Kameyama, A.
Osaka J. Math.
38 (2001), 565–610

THE THURSTON EQUIVALENCE FOR POSTCRITICALLY FINITE BRANCHED COVERINGS

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(Received August 3, 1998)

1. Introduction

In this paper we investigate the 'homotopical' dynamics of branched coverings on S^2. Some branched coverings are expressed by the forms of rational functions on Riemann sphere, the dynamics of which have been deeply studied as the holomorphic dynamics. We will discuss not only rational maps but topological branched coverings from a homotopical viewpoint.

A real rational function is considered as a piecewise-monotone mapping on $\mathbb{R}$. As to the dynamical system of a piecewise-monotone mapping on $\mathbb{R}$, we have a powerful invariant, the kneading sequence. The real line is divided into intervals by the turning points (i.e. points at which the sign of the derivative changes); the mapping is monotone on each interval. A point in $\mathbb{R}$ visits the intervals by iteration of the mapping. Roughly speaking, whole dynamics are determined by the behavior of the turning points. The kneading sequence is defined as the sequences of intervals which the orbits of the turning points visit (in this paper the exact definition is not necessary. For example, the reader may refer to [9]).

The classification by the kneading sequences is weaker than the conjugacy classification. In fact, one can consider the kneading sequence as a 'homotopical invariant', and there exist two maps which are not conjugate to each other but which have the common kneading sequence. For simplicity, let us consider the case that all turning points are eventually periodic. We define an equivalence relation as follows: Let f_1 and f_2 be continuous piecewise-monotone maps on $\mathbb{R}$, and we denote the sets of the turning points by C_{f_1} and C_{f_2}. Since the turning points are eventually periodic, the forward orbit $P_{f_i} = \{f_i^n(c) \,|\, c \in C_{f_i}, n > 0\}$ is a finite set. We say f_1 and f_2 are equivalent if there exist order-preserving homeomorphisms $\phi_1, \phi_2 : \mathbb{R} - P_{f_1} \to \mathbb{R} - P_{f_2}$ such that $f_2 \circ \phi_1 = \phi_2 \circ f_1$. Then it is easily seen that f_1 and f_2 are equivalent if and only if their kneading sequences agree.

In the case of branched coverings on S^2, we can generalize the equivalence relation, though we do not have a good invariant. In [12], Thurston introduced the equivalence relation, and showed a topological condition that a given branched covering is equivalent to a rational map ([4]). The equivalence relation, which we call the Thurston equivalence, is the main object in the study of this paper.

Throughout this paper, all branched coverings and homeomorphisms on S^2 are supposed to be orientation-preserving.

DEFINITION. Let $f : S^2 \to S^2$ be a branched covering on the 2-dimensional sphere. By C_f, we denote the *critical set* of f, or the set of critical points of f. A successor of a critical point is said to be a *postcritical point*. The set of postcritical points is called a *postcritical set*:

$$P_f = \{f^n(c) \mid c \in C_f, n > 0\}.$$

We say f is *postcritically finite* if $\#P_f < \infty$.

From now on, we consider the case where f is postcritically finite.

DEFINITION. Let f and g be postcritically finite branched coverings. We say f and g are *equivalent* if there exist homeomorphisms ϕ_1, ϕ_2 on S^2 such that $\phi_i(P_f) = P_g$ $(i = 1, 2)$, ϕ_1 and ϕ_2 are isotopic relative to P_f, and

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{f} & (S^2, P_f) \\ {\scriptstyle \phi_1}\big\downarrow & & \big\downarrow{\scriptstyle \phi_2} \\ (S^2, P_g) & \xrightarrow[g]{} & (S^2, P_g) \end{array}$$

commutes. This equivalence relation is called the *Thurston equivalence*. (We will give an extended definition later.)

A simple question: Can we decide whether given two postcritically finite branched coverings are equivalent or not?

