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## LIMITING ABSORPTION PRINCIPLE FOR DIRAC OPERATOR WITH CONSTANT MAGNETIC FIELD AND LONG-RANGE POTENTIAL

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### 1. Introduction

The Dirac Hamiltonian with magnetic vector potential  $\mathbf{a} = (a_j(x))_{j=1,\dots,d}$  is expressed by the following form

$$(1.1) \quad H(\mathbf{a}) = \sum_{j=1}^d \gamma_j (P_j - a_j) + m\gamma_{d+1} + V,$$

where  $P_j = 1/i\partial_{x_j}$ ,  $V$  is a multiplication of an Hermitian matrix  $V(x)$ .  $m$  is the mass of electron. The matrices  $\{\gamma_j\}$  satisfy the following relations

$$(1.2) \quad \gamma_j \gamma_k + \gamma_k \gamma_j = 2\delta_{jk} \mathbf{1} \quad (j, k = 1, \dots, d+1).$$

Here  $\delta_{jk}$  is Kronecker's delta and  $\mathbf{1}$  is an identity matrix. We assume that the speed of the light  $c = 1$ . When  $V \equiv 0$ , the square of  $H(\mathbf{a})$  has the form

$$(1.3) \quad H(\mathbf{a})^2 = \sum_{j=1}^d (P_j - a_j)^2 + m^2 + \frac{1}{i} \sum_{1 \leq j < k \leq d} b_{jk}(x) \gamma_j \gamma_k,$$

where

$$(1.4) \quad b_{jk}(x) = \partial_{x_k} a_j(x) - \partial_{x_j} a_k(x).$$

It is called Pauli's Hamiltonian. The skew symmetric matrix  $(b_{jk}(x))$  is the magnetic field associated with  $\mathbf{a}$ . We say the magnetic field is asymptotically constant if it satisfies the following conditions as  $|x| \rightarrow \infty$  :

$$(1.5) \quad b_{jk}(x) \rightarrow \exists \Lambda_{jk} \quad (1 \leq j, k \leq d),$$

where  $(\Lambda_{jk})_{j,k}$  is a constant matrix.

The aim of this paper is to prove the limiting absorption principle for  $H(\mathbf{a})$  with a constant magnetic field  $(b_{jk}(x))$  and a long-range electric potential  $V(x)$  when  $d = 3$ .

Let us recall some known facts about the Dirac Hamiltonian with a constant magnetic field for  $d = 2, 3$ . As can be inferred from (1.3), the spectrum of  $H(\mathbf{a})$  is closely related with that of magnetic Schrödinger operator appearing in the right hand side of (1.3), which depends largely on the space dimension. Suppose  $d = 2$  at first. For simplicity we consider the case that the magnetic field  $b(x) = \partial_{x_2}a_1(x) - \partial_{x_1}a_2(x) = \lambda > 0$ . In this case, the Dirac Hamiltonian  $h(\lambda)$  is represented by

$$(1.6) \quad h(\lambda) = \sigma_1 \left( P_1 + \frac{\lambda}{2}x_2 \right) + \sigma_2 \left( P_2 - \frac{\lambda}{2}x_1 \right) + m\sigma_3,$$

$$\text{with } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They are called Pauli's spin matrices. Obviously  $\{\sigma_j\}$  satisfy the relation (1.2) and by an elementary calculus we have

$$(1.7) \quad h(\lambda)^2 = \left( P_1 + \frac{\lambda}{2}x_2 \right)^2 + \left( P_2 - \frac{\lambda}{2}x_1 \right)^2 + m^2 - \lambda\sigma_3.$$

The right hand side is a de-coupled 2 dimensional magnetic Schrödinger operator. So it suggests that the spectrum of  $h(\lambda)$  is discrete and

$$\sigma(h(\lambda)) \subset \left\{ \pm\sqrt{2\lambda n + m^2} \mid n = 0, 1, 2, \dots \right\}.$$

In fact we have

$$\sigma(h(\lambda)) = \left\{ \sqrt{2\lambda n + m^2}, \quad -\sqrt{2\lambda(n+1) + m^2} \mid n = 0, 1, 2, \dots \right\}$$

by using Foldy-Wouthuysen transform. (See 7.1.3 in [10].) Therefore the spectrum of  $h(\lambda)$  is of pure point with infinite multiplicities.

Next we consider the case of  $d = 3$ . We assume

$$\mathbf{a}_0(x) = \left( -\frac{\lambda x_2}{2}, \frac{\lambda x_1}{2}, 0 \right) \quad (\lambda > 0).$$

Then the associated magnetic field is constant along  $x_3$ -axis :

$$B(x) = (b_{32}(x), b_{13}(x), b_{21}(x)) = (0, 0, \lambda).$$

We denote the associated Dirac Hamiltonian as  $H_0(\lambda)$ . It is the following operator acting on  $\mathbb{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$  :

$$(1.8) \quad H_0(\lambda) = \alpha_1 \left( P_1 + \frac{\lambda x_2}{2} \right) + \alpha_2 \left( P_2 - \frac{\lambda x_1}{2} \right) + \alpha_3 P_3 + m\beta,$$

where  $\{\alpha_j\}$  and  $\beta$  are  $4 \times 4$  Hermitian matrices such that

$$(1.9) \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

We can easily see that these matrices also satisfy the relation (1.2). It is known that  $H_0(\lambda)$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$ . (See Theorem 4.3 in [10].) Now we consider the spectrum of  $H_0(\lambda)$ . At first we rewrite  $H_0(\lambda)$  as follows.

$$(1.10) \quad H_0(\lambda) = Q_0 + m\beta = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix},$$

with  $D_0 = \sigma \cdot (P - \mathbf{a}_0)$  and  $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ .

By using Foldy-Wouthuysen transform, explained in detail in the following section,  $H_0(\lambda)$  can be diagonalized by a unitary operator  $U_{FW}$ .

$$(1.11) \quad U_{FW}H_0(\lambda)U_{FW}^{-1} = \begin{pmatrix} \sqrt{D_0^2 + m^2} & 0 \\ 0 & -\sqrt{D_0^2 + m^2} \end{pmatrix}.$$

From the commutation relation (1.2) we have

$$(1.12) \quad D_0^2 = \left(P_1 + \frac{\lambda x_2}{2}\right)^2 + \left(P_2 - \frac{\lambda x_1}{2}\right)^2 + P_3^2 - \lambda\beta.$$

We can easily see that  $\sigma(D_0^2) = [0, \infty)$ . So we have

$$\sigma(H_0(\lambda)) = (-\infty, -m] \cup [m, \infty).$$

Therefore in the 3 dimensional case, the spectrum of  $H_0(\lambda)$  is absolutely continuous.

