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# ON PLURICANONICAL MAPS FOR THREEFOLDS OF GENERAL TYPE, II

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## 1. Introduction

This paper is a continuation of [4, 9, 13]. To classify algebraic varieties is one of the goals in algebraic geometry. One way to study a given variety is to understand the behavior of its pluricanonical maps. The objects concerned here are complex projective 3-folds of general type over $\mathbb{C}$. Let $X$ be such an object and denote by $\phi_m$ the m-th pluricanonical map of $X$, which is the rational map associated with the $m$-canonical system $|mK_X|$. The very natural question is when $|mK_X|$ gives a birational map, a generically finite map, $\cdots$, etc. According to [2, 4, 9, 12, 13], one has the following

**Theorem 0.** *Let $X$ be a complex projective 3-fold of general type with the canonical index $r$. Then*
(i) *when $r = 1$, $\phi_m$ is a birational morphism onto its image for $m \geq 6$;*
(ii) *when $r \geq 2$, $\phi_m$ is a birational map onto its image for $m \geq 4r + 5$.*

In this paper, we give our results on the generic finiteness of $\phi_m$. By a delicate use of the Kawamata-Viehweg vanishing theorem, we reduce the problem to a parallel one for adjoint systems on some smooth surface. Reider's results as well as other theorems on surfaces make it possible for us to go on a detailed argument.

**Theorem 1.** *Let $X$ be a projective 3-fold of general type with the canonical index $r \geq 2$. Then $\phi_m$ is generically finite for $m \geq m(r)$, where $m(r)$ is a function as follows:*
$m(2) = 11$;
$m(r) = 2r + 8$, *for* $3 \leq r \leq 5$;
$m(r) = 2r + 6$, *for* $r \geq 6$.

**Theorem 2.** *Let $X$ be a projective minimal Gorenstein 3-fold of general type. Then*

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(1) $\phi_5$ *is birational except for some* 3*-folds with* $K_X^3 = 2$ *and* $p_g(X) \leq 2$; $\phi_5$ *is generically finite of degree* $\leq 8$.
(2) $\phi_4$ *is birational if* $K_X^3 > 2$ *and* $\dim \phi_1(X) = 3$; $\phi_4$ *is generically finite except for some* 3*-folds with* $K_X^3 = 2$, $p_g(X) \leq 1$ *and* $\chi(\mathcal{O}_X) = -1$.
(3) $\phi_3$ *is generically finite if* $p_g(X) \geq 39$.

For a nonsingular projective minimal 3-fold $X$ of general type, Benveniste ([2]) proved that $\dim \phi_m(X) \geq 2$ for $m \geq 4$, i.e. $|4K_X|$ can not be composed of a pencil. Recently, it has been proved ([5]) that $|3K_X|$ also can not be composed of a pencil. (Actually, the method is also effective for Gorenstein 3-folds of general type.) Thus it is interesting whether $|2K_X|$ can be composed of a pencil and like what a bicanonical pencil behaves. So in Section 4, we study the bicanonical pencil of a Gorenstein 3-fold of general type. According to the 3-dimensional MMP, we can suppose that $X$ is a minimal locally factorial Gorenstein 3-fold of general type. Take a birational modification $\pi : X' \longrightarrow X$ such that $X'$ is smooth, $|\pi^*(2K_X)|$ gives a morphism and $\pi^*(2K_X)$ has supports with only normal crossings. This is possible because of Hironaka's big theorem. Let $W := \overline{\phi_2(X)}$ and take the Stein factorization

$$\phi_2 \circ \pi : X' \xrightarrow{f} C \xrightarrow{s} W.$$

Then $f$ is a fibration onto the nonsingular curve $C$, we call $f$ a *derived fibration* of $\phi_2$. Denote by $F$ a general fibre of $f$. Then $F$ is a nonsingular surface of general type by virtue of the Bertini theorem. Also set $b := g(C)$, the geometric genus of $C$. From [7], we know that $0 \leq b \leq 1$. We shall prove the following

**Theorem 3.** *Let* $X$ *be a projective minimal Gorenstein* 3*-fold of general type and suppose that* $|2K_X|$ *is composed of a pencil. Let* $f$ *be the derived fibration of* $\phi_2$ *and* $F$ *be a general fibre of* $f$. *Then we have* $p_g(F) = 1$ *and* $K_{F_0}^2 \leq 3$, *where* $F_0$ *is the minimal model of* $F$.

As an application of our method, we shall present a corollary on surfaces of general type which somewhat simplifies Xiao's theorem for the bicanonical finiteness.

## 2. Proof of Theorem 1

Throughout our argument, the Kawamata-Viehweg vanishing theorem is always employed as a much more effective tool. We use it in the following form.

**K-V Vanishing Theorem** ([10] or [17]). *Let* $X$ *be a nonsingular complete variety,* $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$. *Assume the following two conditions*:
(1) $D$ *is nef and big*;
(2) *the fractional part of* $D$ *has the support with only normal crossings.*

*Then* $H^i(X, \mathcal{O}_X(\ulcorner D \urcorner + K_X)) = 0$ *for* $i > 0$, *where* $\ulcorner D \urcorner$ *is the round-up of* $D$, *i.e. the minimum integral divisor with* $\ulcorner D \urcorner - D \geq 0$.

**Lemma 2.1** (Corollary 2 of [16]). *Let* $S$ *be a nonsingular algebraic surface,* $L$ *be a nef divisor on* $S$, $L^2 \geq 10$ *and let* $\phi$ *be a map defined by* $|L + K_S|$. *If* $\phi$ *is not birational, then* $S$ *contains a base point free pencil* $E'$ *with* $L \cdot E' = 1$ *or* $L \cdot E' = 2$.

**Lemma 2.2.** *Let* $X$ *be a nonsingular variety of dimension* $n$, $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$ *be a* $\mathbb{Q}$*-divisor on* $X$. *Then we have the following*:
(i) *if* $S$ *is a smooth irreducible divisor on* $X$, *then* $\ulcorner D \urcorner|_S \geq \ulcorner D|_S \urcorner$;
(ii) *if* $\pi : X' \longrightarrow X$ *is a birational morphism, then* $\pi^*(\ulcorner D \urcorner) \geq \ulcorner \pi^*(D) \urcorner$.

Proof. We can write $D$ as $G + \sum_{i=1}^{t} a_i E_i$, where $G$ is a divisor, the $E_i$ are effective divisors for each $i$ and $0 < a_i < 1$, $\forall i$. So we only have to prove the lemma for effective $\mathbb{Q}$-divisors. That is easy to check. □

**Lemma 2.3** (Lemma 2.3 of [9]). *Let* $X$ *be a minimal threefold of general type with canonical index* $r$. *Then we have the plurigenus formula*

$$\begin{aligned} &h^0(X, \omega_X^{[mr+s]}) \\ &= \frac{1}{12}(mr + s)(mr + s - 1)(2mr + 2s - 1)(K_X^3) + am + c_s \end{aligned}$$

*for* $0 \leq s < r$, $mr + s \geq 2$, *where* $a$ *is a constant and* $c_s$ *is a constant only relating to* $s$.

