G,\mathbb{K})$ to the almost simple case:

Theorem 3.5. *Consider $C \in \mathrm{n}_0(G)$ and $A/B \in \mathrm{a}(C)$. Then the map*

$$\uparrow^G : \{W \in \mathcal{C}_0(A,\mathbb{K}); \mathrm{C}_A(W) = B\} \to \{V \in \mathcal{C}_0(G,\mathbb{K}); \mathrm{C}_G(V) = C\}$$

is bijective. Moreover $\ell_2^{A/B}(W) = \ell_2^{G/C}(W\uparrow^G)$, $\ell_2^A(W) = \ell_2^G(W\uparrow^G)$ *and* $\mathrm{cm}^A(W) = \mathrm{cm}^G(W\uparrow^G)$.

Proof. By Lemma 3.3, (a) (b) we have a well-defined injective map. It is surjective by Theorem 3.4. □

Now we can give the following first explicit description of eJ/eJ^2.

Theorem 3.6. *Let* $C \in \mathrm{n}_0(G)$ *and* $A/B \in \mathrm{a}(C)$. *Let* $\{W_1 \cdots W_m\}$ *be a complete set of representatives of the isomorphism classes of faithful modules in* $\mathcal{C}(A/B, \mathbb{K})$. *We set*:

$$\begin{aligned} \mathrm{M}(C) &:= \ell_2^{A/B}(W_1)\cdot W_1\uparrow^G \oplus \cdots \oplus \ell_2^{A/B}(W_m)\cdot W_m\uparrow^G \\ \mathrm{R}(C) &:= \ell_2^{A}(W_1)\cdot W_1\uparrow^G \oplus \cdots \oplus \ell_2^{A}(W_m)\cdot W_m\uparrow^G . \end{aligned}$$

Then we have:

$$\begin{aligned} eJ/eJ^2 &\cong \left(\bigoplus_{V\in\mathcal{C}(G,\mathbb{K})} \mathrm{cm}^G(V)\cdot V\right) \oplus \left(\bigoplus_{C\in\mathrm{n}_0(G)} \mathrm{M}(C)\right) \\ &\cong \left(\bigoplus_{V\in\mathcal{C}(G,\mathbb{K})\setminus\mathcal{C}_0(G,\mathbb{K})} \mathrm{cm}^G(V)\cdot V\right) \oplus \left(\bigoplus_{C\in\mathrm{n}_0(G)} \mathrm{R}(C)\right). \end{aligned}$$

Proof. By Corollary 2.3,

$$\begin{aligned} eJ/eJ^2 &\cong \bigoplus_{V\in\mathcal{C}(G,\mathbb{K})} \ell_2^G(V)\cdot V \\ &\cong \left(\bigoplus_{V\in\mathcal{C}(G,\mathbb{K})} \mathrm{cm}^G(V)\cdot V\right) \oplus \left(\bigoplus_{V\in\mathcal{C}(G,\mathbb{K})} \ell_2^{G/\mathrm{C}_G(V)}(V)\cdot V\right). \end{aligned}$$

Now,

$$\bigoplus_{V\in\mathcal{C}(G,\mathbb{K})} \ell_2^{G/\mathrm{C}_G(V)}(V)\cdot V \cong \bigoplus_{V\in\mathcal{C}_0(G,\mathbb{K})} \ell_2^{G/\mathrm{C}_G(V)}(V)\cdot V$$

(by the definition of $\mathcal{C}_0(G, \mathbb{K})$)

$$\cong \bigoplus_{C\in\mathrm{n}_0(C)} \left(\bigoplus_{\substack{V\in\mathcal{C}_0(G,\mathbb{K}) \\ \mathrm{C}_G(V)=C}} \ell_2^{G/C}(V)\cdot V \right)$$

(as $\mathcal{C}_0(G,\mathbb{K}) = \bigcup_{C\in \mathrm{n}_0(G)}\{V; V\in\mathcal{C}_0(G,\mathbb{K}),\ \mathrm{C}_G(V)=C\}$ by Corollary 3.2)

$$\cong \bigoplus_{C\in \mathrm{n}_0(G)} \mathrm{M}(C)$$

X(by Theorem 3.5).

On the other hand, if $V\in\mathcal{C}(G,\mathbb{K})\setminus\mathcal{C}_0(G,\mathbb{K})$, then $\ell_2^G(V)=\mathrm{cm}^G(V)$. Therefore

$$eJ/eJ^2 \cong \left(\bigoplus_{V\in\mathcal{C}(G,\mathbb{K})\setminus\mathcal{C}_0(G,\mathbb{K})} \mathrm{cm}^G(V)\cdot V\right) \oplus \left(\bigoplus_{V\in\mathcal{C}_0(G,\mathbb{K})} \ell_2^G(V)\cdot V\right)$$

and

$$\bigoplus_{V\in\mathcal{C}_0(G,\mathbb{K})} \ell_2^G(V)\cdot V \cong \bigoplus_{C\in\mathrm{n}_0(G)}\left(\bigoplus_{\substack{V\in\mathcal{C}_0(G,\mathbb{K})\\ \mathrm{C}_G(V)=C}} \ell_2^G(V)\cdot V\right) \cong \bigoplus_{C\in\mathrm{n}_0(G)} \mathrm{R}(C). \qquad \square$$

If $H\leq G$, then we put

$$\mathrm{h}_G(H) = e\,\mathrm{I}(H)\mathbb{K}G + eJ^2,$$

where $\mathrm{I}(H)=\{\sum_{h\in H} a_h h; \sum_{h\in H} a_h = 0,\ a_h\in\mathbb{K}\}$ is the augmentation ideal of $\mathbb{K}H$.