As mentioned above, in the 1-dimensional case, the kneading sequence is a good invariant. Unfortunately, however, we cannot use the kneading sequence in our case. Obviously, we have trivial invariants: the degree of a branched covering and the number of the postcritical points. Moreover, 'the local dynamics' of $C_f \cup P_f$ is one of simple invariants. For example, let us consider the 1-parameter family $f_c(z) = z^2 + c$ $(c \in \mathbb{C})$. The critical set is $C_{f_c} = \{0, \infty\}$. The infinity is superattracting fixed point. If $c = -2$, then 0 is strictly preperiodic and $f_{-2}^2(0) = 2$ is a fixed point: $0 \mapsto -2 \mapsto 2 \mapsto 2$. If $c = -1$, then 0 is 2-periodic: $0 \mapsto -1 \mapsto 0$. Thus their local dynamics are different, and f_{-2} is not equivalent to f_{-1}. Besides, we have parameters at which the maps have the identical local dynamics. Indeed, there exist three parameters $a, b, \bar{b}$ such that 0 is 3-periodic: $0 \mapsto c \mapsto f_c(c) \mapsto 0$; one parameter a is real and the other two are complex conjugate. Are these polynomial f_a, f_b and $f_{\bar{b}}$ equivalent to one another? The negative answer is obtained from Thurston's theory ([12], [4]) via the Teichmüller space. Furthermore, we can answer that by seeing the shape of their Julia sets ([8]). But these approaches are not so direct, and are not useful for branched coverings not equivalent to rational maps. The aim of this paper is to give a direct proof from a purely topological standpoint. Moreover, our purpose includes finding an algorithm to check the Thurston equivalence. To this end, we need a presentation of a branched covering, by which we carry out a calculation.

We give an easier example.

EXAMPLE. Consider two polynomials

$$g_1(z) = 3\sqrt{-3}\left(z - \frac{\exp\left(\pi\sqrt{-1}/6\right)}{\sqrt{3}}\right)^3$$
$$g_2(z) = -3\sqrt{-3}\left(z - \frac{\exp\left(-\pi\sqrt{-1}/6\right)}{\sqrt{3}}\right)^3.$$

The postcritical sets are $P_{g_1} = P_{g_2} = \{0, 1, \infty\}$. Their local dynamics are identical:

$$\infty \mapsto \infty, \ \ c_i \mapsto 0 \mapsto 1 \mapsto 1,$$

where $c_i = \exp\left(\pm\pi\sqrt{-1}/6\right)/\sqrt{3}$.

Problem: Are g_1 and g_2 equivalent to each other?

The answer is negative. In fact, we show a stronger statement: g_1 and g_2 are not weakly equivalent, that is, there does not exist homeomorphisms ϕ_1, ϕ_2 such that $\phi_i(P_{g_1}) = P_{g_2}$ and $g_2 \circ \phi_1 = \phi_2 \circ g_1$. Suppose ϕ_1, ϕ_2 are homeomorphisms such that $\phi_i(x) = x$ for $x \in \{0, 1, \infty\}$. Let γ be a simple path between 0 and 1 in $\Sigma = \hat{\mathbb{C}} - \{0, 1, \infty\}$. The path γ is unique up to homotopy in Σ. Then $\phi_2(\gamma)$ is also a simple path between 0 and 1, which is unique up to homotopy. Each of the inverse images $L_1 = g_1^{-1}(\gamma)$ and $L_2 = g_2^{-1}(\phi_2(\gamma))$ is a topological tree with three endpoints $0, 1, b_i$ and one 3-branch point c_i, where $b_i = \exp\left(\pm\pi\sqrt{-1}/3\right)$. If $g_2 \circ \phi_1 = \phi_2 \circ g_1$, then $\phi_1(L_1) = L_2$. But since L_1 and L_2 have the reverse orientations, it is impossible for an orientation-preserving homeomorphism.

This way is not valid for the example given earlier. Indeed, the three polynomials are weakly equivalent to one another, that is, there exist homeomorphisms ψ_1, ψ_2 of S^2 to itself which fix P_{f_a} such that $f_a \circ \psi_1$ and $f_a \circ \psi_2$ are equivalent to f_b and $f_{\bar{b}}$ respectively. Then a new problem comes upon us: Find a polynomial equivalent to $f_a \circ \psi_1 \circ \psi_1$, a polynomial equivalent to $f_a \circ \psi_1 \circ \psi_2$, a polynomial equivalent to $f_a \circ \psi_2 \circ \psi_1$, a polynomial equivalent to $f_a \circ \psi_2 \circ \psi_2$, a polynomial equivalent to $f_a \circ \psi_1 \circ \psi_1 \circ \psi_1$ and so on. When we work on this problem, it is efficient to consider the set

$$\hat{\Omega}_{f_a} = \{\psi_1 \circ f_a \circ \psi_2 \,|\, \psi_1, \psi_2 : (S^2, P_{f_a}) \to (S^2, P_{f_a}) \text{ homeomorphisms}\}.$$