Let us consider the perturbation of  $H_0(\lambda)$  : We put

$$(1.13) \quad H(\lambda) = H_0(\lambda) + V.$$

Our aim is to show the so-called limiting absorption principle, namely the existence of the boundary value of the resolvent  $(z - H(\lambda))^{-1}$  on the real axis. The precise assumption on  $V$  will be given in Section 3. It is closely related to the absence of singular continuous spectrum of the operator and the asymptotic completeness of the wave operator associated with  $H_0(\lambda)$  and  $H(\lambda)$ . To prove the limiting absorption principle, we use Mourre's commutator method, which makes great progress for various Schrödinger operators. (For example, see [8].)

Suppose we are given a self-adjoint operator  $H$  on a separable Hilbert space. For a closed interval  $I \subset \mathbb{R}$  we denote the spectral measure, corresponding to the interval

as  $E_I(H)$ . Once we find some self-adjoint operator  $A$  satisfying the following inequality, we obtain many informations about  $H$  :

$$(1.14) \quad E_I(H)i[H, A]E_I(H) \geq \alpha E_I(H) + K,$$

where  $\alpha$  is a positive number and  $K$  is a compact operator. To be accurate, we can see the following properties hold.

- (i)  $\sigma_{pp}(H) \cap I$ , the eigenvalues of  $H$  in  $I$ , are discrete.
- (ii) The boundary value of the resolvent on  $I \setminus \sigma_{pp}(H)$  exists in some weighted Hilbert space. (**limiting absorption principle**)

For example, we consider a usual Schrödinger operator

$$H = -\Delta + V(x).$$

Here  $V(x)$  is a real valued function, which is decaying as  $|x| \rightarrow \infty$ . In this case we choose  $A = 1/(2i)\{x \cdot \nabla_x + \nabla_x \cdot x\}$  as the conjugate operator. Then the Mourre's inequality holds for any compact interval  $I \subset \mathbb{R} \setminus \{0\}$ . As a result one can show that  $\sigma_{pp}(H)$  is discrete with no accumulation point except  $\{0\}$ . We denote  $\langle \cdot \rangle = (|\cdot|^2 + 1)^{1/2}$ . Then we can also see the boundary values

$$\langle x \rangle^{-s}(H - \mu \mp i0)^{-1}\langle x \rangle^{-s}$$

exist in the operator norm for  $s > 1/2$  and  $\mu \in \mathbb{R} \setminus (\{0\} \cup \sigma_{pp}(H))$ .

As for the Schrödinger operator with constant magnetic field, Iwashita [6] shows the limiting absorption principle for long-range potential by using commutator method. In [6] the following self-adjoint operator is considered.

$$(1.15) \quad \tilde{H} = \left(P_1 + \frac{\lambda x_2}{2}\right)^2 + \left(P_2 - \frac{\lambda x_1}{2}\right)^2 + P_3^2 + V(x).$$

A self-adjoint operator  $A = 1/2(P_3 \cdot x_3 + x_3 \cdot P_3)$  is used as the conjugate operator. As a result, the existence of the boundary values

$$\langle x_3 \rangle^{-s}(\tilde{H} - \mu \mp i0)^{-1}\langle x_3 \rangle^{-s}$$

is proved for  $s > 1/2$  and  $\mu \in \mathbb{R} \setminus (\{\lambda(2n+1) | n = 0, 1, 2, \dots\} \cup \sigma_{pp}(\tilde{H}))$ .

Commutator method is also used for the free Dirac Hamiltonian and that with a scalar potential, which is decaying as  $|x| \rightarrow \infty$ . (See [2].) As for the electromagnetic Dirac Hamiltonian, asymptotic behavior of the solution of the Dirac equation is investigated in [3]. In their paper, the time dependent electromagnetic field  $a_j(x, t)$  is required to satisfy the following properties.

- (i) Each  $a_j(x, t)$  satisfies the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) a_j(x, t) = 0.$$

(ii) The initial data  $a_j(x, 0)$  and  $\partial_t a_j(x, 0)$  are compactly supported in  $\mathbb{R}^3$ . Hachem [5] showed the limiting absorption principle for the following electromagnetic Dirac Hamiltonian with a short-range potential  $V(x)$ .

$$H = \alpha_1 P_1 + \alpha_2(P_2 + \lambda x_1) + \alpha_3 P_3 + m\beta + V(x).$$

His idea is roughly as follows. First let us consider the case  $V \equiv 0$ . By passing to the Fourier transformation with respect to  $x_2, x_3$ -variables we denote  $F_0 H F_0^*$  as  $D(p)$  ( $p = (p_2, p_3)$ ). Then we have

$$(1.16) \quad D(p)^2 = A_+(p) \oplus A_-(p) \oplus A_+(p) \oplus A_-(p),$$

where  $A_{\pm}(p)$  are harmonic oscillators defined as follows.

$$(1.17) \quad A_{\pm}(p) = -\frac{d^2}{dx_1^2} + (\lambda x_1 + p_2)^2 \pm \lambda + p_3^2 + m^2$$

He then switched on the short-range potential  $V(x)$  by perturbative argument. Roughly speaking, his assumption means that the absolute value of each component of  $V$  is dominated from above by  $C\langle x' \rangle^{-1-\epsilon}\langle x \rangle^{-\epsilon}$  ( $x' = (x_2, x_3)$ ) for sufficiently large  $x$ . We remark that  $\epsilon > 0$  is used as a sufficiently small parameter throughout this paper. To be accurate,  $\langle x' \rangle^{1+\epsilon}V(x)$  is required to be a  $H_0(\lambda)$ -compact operator.

In this paper we treat directly the following operator

$$(1.18) \quad H(\lambda) = \alpha_1 \left( P_1 + \frac{\lambda}{2} x_2 \right) + \alpha_2 \left( P_2 - \frac{\lambda}{2} x_1 \right) + \alpha_3 P_3 + m\beta + V(x),$$

where  $V(x)$  is a matrix potential. Our strategy is to apply Mourre’s commutator method directly to this operator, which enables us to include the long-range diagonal components for  $V(x)$ . In this case it seems that an appropriate choice of the conjugate operator is

$$\frac{P_3}{\langle P_3 \rangle} \cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3 \rangle},$$

which is inspired by [11], when we proved the limiting absorption principle for time-periodic Schrödinger operator. In fact the method of the proof shares many ideas in common with [11]. Namely we rewrite  $H_0(\lambda)$  by a direct integral and the conjugate operator  $A$  acts on each space of fiber. Our main results are Theorem 3.4 and Corollary 3.7.