DEFINITION 2.4. Let $X$ be a nonsingular projective variety of dimension $\geq 2$. Suppose $|M|$ is a base-point-free system on $X$, *a general irreducible element* $S$ *of* $|M|$ means the following:
(i) if $\dim \Phi_{|M|}(X) \geq 2$, then $S$ is just a general member of $|M|$;
(ii) if $\dim \Phi_{|M|}(X) = 1$, taking the Stein factorization of $\Phi_{|M|}$, then we obtain a fibration $f : X \longrightarrow C$ onto a curve $C$. We mean $S$ a general fibre of $f$.

**Proposition 2.5** (Lemma 3.2 of [9]). *Let* $X$ *be a minimal threefold of general type with canonical index* $r \geq 2$. *Then* $\dim \phi_{mr+s}(X) \geq 2$ *in one of the following cases*:
(i) $r = 2$ *and* $m \geq 3$;
(ii) $r = 3$ *and* $m \geq 2$;
(iii) $r = 4, 5$, $0 \leq s \leq 2$ *and* $m \geq 2$; $r = 4, 5$, $s \geq 3$ *and* $m \geq 1$;
(iv) $r \geq 6$, $0 \leq s \leq 1$ *and* $m \geq 2$; $r \geq 6$, $s \geq 2$ *and* $m \geq 1$.

Now we modify Proposition 2.5 by virtue of Hanamura's method in order to prove our Theorem 1. The proof is due to Hamamura ([9]).

**Proposition 2.6.** *Let $X$ be a minimal threefold of general type with canonical index $r \geq 2$. Then $h^0(\omega_X^{[mr+s]}) \geq 3$ in one of the following cases*:
(i) $r = 2$ *and* $m \geq 2$;
(ii) $r \geq 3$, $s = 0, 1$ *and* $m \geq 2$; $r \geq 3$, $s \geq 2$ *and* $m \geq 1$.

Proof. From Lemma 2.3, we can put

$$P(mr+s) = \frac{1}{12}(mr+s)(mr+s-1)(2mr+2s-1)(K_X^3) + am + c_s \tag{2.1}$$

where $a$ and $c_s$ are constants for $0 \leq s < r$. We consider the right handside of (2.1) as a polynomial in $m$ and denote it by $P_s(m)$. Let $Q_s(m)$ be the first term of $P_s(m)$. We have

$$P_s(m) = Q_s(m) + am + c_s.$$

We see that, for $m \geq 1$ or $m = 0$ and $s \geq 2$,

$$P_s(m) \geq 0. \tag{2.2}$$

By Kollár's result ([11]) that the $\omega_X^{[mr+s]}$ are Cohen-Macaulay, using the Grothendieck duality, one can see that, for $m \leq -1$,

$$P_s(m) \leq 0. \tag{2.3}$$

Now we want to estimate both $a$ and $c_s$. For any $r$ and $s$, by (2.2) and (2.3), we have

$$Q_s(1) + a + c_s \geq 0 \tag{2.4}$$

$$-Q_s(-1) + a - c_s \geq 0. \tag{2.5}$$

Which induces

$$\begin{aligned} a &\geq \frac{1}{2}\Big\{Q_s(-1) - Q_s(1)\Big\} \\ &= -\frac{1}{12}\Big\{2r^2 + (6s^2 - 6s + 1)\Big\}(rK_X^3). \end{aligned} \tag{2.6}$$

When $r \geq 3$ and $s \geq 2$, we have

$$Q_s(0) + c_s \geq 0. \tag{2.7}$$

By (2.5) and (2.7), we get

$$a \geq -Q_s(0) + Q_s(-1) \tag{2.8}$$

$$= \frac{1}{12}\left\{-2r^2 + (6s-3)r - (6s^2 - 6s + 1)\right\}(rK_X^3).$$

Explicitly, we have

$$a \geq \frac{1}{12}\left\{-\frac{1}{2}r^2 + \frac{1}{2}\right\}(rK_X^3) \text{ if } r \text{ is odd} \tag{2.9}$$

$$a \geq \frac{1}{12}\left\{-\frac{1}{2}r^2 - 1\right\}(rK_X^3) \text{ if } r \text{ is even.} \tag{2.10}$$

Now we can calculate the $P(mr+s)$ case by case.

CASE 1. $r \geq 3$ and $s \geq 2$.

When $r$ is odd, from (2.7) and (2.9), we have

$$\begin{aligned} P(mr+s) &\geq Q_s(m) - \frac{1}{12}m\left(\frac{1}{2}r^2 - \frac{1}{2}\right)(rK_X^3) - Q_s(0) \\ &= \frac{1}{12}\left\{(mr+s)(mr+s-1)(2mr+2s-1) + m\left(-\frac{1}{2}r^3 + \frac{1}{2}r\right)\right. \\ &\qquad \left. - s(s-1)(2s-1)\right\}(K_X^3). \end{aligned}$$

We get $P(mr+s) \geq 7$ for $m \geq 1$.

When $r$ is even, from (2.7) and (2.10), we have

$$\begin{aligned} P(mr+s) &\geq Q_s(m) - \frac{1}{12}m\left(\frac{1}{2}r^2 + 1\right)(rK_X^3) - Q_s(0) \\ &= \frac{1}{12}\left\{2r^2m^3 + (6s-3)rm^2 + \left(6s^2 - 6s - \frac{1}{2}r^2\right)m\right\}(rK_X^3). \end{aligned}$$

We get $P(mr+s) \geq 5$ for $m \geq 1$.

CASE 2. $s = 1$.

From (2.4) and (2.5), we have

$$P(mr+1) \geq \frac{1}{12}r(m^2-1)(2rm+3)(rK_X^3).$$

We get $P(mr+1) \geq 6$ for $m \geq 2$.

CASE 3. $s = 0$.

By (2.4) and (2.5), we have

$$P(mr) \geq \frac{1}{12}r(m^2-1)(2rm-3)(rK_X^3).$$

We get $P(mr) \geq 3$ for $m \geq 2$. Thus we complete the proof. □

In what follows we can get an improved version of Hanamura's theorem.