Additionally, the difference between the two examples also comes from the structures of the mapping class groups. For a finite set $A \subset S^2$, we denote, by $M(A)$, the mapping class group, i.e. the group of isotopy classes of homeomorphisms of $S^2 - A$ to itself. A subgroup $M^0(A) \subset M(A)$ is defined as the subgroup of isotopy classes of homeomorphisms by which each point of A is fixed. Then $M^0(A)$ is trivial if $\#A = 3$, and $M^0(A)$ is not trivial if $\#A = 4$. Therefore, in the case $\#A = 4$, a path with endpoints in A is not unique up to homotopy. In order to study the Thurston equivalence in this case, we have to use some more structure of the mapping class group. We introduce the mapping class semigroup, which is an extension of the mapping class group. The mapping class semigroup is divided into subsets which are written as $\Omega_f = \{\phi_1 f \phi_2 \mid \phi_1, \phi_2 \in M^0(A)\}$. We will investigate the structure of Ω_f. In particular, we obtain a complete classification in the case f is of degree two with $\#A = 4$, and in the case f has $(2, 2, 2, 2)$-orbifolds.

In Section 2 we will define three equivalence relations: the Thurston equivalence, the weak equivalence and the local equivalence, which are the main objects of this paper.

Section 3 gives the definition of the branch group and the induced homomorphism. The branch group is a generalization of the fundamental group. For a universal covering $\rho : U \to X$, the branch group $G(X)$ of degree d is defined as the group of covering transformations of $\bigsqcup_{i=1}^{d} U_i \to X$, where $\bigsqcup_{i=1}^{d} U_i$ is the disjoint union of d copies of U. A branched covering $f : S^2 \to S^2$ of degree d induces a homomorphism $f_\dagger : \pi_1(S^2 - P_f, x) \to G(S^2 - P_f)$. We will explain why the homomorphism is considered as a presentation of the branched covering f.

In Section 4 we study the Thurston equivalence by using the mapping class group. This is applied to special cases in Section 5.

REMARK. After writing this paper, the author discovered the result of Brezin et al. ([3]). They enumerated hyperbolic nonpolynomial rational maps of degree two or three with four or fewer postcritical points.

As well as the enumerating problem, Pilgrim recently developed a general combinatorial theory of branched coverings ([10]).

2. Basic definitions

In this paper, we assume mappings on S^2 to be orientation-preserving.

DEFINITION. Let f be a postcritically finite branched covering. Suppose A is a finite subset of S^2 including P_f such that $f(A) \subset A$. Then we say A is a *generalized postcritical set* of f, and a pair (f, A) is a *furnished branched covering*.

Proposition 2.1. *Let (f, A_1) and (g, A_2) be furnished branched coverings. Suppose that there exist homeomorphisms ϕ_1, ϕ_2 on S^2 such that $\phi_i(A_1) = (A_2)$ $(i = 1, 2)$*

and $g \circ \phi_1 = \phi_2 \circ f$, namely, the following diagram commutes:

$$\begin{array}{ccc} (S^2, A_1) & \xrightarrow{f} & (S^2, A_1) \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ (S^2, A_2) & \xrightarrow[g]{} & (S^2, A_2) \end{array}$$

If ϕ'_2 is a homeomorphism isotopic to ϕ_2 relative to A_1, then there exists a homeomorphism ϕ'_1 isotopic to ϕ_1 relative to A_1 such that $g \circ \phi'_1 = \phi'_2 \circ f$.

Proof. Let $H : S^2 \times [0,1] \to S^2$ denote an isotopy between ϕ_2 and ϕ'_2. Take a point x in $S^2 - A_1$. Then $\gamma = H(\{f(x)\} \times [0,1])$ is a curve joining $\phi_2(f(x))$ and $\phi'_2(f(x))$. There is a component of $g^{-1}(\gamma)$ which has an endpoint $\phi_1(x)$. We denote the other endpoint by $\phi'_1(x)$, and the correspondence $x \mapsto \phi'_1(x)$ is the required homeomorphism. □

DEFINITION. Let (f, A_1) and (g, A_2) be furnished postcritically finite branched coverings. We say (f, A_1) and (g, A_2) are *equivalent* if there exist homeomorphisms ϕ_1, ϕ_2 on S^2 such that $\phi_i(A_1) = A_2$ $(i = 1, 2)$, ϕ_1 and ϕ_2 are isotopic relative to A_1, and $g \circ \phi_1 = \phi_2 \circ f$. This equivalence relation is called the *Thurston equivalence*.