## 2. Conjugate operator

Let us recall

$$(2.1) \quad Q_0 = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix}, \quad D_0 = \sigma(P - \mathbf{a}_0),$$

with

$$(2.2) \quad \mathbf{a}_0(x) = \left( -\frac{\lambda x_2}{2}, \frac{\lambda x_1}{2}, 0 \right).$$

The Dirac Hamiltonian  $Q_0+m\beta$  can be diagonalized by sandwiching it between a unitary operator  $U$  and  $U^* = U^{-1}$ . In the beginning of this section we introduce a unitary operator which diagonalizes the self-adjoint operator  $H_0(\lambda)$ . Secondly we give a conjugate operator associated with the diagonalized Dirac Hamiltonian. Finally we show Mourre’s inequality for original Hamiltonians  $H_0(\lambda)$  and  $H(\lambda)$ .

Let  $Q_0$  be the self-adjoint operator as in (2.1) and  $|Q_0| = \sqrt{Q_0^2}$ ,  $|H_0(\lambda)| = \sqrt{H_0(\lambda)^2}$ . We define a unitary operator  $U_{FW}$ , which diagonalizes  $H_0(\lambda)$ , in the following way.

DEFINITION 2.1. (i) At first we define a signature function associated with  $Q_0$  by

$$(2.3) \quad \text{sgn } Q_0 = \begin{cases} \frac{Q_0}{|Q_0|}, & \text{on } (\ker Q_0)^\perp \\ 0, & \text{on } (\ker Q_0) \end{cases}$$

We note that  $\text{sgn } Q_0$  is isometry on  $(\ker Q_0)^\perp$ .

(ii) We can easily see that  $m/|H_0(\lambda)| \leq 1$ . So we denote the square root of  $1/2(1 \pm m/|H_0(\lambda)|)$  as  $a_\pm$ . i.e.

$$(2.4) \quad a_\pm = \frac{1}{\sqrt{2}} \sqrt{1 \pm \frac{m}{|H_0(\lambda)|}}.$$

(iii) Combining these operators we define the operator  $U_{FW}$  as

$$(2.5) \quad U_{FW} = a_+ + \beta(\text{sgn } Q_0)a_-.$$

**Lemma 2.2.** (i)  $U_{FW}$  is a unitary operator on  $L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ .

Further,

$$(2.6) \quad U_{FW}^* = U_{FW}^{-1} = a_+ - \beta(\text{sgn } Q_0)a_-.$$

(ii)  $H_0(\lambda)$  can be diagonalized by  $U_{FW}$  as follows.

$$(2.7) \quad U_{FW}H_0(\lambda)U_{FW}^{-1} = |H_0(\lambda)|\beta = \begin{pmatrix} \sqrt{D_0^2 + m^2} & 0 \\ 0 & -\sqrt{D_0^2 + m^2} \end{pmatrix}.$$

Proof. See 5.6.1 in [10]. □

We denote the diagonalized Dirac Hamiltonian as  $\hat{H}_0(\lambda)$ . i.e.

$$\hat{H}_0(\lambda) = U_{FW} H_0(\lambda) U_{FW}^{-1}.$$

We rewrite (1.12) as follows.

$$D_0^2 = \begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix}.$$

Here  $D_{\pm}$  are the operators acting on  $L^2(\mathbb{R}^3)$  such that

$$D_{\pm} = \left( P_1 + \frac{\lambda}{2} x_2 \right)^2 + \left( P_2 - \frac{\lambda}{2} x_1 \right)^2 + P_3^2 \pm \lambda.$$

It is well-known that  $(P_1 + \lambda/2x_2)^2 + (P_2 - \lambda/2x_1)^2$  has eigenvalues

$$\{ \lambda(2n + 1) \mid n = 0, 1, 2, \dots \}.$$

We denote the eigenprojection on each eigenspace as  $\Pi_n$ . With these projections,  $\sqrt{D_0^2 + m^2}$  can be rewritten as follows.

$$(2.8) \quad \sqrt{D_0^2 + m^2} = \sum_{n=0}^{\infty} \begin{pmatrix} d_n \otimes \Pi_n & 0 \\ 0 & d_{n+1} \otimes \Pi_n \end{pmatrix},$$

with  $d_n = d_n(P_3) = \sqrt{2\lambda n + P_3^2 + m^2}$ .

Combining (2.7) and (2.8), we have

$$f(\hat{H}_0(\lambda)) = \sum_{n=0}^{\infty} \begin{pmatrix} f(d_n) \otimes \Pi_n & & & \\ & f(d_{n+1}) \otimes \Pi_n & & \\ & & f(-d_n) \otimes \Pi_n & \\ & & & f(-d_{n+1}) \otimes \Pi_n \end{pmatrix},$$

for any Borel function  $f$ .

Now we define the conjugate operator. At first we define

$$(2.9) \quad \hat{A} = \frac{1}{2} \left\{ \frac{P_3}{\langle P_3 \rangle} \cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3 \rangle} \right\}.$$

We note that  $\hat{A}$  is essentially self-adjoint operator on  $D(|x_3|)$ . (It is obtained by use of Nelson’s commutator theorem [9].) The conjugate operator for the Dirac Hamiltonian

associated with constant magnetic field is defined by sandwiching  $\hat{A}\beta$  between  $U_{FW}^{-1}$  and  $U_{FW}$  :

$$(2.10) \quad A = U_{FW}^{-1}(\hat{A}\beta)U_{FW}.$$

Letting  $F$  be a Fourier transformation with respect to  $x_3$ -variable. We define the self-adjoint operator  $A_F$  by  $F\hat{A}F^{-1}$ . Then we have

$$(2.11) \quad (e^{itA_F}\phi)(x_1, x_2, p_3) = \left| \frac{\partial \Gamma_t}{\partial p_3}(p_3) \right|^{1/2} \phi(x_1, x_2, \Gamma_t(p_3)),$$

for  $\phi \in L^2(\mathbb{R}_x^2 \times \mathbb{R}_p)$ . Here  $\Gamma_t$  is a solution of the following equation.