**Theorem 2.7.** *Let $X$ be a projective threefold of general type with the canonical index $r \geq 2$. Then $\phi_m$ is birational onto its image for $m \geq 4r + 3$.*

Proof. We can suppose that $X$ is a minimal 3-fold. For any $m_1 \geq r + 2$, take some blowing-ups $\pi : X' \longrightarrow X$ according to Hironaka such that $X'$ is nonsingular and that the movable part of $|m_1 K_{X'}|$ defines a morphism. Denote by $|M|$ the moving part of $|m_1 K_{X'}|$ and by $S$ a general irreducible element of $|M|$. Then $S$ is a nonsingular projective surface of general type by the Bertini theorem. On $X'$, we consider the system $|K_{X'} + 3\pi^*(rK_X) + S|$. Because $K_{X'} + 3\pi^*(rK_X)$ is effective by Proposition 2.6, so the system can distinguish general irreducible elements of $|M|$. On the other hand, the vanishing theorem gives

$$|K_{X'} + 3\pi^*(rK_X) + S||_S = |K_S + 3L|,$$

where $L := \pi^*(rK_X)|_S$ is a nef and big divisor on $S$ and $L^2 \geq 2$. Reider's result tells that the right system gives a birational map, so does $|K_{X'} + 3\pi^*(rK_X) + S|$. Thus $\phi_m$ is birational for $m \geq 4r + 3$. $\square$

Proof Theorem 1. We can suppose that $X$ is a minimal model. If $r = 2$, then $\phi_m$ is birational for $m \geq 11$ according to Theorem 2.7. From now on, we assume $r \geq 3$ and define

$$m_2 = \begin{cases} r + 3, & \text{for } 3 \leq r \leq 5 \\ r + 2, & \text{for } r \geq 6. \end{cases}$$

Take some blowing-ups $\pi : X' \longrightarrow X$ such that $X'$ is nonsingular, $|m_2 K_{X'}|$ defines a morphism and the fractional part of $\pi^*(K_X)$ has supports with only normal crossings. Denote by $|M_2|$ the moving part of $|m_2 K_{X'}|$ and by $S_2$ a general irreducible element of $|M_2|$. For any $t \in \mathbb{Z}_{>0}$, we consider the system

$$|K_{X'} + \ulcorner (t + m_2)\pi^*(K_X) \urcorner + S_2|,$$

which is a sub-system of $|(t+2m_2+1)K_{X'}|$. Because $K_{X'} + \ulcorner (t + m_2)\pi^*(K_X) \urcorner$ is effective by Proposition 2.6, so the system can distinguish general irreducible elements of $|M_2|$. On the other hand, the K-V vanishing theorem tells that

$$\begin{aligned} &|K_{X'} + \ulcorner (t + m_2)\pi^*(K_X) \urcorner + S_2||_{S_2} \\ = &|G + L|, \end{aligned}$$

where $G := \{K_{X'} + \ulcorner (t + m_2)\pi^*(K_X) \urcorner\}|_{S_2}$ is effective and $L := S_2|_{S_2}$. We can see that

$$G + L \geq K_{S_2} + \ulcorner t\pi^*(K_X) \urcorner|_{S_2} + L.$$

From Proposition 2.5, we have $h^0(S_2, L) \geq 2$. Modulo blowing-ups, actually we can suppose that $|L|$ is free from base points. Let $C$ be a general irreducible element of $|L|$. It is obvious that $|G + L|$ can distinguish gereral irreducible elements of $|L|$. On the other hand, the K-V vanishing theorem gives

$$|K_{S_2} + \ulcorner t\pi^*(K_X)|_{S_2}\urcorner + C||_C = |K_C + D|,$$

where $D := \ulcorner t\pi^*(K_X)|_{S_2}\urcorner|_C$ is a divisor of positive degree. Because $C$ is a curve of genus $\geq 2$, so $h^0(C, K_C + D) \geq 2$ and $|K_C + D|$ gives a finite map. Thus we have $\dim \Phi_{|G+L|}(C) = 1$. Therefore $\phi_m$ is generically finite for $m \geq 2m_2+2$, which completes the proof. □

## 3. On Gorenstein 3-folds of general type

For a minimal threefold $X$ of general type with canonical index 1, we can find certain birational modifications $f : X' \longrightarrow X$ according to [15] such that $c_2(X') \cdot \Delta = 0$, where $\Delta$ is the ramification divisor of $f$. Then we can get the same plurigenus formula as that for a nonsingular minimal threefold, i.e.

$$p(n) := h^0(X, \mathcal{O}_X(nK_X)) = (2n-1)\left[\frac{n(n-1)}{12}K_X^3 - \chi(\mathcal{O}_X)\right],$$

for $n \geq 2$. On the other hand, the Miyaoka-Yau inequality ([14]) shows that $\chi(\mathcal{O}_X) < 0$. From [4] or [12], we know that $\phi_m$ is birational for $m \geq 6$.

**Theorem 3.1.** *Let $X$ be a projective minimal Gorenstein 3-fold of general type. Then*
(1) *$\phi_5$ is birational if either $K_X^3 > 2$ (Ein-Lazarsfeld-Lee) or $p_g(X) > 2$.*
(2) *When $p_g(X) = 2$, then $\phi_5$ is birational except for some 3-folds with $q(X) = h^2(\mathcal{O}_X) = 0$, and $|K_X|$ composed with a rational pencil of surfaces of general type with $(K^2, p_g) = (1, 2)$. In this situation, $\phi_5$ is generically finite of degree 2.*
(3) *$\phi_5$ is birational if $\dim \phi_2(X) = 1$.*

Proof. This is the main theorem in [7]. Though the objects considered there are nonsingular minimal 3-folds, the method is also effective for all Gorenstein 3-folds of general type. □

DEFINITION 3.2. Let $X$ be a projective minimal Gorenstein 3-fold of general type. Suppose $\dim \phi_i(X) \geq 2$ and set $iK_X \sim_{\text{lin}} M_i + Z_i$, where $M_i$ is the moving part and $Z_i$ the fixed one for any integer $i$. We define $\delta_i(X) := K_X^2 \cdot M_i$.

**Proposition 3.3.** *Let $X$ be a projective minimal Gorenstein 3-fold of general type. Suppose $|2K_X|$ is not composed of a pencil and $K_X^3 > 2$. Then $\delta_2(X) \geq 3$.*

Proof. We have $\delta_2(X) \geq 2$ by Proposition 2.2 of [4]. Take a birational modification $f : X' \longrightarrow X$ such that $|2f^*(K_X)|$ defines a morphism. Set $2f^*(K_X) \sim_{\text{lin}} M + Z$, where $M$ is the moving part and $Z$ the fixed one. A general member $S \in |M|$ is an irreducible nonsingular projective surface of general type. Denote $L := f^*(K_X)|_S$. If $L^2 = f^*(K_X)^2 \cdot S = \delta_2(X) = 2$, then we have

$$4 = 2f^*(K_X)^2 \cdot S = f^*(K_X) \cdot S^2 + f^*(K_X) \cdot S \cdot Z.$$

Noting that $S$ is nef and $S \not\approx 0$, we have $f^*(K_X) \cdot S^2 \geq 1$. Therefore four cases occur as follows:
(i) $f^*(K_X) \cdot S^2 = 4$, $f^*(K_X) \cdot S \cdot Z = 0$;
(ii) $f^*(K_X) \cdot S^2 = 3$, $f^*(K_X) \cdot S \cdot Z = 1$;
(iii) $f^*(K_X) \cdot S^2 = 2$, $f^*(K_X) \cdot S \cdot Z = 2$;
(iv) $f^*(K_X) \cdot S^2 = 1$, $f^*(K_X) \cdot S \cdot Z = 3$.