Observe that $\mathrm{h}_G(H)$ is a $\mathbb{K}G$-module and $eJ^2\subseteq \mathrm{h}_G(H)\subseteq eJ$ since $e\,\mathrm{I}(G)=eJ$.

The filtration of eJ/eJ^2 given by Okuyama and Tsushima [10] for $\mathbb{K}=\mathbb{F}_p$ and p-soluble G is a particular case of the following second description we give of eJ/eJ^2:

Theorem 3.7. *Let $1=G_0\leq G_1\leq\cdots\leq G_{n-1}\leq G_n=G$ be a chief series of G and consider the associated filtration of eJ/eJ^2:*

$$eJ^2=\mathrm{h}_G(G_0)\subseteq \mathrm{h}_G(G_1)\subseteq\cdots\subseteq \mathrm{h}_G(G_{n-1})\subseteq \mathrm{h}_G(G_n)=eJ.$$

Then we have:

$$\mathrm{h}_G(G_i)/\mathrm{h}_G(G_{i-1}) \cong \begin{cases} 0 \text{ if } G_i/G_{i-1} \text{ is a } p'\text{-chief factor or a frattini } p\text{-chief factor} \\ (G_i/G_{i-1})_{\mathbb{K}} \text{ if } G_i/G_{i-1} \text{ is a complemented } p\text{-chief factor} \\ \mathrm{M}(\mathrm{C}_G(G_i/G_{i-1})) \text{ otherwise.}\end{cases}$$

Proof. We proceed with the induction on n. If $n=0$, the result is trivial. Assume $n>0$, take $N=G_1$ and consider $\overline{G}=G/N$.

As eJ/eJ^2 is completely reducible, $eJ/eJ^2\cong eJ/\mathrm{h}_G(N)\oplus \mathrm{h}_G(N)/eJ^2$. Now

$$eJ/\mathrm{h}_G(N)=\mathrm{h}_G(G)/\mathrm{h}_G(N)\cong \mathrm{h}_{\overline{G}}(\overline{G})/\mathrm{h}_{\overline{G}}(\overline{N}) = \overline{e}\overline{J}/\overline{e}\overline{J}^2.$$

Therefore

$$(*) \qquad eJ/eJ^2 \cong \overline{eJ}/\overline{eJ}^2 \oplus \mathrm{h}_G(N)/eJ^2.$$

As $\mathrm{h}_{\overline{G}}(\overline{G}_i)/\mathrm{h}_{\overline{G}}(\overline{G}_{i-1}) \cong \mathrm{h}_G(G_i)/\mathrm{h}_G(G_{i-1})$, the result is true by the inductive hypothesis for the factors G_i/G_{i-1}, $i > 1$.

Assume that N is a p-group or a p'-group. Then $N \leq \mathrm{F}_p(G) \leq \mathrm{C}_G(V)$ for each $V \in \mathcal{C}(G, \mathbb{K})$, and hence $\mathcal{C}(G, \mathbb{K}) = \mathcal{C}(\overline{G}, \mathbb{K})$ and $\mathrm{n}_0(G) = \mathrm{n}_0(\overline{G})$.

If N is a frattini p-chief factor or a p'-factor, then $\mathrm{cm}^G(V) = \mathrm{cm}^{\overline{G}}(V)$ for each $V \in \mathrm{Irr}(G, \mathbb{K})$. Then, by Theorem 3.6, we have in this case that $eJ/eJ^2 \cong \overline{eJ}/\overline{eJ}^2$. From $(*)$ we conclude that $\mathrm{h}_G(N)/eJ^2 = 0$.

If N is a complemented p-chief factor, from Theorem 3.6 $eJ/eJ^2 \cong \overline{eJ}/\overline{eJ}^2 \oplus N_{\mathbb{K}}$, and by $(*)$ we have that $\mathrm{h}_G(N)/eJ^2 \cong N_{\mathbb{K}}$.

Assume that N is nonabelian and p is a divisor of $|N|$. Let $C = \mathrm{C}_G(N)$. Then $G/C \in \mathcal{P}_2$, as NC/C is the only minimal normal subgroup of G/C. We have that, if $i > 1$, then $N \leq G_{i-1} \leq \mathrm{C}_G(G_i/G_{i-1})$, and hence $C \neq \mathrm{C}_G(G_i/G_{i-1})$, as N is nonabelian. Therefore $\mathrm{n}_0(G) = \mathrm{n}_0(\overline{G}) \cup \{C\}$. On the other hand $\mathcal{C}(\overline{G}, \mathbb{K}) \subseteq \mathcal{C}(G, \mathbb{K})$, as the inflation map $H^1(\overline{G}, V) \to H^1(G, V)$ is injective, and $\mathrm{cm}^G(V) = \mathrm{cm}^{\overline{G}}(V)$ for each $V \in \mathrm{Irr}(G, \mathbb{K})$. Consequently $eJ/eJ^2 \cong \overline{eJ}/\overline{eJ}^2 \oplus \mathrm{M}(C)$ and so $\mathrm{h}_G(N)/eJ^2 \cong \mathrm{M}(C)$.

As $\mathrm{h}_G(N)/eJ^2 = \mathrm{h}_G(G_1)/\mathrm{h}_G(G_0)$, this completes the proof. □

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J. P. Lafuente Departamento de Matemática
e Informática
Universidad Pública de Navarra
31006 Pamplona, Spain
e-mail: lafuente@unavarra.es

C. Martínez-Pérez
Departamento de Matemáticas
Universidad de Zaragoza,
50009 Zaragoza, Spain
e-mail: conmar@posta.unizar.es