$$(2.12) \quad \begin{cases} \frac{d}{dt}\Gamma_t(p_3) = \langle \Gamma_t(p_3) \rangle^{-1} \Gamma_t(p_3) \\ \Gamma_0(p_3) = p_3 \end{cases}$$

For the proof, see Appendix 1 in [8]. Therefore the unitary group  $e^{it\hat{A}\beta}$  is rewritten

$$(2.13) \quad (Fe^{it\hat{A}\beta}F^{-1}\phi)(x_1, x_2, p_3) = \begin{pmatrix} \left| \frac{\partial \Gamma_t}{\partial p_3}(p_3) \right|^{1/2} \phi_1(x_1, x_2, \Gamma_t(p_3)) \\ \left| \frac{\partial \Gamma_t}{\partial p_3}(p_3) \right|^{1/2} \phi_2(x_1, x_2, \Gamma_t(p_3)) \\ \left| \frac{\partial \Gamma_{-t}}{\partial p_3}(p_3) \right|^{1/2} \phi_3(x_1, x_2, \Gamma_{-t}(p_3)) \\ \left| \frac{\partial \Gamma_{-t}}{\partial p_3}(p_3) \right|^{1/2} \phi_4(x_1, x_2, \Gamma_{-t}(p_3)) \end{pmatrix}.$$

Before we compute the commutator  $i[H(\lambda), A]$ , we have to care that the following matters hold.

- Lemma 2.3.** (i)  $A$  is a self-adjoint operator on  $\mathbb{H}$ .
- (ii)  $e^{-itA}$  leaves  $D(H_0(\lambda))$  invariant, i.e.

$$(2.14) \quad \sup_{|t| \leq 1} \|H_0(\lambda)e^{itA}(H_0(\lambda) + i)^{-1}\phi\|_{\mathbb{H}} < \infty \quad \text{for } \phi \in \mathbb{H},$$

where  $\|\cdot\|_{\mathbb{H}}$  denotes the operator norm on  $\mathbb{H}$ .

Proof. The self-adjointness of  $A$  is easily obtained from that of  $\hat{A}\beta$ . To see the invariance of  $D(H_0(\lambda))$ , it is sufficient to show the following

$$(2.15) \quad \sup_{|t| \leq 1} \|\hat{H}_0(\lambda)e^{it\hat{A}\beta}(\hat{H}_0(\lambda) + i)^{-1}\phi\|_{\mathbb{H}} < \infty \quad \text{for } \phi \in \mathbb{H}.$$



From the arguments (2.11) and (2.13), we have

$$F \hat{H}_0(\lambda) e^{it\hat{A}\beta} (\hat{H}_0(\lambda) + i)^{-1} F^{-1} \phi = \sum_{n=0}^{\infty} \begin{pmatrix} D_{n,t}^+ \left| \frac{\partial \Gamma_t}{\partial p_3}(p_3) \right|^{1/2} \phi_1(x_1, x_2, \Gamma_t(p_3)) \\ D_{n+1,t}^+ \left| \frac{\partial \Gamma_t}{\partial p_3}(p_3) \right|^{1/2} \phi_2(x_1, x_2, \Gamma_t(p_3)) \\ D_{n,-t}^- \left| \frac{\partial \Gamma_{-t}}{\partial p_3}(p_3) \right|^{1/2} \phi_3(x_1, x_2, \Gamma_{-t}(p_3)) \\ D_{n+1,-t}^- \left| \frac{\partial \Gamma_{-t}}{\partial p_3}(p_3) \right|^{1/2} \phi_4(x_1, x_2, \Gamma_{-t}(p_3)) \end{pmatrix}$$

where  $D_{n,\alpha}^\pm = d_n(p_3)(d_n(\Gamma_\alpha(p_3)) \pm i)^{-1} \otimes \Pi_n$ . By integrating (2.12), we have

$$(2.16) \quad |\Gamma_\alpha(p_3) - p_3| \leq 1 \quad (|\alpha| \leq 1).$$

So (2.14) is obtained from the fact that  $D_{n,\alpha}^\pm$  is bounded uniformly for  $n \in \mathbb{N}$  and  $|\alpha| \leq 1$ . □

Before we show Mourre’s inequality, we introduce the usual functional calculus, started by Helffer and Sjöstrand.

Suppose that  $f \in C^\infty(\mathbb{R})$  satisfies the following condition for some  $m_0 \in \mathbb{R}$ .

$$(2.17) \quad |f^{(k)}(t)| \leq C_k(1 + |t|)^{m_0 - k}, \quad \forall k \in \mathbb{N} \cup \{0\}.$$

Then we can construct an almost analytic extension  $\tilde{f}(z)$  of  $f(t)$  having the following properties

$$\begin{aligned} \tilde{f}(t) &= f(t), \quad t \in \mathbb{R}, \\ \text{supp } \tilde{f} &\subset \{z; |\text{Im } z| \leq 1 + |\text{Re } z|\}, \end{aligned}$$

$$(2.18) \quad |\partial_{\bar{z}} \tilde{f}(z)| \leq C_N |\text{Im } z|^N \langle z \rangle^{m_0 - 1 - N}, \quad \forall N \in \mathbb{N}.$$

Then for all  $f$ , satisfying (2.17) for  $m_0 < 0$  and a self-adjoint operator  $H$ , we have

$$(2.19) \quad f(H) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z) (z - H)^{-1} dz \wedge d\bar{z}.$$

With this form, we can compute the commutator of an operator  $P$  and  $g(A)$  in the following way.

For operators  $P$  and  $Q$ , we define  $ad_Q^0(P) = P$  and inductively  $ad_Q^m(P) = [ad_Q^{m-1}(P), Q]$  for  $m \in \mathbb{N}$ .