We also have

$$\begin{aligned} 2K_X^3 &= 2f^*(K_X)^3 = f^*(K_X)^2 \cdot S + f^*(K_X)^2 \cdot Z \\ &= 2 + \frac{1}{2} f^*(K_X) \cdot Z(S+Z) \\ &= 2 + \frac{1}{2} f^*(K_X) \cdot S \cdot Z + \frac{1}{2} f^*(K_X) \cdot Z^2. \end{aligned} \tag{3.1}$$

CASE (i). Noting that $f^*(K_X)$ is nef and big, we see that $mf^*(K_X)$ is linearly equivalent to a nonsingular projective surface of general type according to Kawamata for sufficiently large integer $m$. Then $S|_{mf^*(K_X)}$ is nef and big and, by the Hodge Index Theorem, we have $f^*(K_X) \cdot Z^2 \leq 0$. Thus (3.1) is false and this case does not occur.

CASE (ii). We have $f^*(K_X) \cdot S(S - 3Z) = 0$, then $f^*(K_X)(S - 3Z)^2 \leq 0$, which derives $f^*(K_X) \cdot Z^2 \leq 1/3$, i.e. $f^*(K_X) \cdot Z^2 \leq 0$. (3.1) is also false.

CASE (iii). $f^*(K_X) \cdot S(S - Z) = 0$ induces $f^*(K_X) \cdot Z^2 \leq 2$, then (3.1) becomes $K_X^3 \leq 2$. Thus $K_X^3 = 2$. Actually, in this case, $f^*(K_X) \cdot (S - Z) \sim_{\text{num}} 0$ (as 1-cycle).

CASE (iv). $f^*(K_X) \cdot (3S - Z)^2 \leq 0$ induces $f^*(K_X) \cdot Z^2 \leq 9$. And (3.1) becomes $K_X^3 \leq 4$. If $K_X^3 = 4$, we see that $f^*(K_X) \cdot (3S - Z) \sim_{\text{num}} 0$ as 1-cycle. Now we set $f^*(M_2) = S + E$. Then $Z = f^*(Z_2) + E$. Obviously, we have $f_*(S) = M_2$ and $f_*(Z) = Z_2$. From $f^*(M_2) \cdot f^*(K_X) \cdot (3S - Z) = 0$, we get $3K_X \cdot M_2^2 = K_X \cdot M_2 \cdot Z_2$. Then $4 = 2K_X^2 \cdot M_2 = K_X \cdot M_2^2 + K_X \cdot M_2 \cdot Z_2 = 4K_X \cdot M_2^2$, i.e. $K_X \cdot M_2^2 = 1$. Which derives a contradiction, because $K_X \cdot M_2^2$ is even. Thus $K_X^3 = 2$. □

**Proposition 3.4.** *Let $X$ be a projective minimal Gorenstein 3-fold of general type. Suppose $K_X^3 > 2$ and $\dim \phi_1(X) \geq 2$. Then $\delta_1(X) \geq 3$.*

Proof. As in the proof of the previous proposition, we first take a modification $f : X' \longrightarrow X$. Set $f^*(K_X) \sim_{\text{lin}} M + Z$, where $M$ is the moving part. A general member $S \in |M|$ is a nonsingular projective surface of general type. Also denote $L :=$

$f^*(K_X)|_S$. Then $L^2 = \delta_1(X) \geq 2$ according to Proposition 2.1 of [7]. If $L^2 = 2$, then we have

$$2 = f^*(K_X)^2 \cdot S = f^*(K_X) \cdot S^2 + f^*(K_X) \cdot S \cdot Z.$$

We also have

$$\begin{aligned} K_X^3 &= f^*(K_X)^2 \cdot S + f^*(K_X)^2 \cdot Z \\ &= 2 + f^*(K_X) \cdot S \cdot Z + f^*(K_X) \cdot Z^2. \end{aligned} \tag{3.2}$$

Similarly, $f^*(K_X) \cdot S^2 \geq 1$. If $f^*(K_X) \cdot S^2 = 2$ and $f^*(K_X) \cdot S \cdot Z = 0$, then, by the Hodge Index Theorem, $f^*(K_X) \cdot Z^2 \leq 0$. Then (3.2) becomes $K_X^3 \leq 2$, which says $K_X^3 = 2$. If $f^*(K_X) \cdot S^2 = f^*(K_X) \cdot S \cdot Z = 1$, $f^*(K_X) \cdot S \cdot (S - Z) = 0$ induces $f^*(K_X) \cdot Z^2 \leq 1$. By (3.2), we get $K_X^3 \leq 4$. If $K_X^3 = 4$, then we can see $f^*(K_X) \cdot (S - Z) \sim_{\text{num}} 0$. By the same argument as in the case (iv) of the proof of Proposition 3.3, we have $f^*(M_1) \cdot f^*(K_X) \cdot (S - Z) = 0$, i.e. $K_X \cdot M_1^2 = K_X \cdot M_1 \cdot Z_1$. We have $2 = K_X^2 \cdot M_1 = K_X \cdot M_1^2 + K_X \cdot M_1 \cdot Z_1 = 2K_X \cdot M_1^2$. Therefore $K_X \cdot M_1^2 = 1$, which is impossible. Thus $K_X^3 = 2$. □

**Theorem 3.5.** *Let $X$ be a projective minimal Gorenstein 3-fold of general type. Then $\phi_5$ is generically finite of degree $\leq 8$. If $\deg(\phi_5) > 2$, then $K_X^3 = 2$, $\chi(\mathcal{O}_X) = -1$ and $p_g(X) = 0, 1$.*

Proof. According to Theorem 3.1, we only have to study the case when $|2K_X|$ is not composed of a pencil. Take a modification $f : X' \longrightarrow X$ according to Hironaka such that $|2f^*(K_X)|$ defines a morphism. Set $2f^*(K_X) \sim_{\text{lin}} M + Z$, where $M$ is the moving part and $Z$ the fixed one. A general member $S \in |M|$ is a nonsingular projective surface of general type by the Bertini Theorem. We have

$$|K_{X'} + 2f^*(K_X) + S| \subset |5K_{X'}|.$$

Because $K_{X'} + 2f^*(K_X)$ is effective, the left system can distinguish general members of $|M|$. Denote $L := f^*(K_X)|_S$, using the long exact sequence and the vanishing theorem, we have

$$|K_{X'} + 2f^*(K_X) + S||_S = |K_S + 2L|.$$

Obviously, $K_S + 2L = G + H$, where $G := (K_{X'} + 2f^*(K_X))|_S$ is effective and $H := S|_S$. Note that $h^0(S, \mathcal{O}_S(2L)) \geq h^0(S, H) \geq P(2) - 1 \geq 3$. We have two cases.