**Lemma 2.4.** *Let  $A$  and  $P$  be self-adjoint operators on  $\mathbb{H}$ . Suppose that  $ad_A^m(P)(A + i)^{-m}$  extends to a bounded operator for  $1 \leq m \leq n$ . Then for any  $g \in C^\infty(\mathbb{R})$  satisfying (2.17) with  $m_0 < 0$ , we have*

$$(2.20) \quad P g(A) = \sum_{m=0}^{n-1} \frac{g^{(m)}(A)}{m!} ad_A^m(P) + \frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{g}(z) R_{n,A,P}^r(z) dz \wedge d\bar{z}$$

where  $R_{n,A,P}^r(z) = (z - A)^{-n} ad_A^n(P)(z - A)^{-1}$ , and

$$(2.21) \quad g(A)P = \sum_{m=0}^{n-1} ad_A^m(P) \frac{(-1)^m}{m!} g^{(m)}(A) + \frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{g}(z) R_{n,A,P}^l(z) dz \wedge d\bar{z}$$

where  $R_{n,A,P}^l(z) = (z - A)^{-1} ad_A^n(P)(z - A)^{-n}$  and  $\tilde{g}(z)$  denotes an almost analytic extension of  $g(t)$ .

For the proof of above results, see [4].

### 3. Limiting absorption principle for long-range potentials

Now we show the Mourre’s inequality for the Dirac Hamiltonian by choosing  $A$  defined in the previous section as the conjugate operator.

**Lemma 3.1.** *Let  $\mathbb{R}_{\mathbb{N}}$  be the following discrete subset of  $\mathbb{R}$*

$$\mathbb{R}_{\mathbb{N}} = \{ \pm \sqrt{2\lambda n + m^2} \mid n = 0, 1, 2, \dots \} \subset \mathbb{R}.$$

We take a compact interval  $I \subset \mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$  arbitrarily. Then there exists  $\alpha > 0$  such that the following inequality holds for any real valued  $f \in C_0^\infty(I)$

$$(3.1) \quad f(H_0(\lambda))i[H_0(\lambda), A]f(H_0(\lambda)) \geq \alpha f(H_0(\lambda))^2.$$

Proof. By the relations (2.7) and (2.10), it is sufficient to show the inequality

$$(3.2) \quad f(\hat{H}_0(\lambda))i[\hat{H}_0(\lambda), \hat{A}\beta]f(\hat{H}_0(\lambda)) \geq \alpha f(\hat{H}_0(\lambda))^2.$$

We rewrite the commutator as follow.

$$(3.3) \quad i[\hat{H}_0(\lambda), \hat{A}\beta] = \left( i \left[ \sqrt{D_0^2 + m^2}, \hat{A} \right] \quad i \left[ \sqrt{D_0^2 + m^2}, \hat{A} \right] \right).$$

We proceed the calculus more precisely to see that

$$(3.4) \quad i \left[ \sqrt{D_0^2 + m^2}, \hat{A} \right] = \sum_{n=0}^{\infty} \left( i [d_n, \hat{A}] \otimes \Pi_n \quad i [d_{n+1}, \hat{A}] \otimes \Pi_n \right)$$

by (2.8). From (3.3) and (3.4) the left hand side of (3.2) is rewritten as

$$(3.5) \quad f(\hat{H}_0(\lambda))i[\hat{H}_0(\lambda), \hat{A}\beta]f(\hat{H}_0(\lambda)) = \begin{pmatrix} I_1 & & & \\ & I_2 & & \\ & & I_3 & \\ & & & I_4 \end{pmatrix}$$

where

$$\begin{aligned} I_1 &= \sum_{n=0}^{\infty} f(d_n)i[d_n, \hat{A}]f(d_n) \otimes \Pi_n, \\ I_2 &= \sum_{n=0}^{\infty} f(d_{n+1})i[d_{n+1}, \hat{A}]f(d_{n+1}) \otimes \Pi_n, \\ I_3 &= \sum_{n=0}^{\infty} f(-d_n)i[d_n, \hat{A}]f(-d_n) \otimes \Pi_n, \\ I_4 &= \sum_{n=0}^{\infty} f(-d_{n+1})i[d_{n+1}, \hat{A}]f(-d_{n+1}) \otimes \Pi_n. \end{aligned}$$

We note that all the sum in  $I_1, \dots, I_4$  are finite since  $f$  is a compactly supported function. By an elementary calculus, we have

$$(3.6) \quad i[d_l, \hat{A}] = \frac{P_3^2}{\sqrt{2\lambda l + P_3^2 + m^2} \langle P_3 \rangle} \quad (l \in \mathbb{N} \cup \{0\}).$$

Since  $\text{supp } f \subset I \subset \mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$ ,  $P_3$  is away from zero when  $P_3 \in \text{supp } f(d_l(P_3))$  or  $P_3 \in \text{supp } f(-d_l(P_3))$ . So there exist  $C_l > 0$  such that

$$\begin{aligned} f(d_l)i[d_l, \hat{A}]f(d_l) \otimes \Pi_l &\geq C_l f(d_l)^2 \otimes \Pi_l, \\ f(-d_l)i[d_l, \hat{A}]f(-d_l) \otimes \Pi_l &\geq C_l f(-d_l)^2 \otimes \Pi_l. \end{aligned}$$

Since only a finite number of  $l = l_j \quad (j = 1, \dots, N)$  is concerned, we have (3.2) with  $\alpha = \inf_{j=1, \dots, N} C_{l_j}$ . □

Now we give the assumption for the potential, which is necessary to prove Mourre's inequality associated to  $H(\lambda)$ . After that we give an example of  $V$  satisfying this assumption. The potential  $V$  consists of a sum of long-range part and short-range

part. In our case short-range potential means  $V(x) = O(\langle x \rangle^{-\epsilon} \langle x_3 \rangle^{-1-\epsilon})$  as  $|x| \rightarrow \infty$ . And long-range part is a multiplication of a real valued function  $\varphi(x)$  such that  $\varphi(x) = O(\langle x \rangle^{-\epsilon})$  as  $|x| \rightarrow \infty$ . More precisely we assume that  $V$  satisfies the following.

ASSUMPTION 3.2.  $V = V(x)$  is a multiplicative operator of a  $4 \times 4$  Hermitian matrix satisfying the following properties.

- (i)  $V$  is a  $H_0(\lambda)$ -compact operator.
- (ii) The form  $[V, A]$  can be extended to a  $H_0(\lambda)$ -compact operator.

For example a  $4 \times 4$  matrix  $V(x)$  satisfying the following inequality is  $H_0(\lambda)$ -compact.

$$(3.7) \quad |V(x)| \leq C \langle x \rangle^{-\epsilon} \quad (x \in \mathbb{R}^3).$$

It is owing to the fact that  $V(x)(-\Delta_x + 1)^{-1}$  is compact. (It is due to Theorem 2.6 in [1].) Under this assumption we show Mourre’s inequality for  $H(\lambda)$ .

**Lemma 3.3.** *Suppose  $V$  satisfies Assumption 3.2.*

- (i) *We take  $\mu \in \mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$  and  $\delta > 0$  so that the closed interval  $I \equiv [\mu - \delta, \mu + \delta] \subset \mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$ . There exist  $\alpha > 0$  and a compact operator  $K$  such that the following inequality holds for all  $f \in C_0^\infty(I)$ .*

$$(3.8) \quad f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \geq \alpha f(H(\lambda))^2 + K.$$

- (ii) *There is no accumulation point of  $\sigma_{pp}(H(\lambda))$  in  $\mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$ . For  $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$ , there exist  $\delta_0 > 0$  and  $\alpha_0 > 0$  such that the following inequality holds for all  $f \in C_0^\infty([\mu - \delta_0, \mu + \delta_0])$ .*

$$(3.9) \quad f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \geq \alpha_0 f(H(\lambda))^2.$$

Proof. From (2.19) we have

$$f(H(\lambda)) - f(H_0(\lambda)) = \frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} f(z)(z - H(\lambda))^{-1} V(z - H_0(\lambda))^{-1} dz \wedge d\bar{z}$$

for  $f \in C_0^\infty(\mathbb{R})$ . We can easily see that  $f(H(\lambda)) - f(H_0(\lambda))$  is a compact operator since  $V(H_0(\lambda) + i)^{-1}$  is compact. Combing this fact and (3.1), we have (3.8) by replacing  $f(H_0(\lambda))$  in (3.1) by  $f(H(\lambda))$ . As for the non-existence of the accumulation point of  $\sigma_{pp}(H(\lambda))$ , see Theorem 2.2 in [7]. (3.9) follows from the arguement in [8] □

With this inequality we have the limiting absorption principle for the Dirac Hamiltonian.

**Theorem 3.4.** *Let  $s > 1/2$ . Suppose  $V$  satisfies Assumption 3.2. Then for  $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$ , the following limits*

$$(3.10) \quad R^\pm(\mu) = \lim_{\epsilon \downarrow 0} \langle x_3 \rangle^{-s} (H(\lambda) - \mu \mp i\epsilon)^{-1} \langle x_3 \rangle^{-s}$$

*exist and  $R^\pm(\mu)$  are continuous with respect to  $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$ .*

*Sketch of proof*

From (3.9) and Theorem 2.2 in [7], we can see that the boundary value  $\langle A \rangle^{-s} (H(\lambda) - \mu \mp i0)^{-1} \langle A \rangle^{-s}$  exist for  $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$ . To see the existence of (3.10), it is sufficient to show the boundedness of  $\langle A \rangle^s \langle x_3 \rangle^{-s}$ . Since  $\langle \hat{A} \rangle^s \langle x_3 \rangle^{-s}$  is bounded, it is sufficient to show  $\langle x_3 \rangle^s U_{FW} \langle x_3 \rangle^{-s}$  is bounded. We prove it in the following Lemma. Before that we introduce smooth functions. Let  $\chi(t) \in C^\infty(\mathbb{R})$  such that

$$(3.11) \quad \chi(t) = \begin{cases} \frac{1}{\sqrt{2}} & \left( t > -\frac{m^2}{3} \right) \\ 0 & \left( t < -\frac{2m^2}{3} \right). \end{cases}$$

With this function we define  $F_\pm(t)$  and  $F_{\chi,\pm}$  as follows.

$$\begin{aligned} F_+(t) &= \chi(t) \sqrt{1 + \frac{m}{\sqrt{t+m^2}}} \\ F_-(t) &= \chi(t) \left( \sqrt{1 + \frac{m}{\sqrt{t+m^2}}} \right)^{-1} \frac{1}{\sqrt{t+m^2}} \\ F_{\chi,+}(t) &= F_+(t) - \chi(t) \\ F_{\chi,-}(t) &= \sqrt{t+m^2} F_-(t) - \chi(t) \end{aligned}$$

Then we can easily verify that

$$\begin{aligned} a_+ &= F_+(Q_0^2), \\ a_- \operatorname{sgn} Q_0 &= F_-(Q_0^2) Q_0 = Q_0 F_-(Q_0^2). \end{aligned}$$

Obviously  $[Q_0, F_-(Q_0^2)] = 0$ . By the construction of these functions, we can also see that  $F_{\chi,\pm}(t)$  satisfy (2.17) with  $m_0 < 0$ . So we apply the functional calculus in Section 2 to  $F_{\chi,\pm}(t)$  and see the following properties hold.

**Lemma 3.5.** *Suppose  $0 \leq s \leq 2$  and  $z \in \mathbb{C} \setminus \mathbb{R}$ . Then*

(i) *For  $0 < s \leq 1$ , there exists  $C_s > 0$  such that*

$$(3.12) \quad \|\langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s}\|_{\mathbb{H}} \leq C_s (|\operatorname{Im} z|^{-1} + |\operatorname{Im} z|^{-2} \langle z \rangle).$$

(ii) For  $1 < s \leq 2$ , there exists  $C'_s > 0$  such that

$$(3.13) \quad \|\langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s}\|_{\mathbb{H}} \leq C'_s (|\operatorname{Im} z|^{-1} + |\operatorname{Im} z|^{-2} \langle z \rangle + |\operatorname{Im} z|^{-3} \langle z \rangle^2).$$

(iii)  $\langle x \rangle^s F_+(Q_0^2) \langle x \rangle^{-s}$  and  $\langle x \rangle^s F_-(Q_0^2) Q_0 \langle x \rangle^{-s}$  are bounded operators.