CASE 1. $|H|$ is composed of a pencil. Taking a birational modification to $S$ if necessary, we can suppose $|H|$ is free from base points. Denote $H \sim_{\text{lin}} \sum_{i=1}^{a} C_i + E$, where $E$ is the fixed part. In general position, $\sum_{i=1}^{a} C_i$ can be a disjoint union of nonsingular curves in a family. We have $a \geq 2$. Thus $L \sim_{\text{num}} (a/2)C + E_0$, where

$E_0 \geq (1/2)E$ is an effective $\mathbb{Q}$-divisor. If $p_g(S) = 0$, then $q(S) = 0$ and then we can see by the long exact sequence that $|K_S + H|$ can distinguish $C_i$'s and that $|K_S + \sum_{i=1}^{a} C_i||_{C_i} = |K_{C_i}|$, which means $|K_S + 2L|$ gives at worst a generically finite map of degree 2 and so does $\phi_5$. If $p_g(S) > 0$, it is obvious that $|K_S + 2L|$ can distinguish $C_i$'s. For a general curve $C$ which is algebraically equivalent to $C_i$, we consider the $\mathbb{Q}$-divisor $G := K_S + 2L - (1/2)\sum_{i=3}^{a} C_i - E_0$. We have $\ulcorner G \urcorner \leq K_S + 2L$. On the other hand, $G - C - K_S$ is nef and big, thus by the K-V vanishing we have $|\ulcorner G \urcorner||_C = |K_C + \ulcorner E_0 \urcorner|_C|$. Because $\ulcorner E_0 \urcorner|_C$ is effective, $\Phi_{|K_S+2L|}$ is at worst a generically finite map of degree 2 and so is $\phi_5$ of $X$.

CASE 2. $|H|$ is not composed of a pencil, so neither is $|2L|$. Similarly, we can suppose $|2L|$ is base point free. If $p_g(S) = 0$, we can use a parallel discussion to that of Case 1 to see that $\phi_5$ is at worst a generically finite map of degree 2. If $p_g(S) > 0$, then $\Phi_{|K_S+2L|}$ is obviously generically finite. We know that $L^2 \geq 2$ from Proposition 2.2 of [4]. If $\Phi_{|K_S+2L|}$ is not birational and $L^2 \geq 3$, then according to Lemma 2.1, there is a free pencil on $S$ with a general member $C$ such that $C^2 = 0$ and $L \cdot C = 1$. Since $\dim \Phi_{|2L|}(C) = 1$, then $h^0(2L|_C) \geq 2$ and then, by the Clifford theorem, we see that $C$ is a curve of genus 2 and $2L|_C \sim_{\text{lin}} K_C$. Finally we can see that $|2L||_C = |K_C|$. Therefore $\Phi_{|K_S+2L|}$ is a generically finite map of degree 2. Therefore $\phi_5$ is generically finite with $\deg(\phi_5) \leq 2$. If $L^2 = 2$, then $K_X^3 = 2$ by the proof of Proposition 3.3. On the surface $S$, set $2L \sim_{\text{lin}} C_1 + E_1$, where $C_1$ is the moving part. We easily get

$$8 = (2L)^2 \geq C_1^2 \geq d(h^0(2L) - 2) \geq d(P(2) - 3).$$

Therefore we have

$$d \leq \frac{8}{P(2) - 3} = \frac{8}{-3\chi(\mathcal{O}_X) - 2}.$$

If $d > 2$, then $\chi(\mathcal{O}_X) = -1$. □

For the 4-canonical map of $X$, it is obvious that $\phi_4$ is not birational if $X$ admits a pencil of surfaces of general type with $(K^2, p_g) = (1, 2)$. Therefore it is pessimistic for us to obtain an effective sufficient condition for the birationality of $\phi_4$. We have a partial result as follows.

**Theorem 3.6.** *Let $X$ be a projective minimal Gorenstein 3-fold of general type. Suppose $K_X^3 > 2$ and $\dim \phi_1(X) = 3$. Then $\phi_4$ is a birational map onto its image.*

*Proof.* Take a birational modification $f : X' \longrightarrow X$ such that the movable part of $|f^*(K_X)|$ is base point free. Set $f^*(K_X) \sim_{\text{lin}} S + Z$, where $S$ is the moving part and $Z$ the fixed one. A general member $S$ is a nonsingular projective surface of general type. We have $|K_{X'} + 2f^*(K_X) + S| \subset |4K_{X'}|$. Using the vanishing theorem, we have

$$|K_{X'} + 2f^*(K_X) + S||_S = |K_S + 2L|,$$

where $L := f^*(K_X)|_S$ is a nef and big divisor on $S$. By Proposition 3.4, we see that $L^2 \geq 3$ under the condition $K_X^3 > 2$. If $\Phi_{|K_S+2L|}$ is not birational, then, by Lemma 2.1, there is a free pencil with a general member $C$ such that $C^2 = 0$ and $L \cdot C = 1$. Because $\dim \Phi_{|L|}(S) = 2$, $h^0(C, \mathcal{O}_C(L|_C)) \geq 2$. Therefore, by the Clifford theorem, we see that $\deg(L|_C) \geq 2h^0(L|_C) - 2 \geq 2$. This is a contradiction. Therefore $\Phi_{|K_S+2L|}$ is birational and so is $\phi_4$. □

EXAMPLE 3.7. We give an example which shows that $\phi_4$ is not birational when $K_X^3 = 2$ and $\dim \phi_1(X) = 3$. On $\mathbb{P}^3(\mathbb{C})$, take a smooth hypersurface $S$ of degree 10, $S \sim_{\text{lin}} 10H$. Let $X$ be a double cover of $\mathbb{P}^3$ with branch locus along $S$. Then $X$ is a nonsingular canonical model, $K_X^3 = 2$ and $p_g(X) = 4$ and $\phi_1$ is a finite morphism onto $\mathbb{P}^3$ of degree 2. One can easily check that $\phi_4$ is also a finite morphism of degree 2.

**Theorem 3.8.** *Let $X$ be a projective minimal Gorenstein 3-fold of general type. Then $\phi_4$ is generically finite when $p_g(X) \geq 2$ or when $K_X^3 > 2$ or when $\chi(\mathcal{O}_X) \neq -1$.*

Proof. PART I: $p_g(X) \geq 2$.

First we make a modification $f : X' \longrightarrow X$ such that the movable part of $|f^*(K_X)|$ is free from base points and that $f^*(K_X)$ has support with only normal crossings. Set $f^*(K_X) \sim_{\text{lin}} M + Z$, where $M$ is the moving part and $Z$ the fixed one.