Proof. For the proof of (i) and (ii), we use the resolvent equation. Suppose  $0 < s \leq 1$ . Then

$$(3.14) \quad \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s} = (z - Q_0^2)^{-1} + (z - Q_0^2)^{-1} (Q_0^2 + 1)$$

$$(3.15) \quad \times (Q_0^2 + 1)^{-1} [\langle x \rangle^s, Q_0^2] (z - Q_0^2)^{-1} \langle x \rangle^{-s}.$$

From the boundedness of  $(Q_0^2 + 1)^{-1} [\langle x \rangle^s, Q_0^2]$  and the following estimate

$$(3.16) \quad \|(z - Q_0^2)^{-1} (Q_0^2 + 1)\|_{\mathbb{H}} \leq C (|\operatorname{Im} z|^{-1} \langle z \rangle + 1),$$

we obtain (i). As for the case  $1 < s \leq 2$ , we rewrite the last term  $(z - Q_0^2)^{-1} [\langle x \rangle^s, Q_0^2] (z - Q_0^2)^{-1} \langle x \rangle^{-s}$  as

$$(3.17) \quad (z - Q_0^2)^{-1} (Q_0^2 + 1) (Q_0^2 + 1)^{-1} [\langle x \rangle^s, Q_0^2] \langle x \rangle^{-s+1}$$

$$(3.18) \quad \times \langle x \rangle^{s-1} (z - Q_0^2)^{-1} \langle x \rangle^{-s+1} \langle x \rangle^{-1}.$$

By using the result for  $0 < s \leq 1$ , we have the inequality for  $1 < s \leq 2$ .

With these estimates, we prove (iii). Since  $\chi(Q_0^2) \equiv 1$ , we can easily see that

$$(3.19) \quad \langle x \rangle^s F_+(Q_0^2) \langle x \rangle^{-s} = \langle x \rangle^s F_{\chi,+}(Q_0^2) \langle x \rangle^{-s} + I.$$

Since  $F_{\chi,+}(t)$  satisfies (2.18) for  $m_0 = -1/2$ ,  $F_{\chi,+}(Q_0^2)$  can be rewritten as follows.

$$(3.20) \quad \frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{F}_{\chi,+}(z) \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s} dz \wedge d\bar{z}.$$

From this formula and (i) (ii) we have

$$\begin{aligned} & \|\partial_{\bar{z}} \tilde{F}_{\chi,+}(z) \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s}\|_{\mathbb{H}} \\ & \leq C |\partial_{\bar{z}} \tilde{F}_{\chi,+}(z)| (|\operatorname{Im} z|^{-1} + |\operatorname{Im} z|^{-2} \langle z \rangle + |\operatorname{Im} z|^{-3} \langle z \rangle^2). \end{aligned}$$

From (2.18) we have

$$\|\partial_{\bar{z}} \tilde{F}_{\chi,+}(z) \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s}\|_{\mathbb{H}} \leq C \langle z \rangle^{-5/2}.$$

This implies the boundedness of  $\langle x \rangle^s F_+(Q_0^2) \langle x \rangle^{-s}$ .

In a similar way, we rewrite  $\langle x \rangle^s F_-(Q_0^2)Q_0 \langle x \rangle^{-s}$  as

$$(3.21) \quad \langle x \rangle^s F_{\chi, -(Q_0^2)} \langle x \rangle^{-s} \langle x \rangle^s \frac{Q_0}{\sqrt{Q_0^2 + m^2}} \langle x \rangle^{-s} + \langle x \rangle^s \frac{Q_0}{\sqrt{Q_0^2 + m^2}} \langle x \rangle^{-s}.$$

It is sufficient to show the boundedness of  $\langle x \rangle^s Q_0 / \sqrt{Q_0^2 + m^2} \langle x \rangle^{-s}$ . To see this, we denote  $\chi(t) / \sqrt{t + m^2} \in C^\infty(\mathbb{R}^3)$  as  $S(t)$  and its almost analytic extension as  $\tilde{S}(z)$ . We can easily see that  $S(Q_0^2) \langle x \rangle^s Q_0 \langle x \rangle^{-s}$  is bounded. So we obtain the boundedness of  $\langle x \rangle^s F_-(Q_0^2)Q_0 \langle x \rangle^{-s}$  if we show that  $[\langle x \rangle^s, S(Q_0^2)]Q_0 \langle x \rangle^{-s}$  is bounded. We rewrite it as follows.

$$\frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{S}(z) \frac{Q_0^2 + 1}{z - Q_0^2} (Q_0^2 + 1)^{-1} [\langle x \rangle^s, Q_0^2] Q_0 \langle x \rangle^{-s} \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s} dz \wedge d\bar{z}.$$

By an elementary calculus, we have  $(Q_0^2 + 1)^{-1} [\langle x \rangle^s, Q_0^2] Q_0 \langle x \rangle^{-s}$  bounded. Combining (i) and (ii), we have

$$\begin{aligned} \|[\langle x \rangle^s, S(Q_0^2)]Q_0 \langle x \rangle^{-s}\|_{\mathbb{H}} &\leq C \int_{\mathbb{C}} |\partial_{\bar{z}} \tilde{S}(z)| \{1 + |\operatorname{Im} z|^{-1}\} \\ &\times \{|\operatorname{Im} z|^{-1} + |\operatorname{Im} z|^{-2} \langle z \rangle + |\operatorname{Im} z|^{-3} \langle z \rangle^2\} dz \wedge d\bar{z} < \infty. \end{aligned}$$

This implies the boundedness of  $\langle x \rangle^s F_-(Q_0^2)Q_0 \langle x \rangle^{-s}$ . □

Next we give an example of  $V$ . It requires smoothness, but allows long-range part in its diagonal components.

**Lemma 3.6.** *Let  $V$  be a  $4 \times 4$  Hermitian matrix of the form*

$$(3.22) \quad V(x) = (v_{ij}(x)) + \varphi(x)I_4 \equiv V_s(x) + V_l(x)$$

where  $V_s(x) = (v_{ij}(x))$  is an Hermitian matrix and  $I_4$  is an identity matrix. Suppose the following conditions hold. Then  $V(x)$  satisfies Assumption 3.2.

There exist  $\delta > 0$  such that the following inequalities hold for all multi-index  $\alpha$ .

$$(3.23) \quad |\partial_x^\alpha v_{ij}(x)| \leq C_\alpha \langle x \rangle^{-\delta - |\alpha|} \langle x_3 \rangle^{-1} \quad (1 \leq i, j \leq 4).$$

$\varphi(x) \in C^\infty(\mathbb{R}^3)$  is real valued and satisfies

$$(3.24) \quad |\partial_x^\alpha \varphi(x)| \leq C'_\alpha \langle x \rangle^{-\delta - |\alpha|}.$$

The relatively compactness of  $V(x)$  itself is clear since  $V$  satisfies (3.7). So we only have to show the relatively compactness of  $[V, A]$ . We prove the relatively compactness of  $[V_s, A] = [V_s, U_{FW}^{-1} \hat{A} \beta U_{FW}]$  at first. From the boundedness of  $\langle x_3 \rangle^{-1} \hat{A} \beta$

and the relatively compactness of  $V_s \langle x_3 \rangle$ , it is sufficient to show that  $\langle x_3 \rangle U_{FW} \langle x_3 \rangle^{-1}$  and  $\langle x_3 \rangle U_{FW}^{-1} \langle x_3 \rangle^{-1}$  are bounded operators in  $\mathbb{H}$ . We have already proved it in Lemma 3.5.

Next we treat the long-range term. The conjugate operator  $A$  can be decomposed into the sum of  $J_1, \dots, J_4$  where

$$\begin{aligned} J_1 &= F_+(Q_0^2) \hat{A} \beta F_+(Q_0^2), \\ J_2 &= F_+(Q_0^2) \hat{A} \beta^2 F_-(Q_0^2) Q_0, \\ J_3 &= \beta F_-(Q_0^2) Q_0 \hat{A} \beta F_+(Q_0^2), \\ J_4 &= \beta F_-(Q_0^2) Q_0 \hat{A} \beta^2 F_-(Q_0^2) Q_0. \end{aligned}$$

We prove that the  $H_0(\lambda)$ -compactness holds for each of  $[V_l, J_1], \dots, [V_l, J_4]$ . To see this we use the functional calculus again and rewrite  $J_1$  as follows.

$$\begin{aligned} F_+(Q_0^2) \hat{A} \beta F_+(Q_0^2) &= \hat{A} \beta F_+(Q_0^2)^2 + [F_{\chi,+}(Q_0^2), \hat{A} \beta] F_+(Q_0^2) \\ &\equiv J'_1 + J''_1 \end{aligned}$$

At first we prove the boundedness of  $J''_1$  and consequently the relatively compactness of  $[V_l, J''_1]$ . By using (2.19), we rewrite  $[F_{\chi,+}(Q_0^2), \hat{A} \beta]$  as follows.

$$(3.25) \quad \frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{F}_{\chi,+}(z) (z - Q_0^2)^{-1} [Q_0^2, \hat{A} \beta] (z - Q_0^2)^{-1} dz \wedge d\bar{z}.$$

From (3.16) we have  $[Q_0^2, \hat{A} \beta] (z - Q_0^2)^{-1}$  is dominated from above by  $C\{1 + |\operatorname{Im} z|\}$ . So we have

$$(3.26) \quad \|[F_{\chi,+}(Q_0^2), \hat{A} \beta]\| \leq C \int_{\mathbb{C}} |\partial_{\bar{z}} \tilde{F}_{\chi,+}(z)| \{|\operatorname{Im} z|^{-1} + |\operatorname{Im} z|^{-2} \langle z \rangle\} dz \wedge d\bar{z}.$$

Since the almost analytic extension  $\tilde{F}_{\chi,+}(z)$  satisfies

$$(3.27) \quad |\partial_{\bar{z}} \tilde{F}_{\chi,+}(z)| \leq C_N |\operatorname{Im} z|^N \langle z \rangle^{-3/2-N} \quad (\forall N \in \mathbb{N}),$$

we have  $[F_{\chi,+}(Q_0^2), \hat{A} \beta]$  is bounded and inductively  $[V_l, J''_1]$  is  $H_0(\lambda)$ -compact. So we only have to show the relatively compactness of  $[V_l, J'_1]$ .

$$(3.28) \quad [V_l, J'_1] = [V_l, \hat{A} \beta] F_+(Q_0^2)^2 + \hat{A} \beta [V_l, F_+(Q_0^2)^2].$$

Clearly  $[V_l, \hat{A} \beta] F_+(Q_0^2)$  is  $H_0(\lambda)$ -compact. Again we rewrite the commutator in the second term, by use of (2.19). Then we have  $\langle x \rangle^{1+\delta} [V_l, F_+(Q_0^2)^2]$  is bounded. Combing these facts, we have the relatively compactness of  $[V_l, J_1]$ .

As for the commutator  $[V_l, J_2], \dots, [V_l, J_4]$  we also replace  $F_{\pm}$  by  $F_{\chi,\pm}$  and use the functional calculus. The proof of relatively compactness of  $[V_l, J_2]$  and  $[V_l, J_3]$  are



almost the same. We only give the proof for  $J_2$ . We also estimate the ‘principle’ part before we compute the commutator with  $V_l$ .

$$(3.29) \quad J_2 = \hat{A}F_+(Q_0^2)F_-(Q_0^2)Q_0 + [F_+(Q_0^2), \hat{A}]F_-(Q_0^2)Q_0$$

It is sufficient to show that  $[V_l, \hat{A}F_+(Q_0^2)F_-(Q_0^2)Q_0]$  is a  $H_0(\lambda)$ -compact operator. We decompose it into the following sum.

$$\begin{aligned} & [V_l, \hat{A}]F_+(Q_0^2)F_-(Q_0^2)Q_0 \\ & \quad + \hat{A}[V_l, F_+(Q_0^2)F_-(Q_0^2)]Q_0 \\ & \quad + \hat{A}F_+(Q_0^2)F_-(Q_0^2)[V_l, Q_0]. \end{aligned}$$

We can easily see that the first and the third term is relatively compact since  $\langle x \rangle^{1+\delta}[V_l, Q_0]$  is bounded. As for the second term, we can also see the relatively compactness in the same argument as we have done in the proof of Lemma 3.5 (iii).

As for  $J_4$ , the proof is similar. We rewrite it as

$$(3.30) \quad \hat{A}F_-(Q_0^2)^2Q_0^2 + [F_-(Q_0^2)Q_0, \hat{A}]F_-(Q_0^2)Q_0$$

We can also obtain the relatively compactness by estimating the term  $[V, \hat{A}F_-(Q_0^2)^2Q_0^2]$ .

**Corollary 3.7.** *Let  $V$  be a  $4 \times 4$  Hermitian matrix and  $s > 1/2$ . Suppose  $V$  satisfies the condition in Lemma 3.6. Then the following limits*

$$(3.31) \quad R^\pm(\mu) = \lim_{\epsilon \downarrow 0} \langle x_3 \rangle^{-s} (H(\lambda) - \mu \mp i\epsilon)^{-1} \langle x_3 \rangle^{-s}$$

exist for  $\mu \in \mathbb{R} \setminus (\mathbb{R}_N \cup \sigma_{pp}(H(\lambda)))$  and  $R^\pm(\mu)$  are continuous with respect to  $\mu$ .

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