If $\dim \phi_1(X) = 2$, then a general member $S \in |M|$ is a nonsingular projective surface of general type. We have

$$|K_{X'} + 2f^*(K_X) + S| \subset |4K_{X'}|.$$

Using the vanishing theorem, we have $|K_{X'} + 2f^*(K_X) + S||_S = |K_S + 2L|$, where $L := f^*(K_X)|_S$ is nef and big effective divisor on $S$. We have $h^0(S, L) \geq 2$. Noting that $p_g(S) > 0$ in this case. And if $|L|$ is not composed of a pencil, then neither is $|K_S + 2L|$. If $|L|$ is composed of a pencil, taking a modification if possible, we can suppose that the movable part of $|L|$ is free from base points. Set $L \sim_{\text{lin}} \sum C_i + Z_0$, we can see $|K_S + L + \sum C_i||_{C_i} = |K_{C_i} + D|$, where $D := L|_{C_i}$ is effective. We easily see that $\Phi_{|K_S+2L|}$ is at worst generically finite of degree $\leq 2$ and so is $\phi_4$.

If $\dim \phi_1(X) = 1$, then $M \sim_{\text{num}} aF$, where $F$ is a nonsingular projective surface of general type. $M_1 \sim_{\text{num}} aF_0$, where $F_0 = f_*(F)$ is irreducible on $X$. If $K_X \cdot F_0^2 = 0$, then, by Lemma 2.3 of [7], we have $\mathcal{O}_F(f^*(K_X)|_F) \cong \mathcal{O}_F(\pi^*(K_0))$, where $\pi$ is the contraction map onto the minimal model and $K_0$ is the canonical divisor of the minimal model of $F$. Obviously, $|K_{X'} + 2f^*(K_X) + M|$ can distinguish general members of $|M|$. Moreover $|K_{X'} + 2f^*(K_X) + M||_F = |K_F + 2\pi^*(K_0)|$, the right system gives a generically finite map and so does $\phi_4$. If $K_X \cdot F_0^2 > 0$, then

$$L^2 = f^*(K_X)^2 \cdot F = K_X^2 \cdot F_0 \geq K_X \cdot F_0^2 \geq 2.$$

It is sufficient to show that $|K_F + 2L|$ gives a generically finite map. We have $K_F + 2L \geq 3L$. If $|3L|$ is not composed of a pencil, then neither is $|K_F + 2L|$. If $|3L|$ is composed of a pencil, we claim that $h^0(F, 3L) \geq 3$. In fact, we have $|K_{X'} + f^*(K_X) + F||_F = |K_F + L|$ and $h^0(F, K_F + L) \geq 3$. Considering the natural map $H^0(X', 3K_{X'}) \overset{\alpha}{\longrightarrow} H^0(F, 3K_F)$, because $K_{X'} + f^*(K_X) + F \leq 3K_{X'}$, we see that $\dim_{\mathbb{C}}(\mathrm{Im}(\alpha)) \geq h^0(K_F + L) \geq 3$. Similarly, considering another natural map $H^0(X', 3f^*(K_X)) \overset{\beta}{\longrightarrow} H^0(F, 3L)$, we have

$$h^0(3L) \geq \dim_{\mathbb{C}}(\mathrm{Im}(\beta)) = \dim_{\mathbb{C}}(\mathrm{Im}(\alpha)) \geq 3.$$

Now we can write $3L \sim_{\mathrm{lin}} \sum_{i=1}^{t} \overline{C_i} + E_0$, where $E_0$ is the fixed part, $t \geq 2$ and the $\overline{C_i}$ are irreducible curves. Denote by $C$ a generic $\overline{C_i}$. Then $2L \sim_{\mathrm{num}} (2/3)tC + (2/3)E_0$ and thus $2L - C - (1/t)E_0$ is a nef and big $\mathbb{Q}$-divisor. Setting $G := 2L - (1/t)E_0$, then we have $K_S + \ulcorner G \urcorner \leq K_S + 2L$. On the other hand, the K-V vanishing gives $|K_S + \ulcorner G \urcorner||_C = |K_C + D|$, where $D$ is a divisor of positive degree. Noting that $C$ is a curve of genus $\geq 2$, so we see that $|K_C + D|$ gives a generically finite map. This means $|K_S + 2L|$ gives a generically finite map.

PART II: $K_X^3 > 2$ or $\chi(\mathcal{O}_X) \neq -1$.

We study $\phi_4$ according to the behavior of $\phi_2$. Of course, first we make a modification $f : X' \longrightarrow X$ such that the movable part of $|2f^*(K_X)|$ is free from base points and that $2f^*(K_X)$ has supports with only normal crossings. Set $2f^*(K_X) \sim_{\mathrm{lin}} \overline{M_2} + \overline{Z_2}$, where $\overline{M_2}$ is the moving part and $\overline{Z_2}$ the fixed one.

If $\dim \phi_2(X) = 1$, then $\overline{M_2} \sim_{\mathrm{num}} a_2 F$, where $F$ is a nonsingular projective surface of general type. We have $\mathcal{O}_F(f^*(K_X)|_F) \cong \mathcal{O}_F(\pi^*(K_0))$ by Lemma 4.2 below in this paper. Because $K_{X'} + f^*(K_X)$ is effective, $|K_{X'} + f^*(K_X) + \overline{M_2}|$ can distinguish general $F$. On the other hand, we have $|K_{X'} + f^*(K_X) + \overline{M_2}||_F = |K_F + \pi^*(K_0)|$. From Theorem 3.1 of [7], we know that $F$ is not a surface with $p_g = q = 0$. Thus $|K_F + \pi^*(K_0)|$ defines a generically finite map according to [19] and so does $\phi_4$.

If $\dim \phi_2(X) \geq 2$, then a general member $S \in |\overline{M_2}|$ is a nonsingular projective surface of general type. We have $|K_{X'} + f^*(K_X) + S||_S = |K_S + L|$, where $L := f^*(K_X)|_S$. Noting that $K_S \geq L$, then we have $K_S + L \geq 2L$. Under our assumption, we have $P(2) \geq 5$. Thus $h^0(2L) \geq 4$. We may suppose that the movable part of $|2L|$ is free from base points. If $|2L|$ is not composed of a pencil, then neither is $|K_S + L|$. Otherwise we can set $2L \sim_{\mathrm{lin}} \sum_{i=1}^{b} C_i + E_1$, where $b \geq 3$ and $E_1$ is the fixed part. We denote by $C$ the general $C_i$. Because $L - C - (1/b)E_1$ is nef and big, therefore

$$\left| K_S + \ulcorner L - \frac{1}{b}E_1 \urcorner \right| \Big|_C = |K_C + D|,$$

where $D$ is a divisor of positive degree. The right system obviously defines a generically finite map. Thus $|K_S + L|$ gives a generically finite map and so does $\phi_4$. □

**Theorem 3.9.** *Let $X$ be a projective minimal Gorenstein 3-fold of general type. Then $\phi_3$ is generically finite when $p_g(X) \geq 39$.*

Proof. First we make a modification $f : X' \longrightarrow X$ such that the movable part of $|f^*(K_X)|$ is free from base points and that $f^*(K_X)$ has support with only normal crossings. Set $f^*(K_X) \sim_{\text{lin}} M + Z$, where $M$ is the moving part and $Z$ the fixed one.

If $\dim \phi_1(X) \geq 2$, then a general member $S \in |M|$ is a nonsingular projective surface of general type. We have $|K_{X'}+f^*(K_X)+S||_S = |K_S+L|$, where $L := f^*(K_X)|_S$. When $p_g(X) \geq 4$, $h^0(S, L) \geq 3$. Noting that $p_g(S) > 0$, if $|L|$ is not composed of a pencil, then nor is $|K_S + L|$. So we may suppose that $|L|$ is composed of a pencil and the movable part of this system is free from base points. Set $L \sim_{\text{lin}} \sum_{i=1}^{a} C_i + E_0$, where we have $a \geq 2$. $|K_S + L|$ can distinguish the $C_i$ generically. On the other hand, $L - C - (1/a)E_0$ is nef and big, we obtain by the Kawamata-Viehweg vanishing that

$$\left| K_S + \ulcorner L - \frac{1}{a}E_0 \urcorner \right| \Big|_C = \left| K_C + \ulcorner \frac{a-1}{a} L \urcorner \Big|_C \right| .$$

The right system defines a generically finite map and so does $\phi_3$.

If $\dim \phi_1(X) = 1$, then $M \sim_{\text{num}} aF$, where $F$ is a nonsingular projective surface of general type. Set $F_0 = f_*(F)$. If $K_X \cdot F_0^2 = 0$, then, by Lemma 2.3 of [7], we have $\mathcal{O}_F(f^*(K_X)|_F) \cong \mathcal{O}_F(\pi^*(K_0))$, where $\pi$ is the contraction onto the minimal model and $K_0$ is the canonical divisor of the minimal model of $F$. We see that $|K_{X'} + f^*(K_X) + M||_F = |K_F + \pi^*(K_0)|$. Because $p_g(F) > 0$, the right system defines a generically finite map and so does $\phi_3$. If $K_X \cdot F_0^2 > 0$, in order to prove the theorem, we have to show the generic finiteness of $\Phi_{|K_F+L|}$, where $L := f^*(K_X)|_F$ is effective. By Theorem 2 of [6], we see that $q(F) \geq 3$ when $p_g(X) \geq 39$. Then $\Phi_{|K_F|}$ is generically finite according to [18]. Therefore under the assumption of the theorem, we can obtain the generic finiteness of $\phi_3$. □

## 4. On bicanonical systems

We suppose that $X$ is a locally factorial Gorenstein minimal 3-fold of general type and that $|2K_X|$ be composed of a pencil. Keep the same notations as in section 1 and let $\pi : X' \longrightarrow X$ be the birational modification and $f : X' \longrightarrow C$ be the derived fibration.

**Lemma 4.1.** *Let $X$ be a projective minimal Gorenstein 3-fold of general type and suppose that $|2K_X|$ is composed of a pencil. Then $q(X) \leq 2$ and $p_g(X) \geq 1$.*

Proof. This is just a generalized version of Corollary 3.1 of [7]. Though the objects considered there are nonsingular minimal 3-folds, the method is also effective for minimal Gorenstein 3-folds. □

**Lemma 4.2.** *Let $X$ be a projective minimal Gorenstein 3-fold of general type, $|2K_X|$ be composed of a pencil, $f : X' \longrightarrow C$ be the derived fibration of $\phi_2$ and $F$ be a general fibre of $f$. Then*

$$\mathcal{O}_F(\pi^*(K_X)|_F) \cong \mathcal{O}_F(\pi_0^*(K_{F_0})),$$

*where $\pi_0 : F \longrightarrow F_0$ is the birational contraction onto the minimal model.*

Proof. This is just a generalized version of Corollary 9.1 of [13]. Though the objects considered there are nonsingular minimal 3-folds, the method is also effective for minimal Gorenstein 3-folds. □

**Lemma 4.3.** *Under the same assumption as in* Lemma 4.2, *we have $K_{F_0}^2 \le 3$ and $1 \le p_g(F) \le 3$.*

Proof. Let $\pi^*(2K_X) \sim_{\mathrm{lin}} g^*(H_2) + Z_2'$, where $g := \phi_2 \circ \pi$, $Z_2'$ is the fixed part and $H_2$ is a general hyperplane section of the closure $W$ of the image of $X$ in $\mathbb{P}^{p(2)-1}$. Obviously we have $g^*(H_2) \sim_{\mathrm{num}} a_2 F$, where $a_2 \ge p(2) - 1$. From Lemma 4.2, we have

$$K_{F_0}^2 = (\pi^*(K_X)|_F)^2 = \pi^*(K_X)^2 \cdot F.$$

Let $2K_X \sim_{\mathrm{lin}} M_2 + Z_2$, where $M_2$ is the moving part and $Z_2$ is the fixed part. We also have $M_2 = \pi_*(g^*(H_2))$. Denote $\overline{F} := \pi_*(F)$, then $M_2 \sim_{\mathrm{num}} a_2\overline{F}$. By the projection formula, we get

$$K_X^2 \cdot \overline{F} = \pi^*(K_X)^2 \cdot F = K_{F_0}^2.$$

Because $K_X$ is nef and big, we have $2K_X^3 \ge a_2 K_X^2 \cdot \overline{F}$. Thus

$$K_X^2 \cdot \overline{F} \le \frac{2}{a_2} K_X^3 \le \frac{4K_X^3}{K_X^3 - 6\chi(\mathcal{O}_X) - 2} \le \frac{4K_X^3}{K_X^3 + 4} < 4,$$

which means $K_{F_0}^2 \le 3$. By Lemma 4.1, the fact that $p_g(X) \ge 1$ induces $p_g(F) > 0$. By the Noether inequality $2p_g(F_0) - 4 \le K_{F_0}^2$, we see that $p_g(F) \le 3$. □

Proof Theorem 3. In order to prove Theorem 3, we shall derive a contradiction under the assumption that $p_g(F) \ge 2$. Obviously, $|2K_{X'}|$ can distinguish general fibres of the morphism $\phi_2 \circ \pi$. We consider the system $|K_{X'} + \pi^*(K_X)|$. Write $2\pi^*(K_X) \sim_{\mathrm{lin}} M_2' + Z_2'$, where $M_2'$ is the moving part and $Z_2'$ is the fixed one. Set $Z_2' = Z_v + Z_h$, where $Z_v$ is the vertical part and $Z_h$ is the horizontal part with respect to the fibration $f : X' \longrightarrow C$. Noting that $\pi^*(K_X)$ is effective by Lemma 4.1, $Z_h$ should be 2-divisible, i.e. $Z_h = 2Z_0$, where $Z_0$ is an effective divisor. Thus we see that $Z_0$ is just the horizontal part of $\pi^*(K_X)$. We know that $a_2 \geq p(2) - 1 \geq 3$ and

$$\pi^*(K_X) \sim_{\text{num}} \frac{a_2}{2}F + \frac{1}{2}Z_2'.$$

Therefore $\pi^*(K_X) - F - (1/a_2)Z_2'$ is a nef and big $\mathbb{Q}$-divisor. Setting $G := \pi^*(K_X) - (1/a_2)Z_2'$, then we have $K_{X'} + \ulcorner G \urcorner \leq K_{X'} + \pi^*(K_X)$. By the Kawamata-Viehweg vanishing theorem, we see that, for a general fibre $F$,

$$|K_{X'} + \ulcorner G \urcorner||_F = |K_F + \ulcorner G \urcorner|_F| \supset |K_F + \ulcorner G|_F \urcorner| = \left| K_F + \ulcorner \frac{a_2 - 2}{a_2} Z_0 \Big|_F \urcorner \right|,$$

where $\ulcorner((a_2 - 2)/a_2)Z_0|_F \urcorner$ is effective on the surface $F$. This means that $\dim \phi_2(F) \geq 1$ under the assumption $p_g(F) \geq 2$ and then $\dim \phi_2(X) \geq 2$, a contradiction. □

The rest of this section is devoted to present an application of our method to bicanonical maps of surfaces of general type.

**Theorem 4.4.** *Let $S$ be a minimal algebraic surface of general type with $p(2) \geq 4$. Then the bicanonical map of $S$ is generically finite.*

Proof. Suppose that $|2K_S|$ is composed of a pencil, we want to derive a contradiction. Taking a birational modification $\pi : S' \longrightarrow S$ such that $|2\pi^*(K_S)|$ defines a morphism and denoting $W := \overline{\phi_2(S)}$, we obtain the following through the Stein factorization:

$$\phi_2 \circ \pi : S' \xrightarrow{f} B \longrightarrow W,$$

where $B$ is a nonsingular curve. Denote by $C$ a general fibre of the derived fibration $f$. We can write

$$\pi^*(2K_S) \sim_{\text{lin}} \sum_{i=1}^{a} C_i + Z,$$

where $a \geq p(2) - 1 \geq 3$ and $Z$ is the fixed part. Considering the system $|K_{S'} + \pi^*(K_S)|$, we can see that the system can distinguish general fibres of $\phi_2$. Setting $G := \pi^*(K_S) - (1/a)Z$, we have $K_S + \ulcorner G \urcorner \leq K_S + \pi^*(K_S)$ and $G - C \sim_{\text{num}} (a - 2/a)\pi^*(K_S)$ is nef and big. Thus, by the K-V vanishing theorem, we have

$$|K_S + \ulcorner G \urcorner||_C = |K_C + D|,$$

where $D := \ulcorner G \urcorner|_C$ is a divisor of positive degree on the curve $C$. Because $g(C) \geq 2$, then $h^0(C, K_C + D) \geq 2$. This means that $|K_S + \pi^*(K_S)|$ gives a generically finite map, a contradiction. □

**Corollary 4.5.** *Let $S$ be a minimal algebraic surface of general type with $p_g \geq 2$. Then the bicanonical map of $S$ is generically finite.*

Proof. If $q = 0$, then $\chi(\mathcal{O}_S) \geq 3$ and $p(2) \geq 4$. If $q > 0$, then $K_S^2 \geq 2p_g \geq 4$ by [8] and then $p(2) \geq 5$. The proof is completed by Theorem 4.4. □

**Corollary 4.6.** *Let $S$ be a minimal algebraic surface of general type with $p(2) = 3$. Then $|2K_S|$ is not composed of an irrational pencil.*

Proof. This is obvious from the proof of Theorem 4.4. The critical point is that we also have $a \geq 3$ in this case. □

The remain cases are like the following:
(I) $K^2 = 1$, $p_g = 1$ and $q = 0$;
(II) $K^2 = 2$ and $p_g = q = 0$;
(III) $K^2 = 2$ and $p_g = q = 1$.

**Proposition 4.7.** *Let $S$ be a minimal algebraic surface of type* (I). *Then the bicanonical map is generically finite.*

Proof. Suppose that $|2K_S|$ is composed of a rational pencil. We write

$$2K_S \sim_{\text{lin}} C_1 + C_2 + Z,$$

where $Z$ is the fixed part. Denote by $C$ a general member which is algebrally equivalent to $C_i$. We have $1 = K_S^2 \geq K_S \cdot C$. On the other hand, $K_S \cdot C + C^2 \geq 2$, which gives $C^2 \geq 1$. Thus $K_S \cdot C = C^2 = 1$, i.e. $C$ is a nonsingular curve of genus two. By the index theorem, we see that $K_S \sim_{\text{num}} C$. But from [3], $\text{Pic}(S)$ is torsion free, then $K_S \sim_{\text{lin}} C$. This is impossible because $h^0(S, C) = 2$. □

**Lemma 4.8** (Lemma 8 of [19]). *Let $S$ be a surface with finite $\pi_1$. Then*

$$H^1(S, \mathcal{O}_S(\mathcal{E})) = 0$$

*for any invertible torsion sheaf $\mathcal{E}$ on $S$.*

**Lemma 4.9.** *Let $S$ be a minimal surface of type* (II) *or* (III). *Suppose that $|2K_S|$ is composed of a rational pencil. Then the moving part of $|2K_S|$ is a free pencil of genus two.*

Proof. We can write $2K_S \sim_{\text{lin}} C_1 + C_2 + Z$, where $Z$ is the fixed part. Denote by $C$ the general member which is algebraically equivalent to $C_i$. If $C^2 > 0$, then

$K_S^2 \geq K_S \cdot C \geq C^2$. On the other hand, the index theorem gives $K_S^2 \times C^2 \leq (K_S \cdot C)^2$. Thus $K_S^2 = K_S \cdot C = C^2 = 2$ and then $K_S \sim_{\text{num}} C$.

If $p_g = 1$, then $Z = 0$. Let $D \in |K_S|$ be the unique effective divisor, then $2D = F_1 + F_2$, where the $F_i$ are two fibres of $\phi_2$. If $F_1 \neq F_2$, then the $F_i$ are multiple fibres and then $D \sim_{\text{num}} 2F_0$, where $F_0$ is a divisor. Which implies $D^2 \geq 4$, a contradiction. If $F_1 = F_2$, then $D = F_1$ and thus $h^0(S, D) = 2$, also a contradiction.

If $p_g = 0$, because the $\pi_1$ of $S$ is a finite group (Corary 5.8 of [1]), then $h^1(S, K_S - C) = 0$ by Lemma 4.8. Whereas we have $h^1(S, K_S - C) = h^1(S, C) = 1$ by R-R, a contradiction. Therefore we have $C^2 = 0$ and then $g(C) = 2$. □

**Proposition 4.10.** *Let S be a minimal surface of type* (II) *or* (III). *Then* $|2K_S|$ *can not be composed of a rational pencil of genus two.*

Proof. We refer to the proof of Proposition 3 and Theorem 3 of [19]. □

Thus we finally arrive at the following theorem of Xiao (Theorem 1 of [19]).

**Theorem 4.11.** *Let S be a projective surface of general type. Then* $\phi_2$ *is generically finite if and only if* $h^0(S, 2K_S) > 2$.

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