REMARK. In the preceding definition we can replace 'isotopic' by 'homotopic' because of the fact that two orientation-preserving homeomorphisms on an orientable surface are homotopic if and only if they are isotopic ([5]).

By Proposition 2.1, if f is equivalent to g, then the iteration f^n is equivalent to g^n.

DEFINITION. Let (f, A_1) and (g, A_2) be furnished postcritically finite branched coverings. We say (f, A_1) and (g, A_2) are *weakly equivalent* if there exist homeomorphisms ϕ_1, ϕ_2 on S^2 such that $\phi_i(A_1) = A_2$ $(i = 1, 2)$ and $g \circ \phi_1 = \phi_2 \circ f$.

DEFINITION. Let (f, A) be a furnished branched covering. For a point x in S^2, the *degree* at x, which we denote by $d(x)$, is the integer n such that f is n-to-1 map on $N - \{x\}$, where N is a small neighborhood of x.

We define a matrix $T_{(f,A)} : A \times (A \cup C_f) \to \{0\} \cup \mathbb{N}$ as

$$\begin{aligned} T_{(f,A)}(x, y) &= 0 \quad \text{if} \quad f(y) \neq x \\ T_{(f,A)}(x, y) &= n \quad \text{if} \quad f(y) = x, d(y) = n, \end{aligned}$$

which is called the *transition matrix* of (f,A). The relative homology group $H_2(S^2, S^2 - A; \mathbb{Z})$ is considered as the free module generated by A. Therefore we consider $H_2(S^2, S^2 - (A \cup C_f); \mathbb{Z})$ is included in $H_2(S^2 - f^{-1}(A); \mathbb{Z})$. The transition matrix is a matrix representation of the induced homomorphism $f_* : H_2(S^2, S^2 - (A \cup C_f); \mathbb{Z}) \subset$

$H_2(S^2, S^2 - f^{-1}(A); \mathbb{Z}) \to H_2(S^2, S^2 - A; \mathbb{Z})$.

The transition matrix is expressed by a directed graph $\mathcal{T}(f, A)$, namely, $\mathcal{T}(f, A) = (C_f \cup A, \{(x, y, T_{(f,A)}(x, y)) | (x, y) \in C_f \cup A\})$ is the pair of the vertex set and the edge set: we consider $(x, y, T_{(f,A)}(x, y))$ as an arrow from y to x with weight $T_{(f,A)}(x, y)$. We say the directed graph is the *local type* of (f, A).

Two furnished branched coverings (f, A) and (g, B) are called *locally equivalent* if they has the same local type, that is, there exists a one-to-one mapping $h : C_f \cup A \to C_g \cup B$ such that $T_{(f,A)}(x, y) = T_{(g,B)}(h(x), h(y))$ for all $x, y \in C_f \cup A$.

EXAMPLE.
(1) $f(z) = z^d$. The critical set is equal to the postcritical set $C_f = P_f = \{0, \infty\}$. The transition matrix of (f, P_f) is $\left(\begin{smallmatrix} d & 0 \\ 0 & d \end{smallmatrix}\right)$.
(2) $f(z) = z^{-d}$. $C_f = P_f = \{0, \infty\}$. The transition matrix of (f, P_f) is $\left(\begin{smallmatrix} 0 & d \\ d & 0 \end{smallmatrix}\right)$.
(3) $f(z) = z^2 + \sqrt{-1}$. The postcritical set is $P_f = \{\sqrt{-1}, -1 + \sqrt{-1}, -\sqrt{-1}, \infty\}$. The transition matrix of (f, P_f) is $\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right)$.

REMARK. Clearly,

$$\text{equivalent} \Rightarrow \text{weakly equivalent} \Rightarrow \text{locally equivalent}.$$

In general, the reverse arrows fail. Some examples will be given later.

The following fact is well-known.

Proposition 2.2. *Let f be a branched covering of degree d. Then*

$$\sum_{y \in f^{-1}(x)} d(y) = d. \tag{A}$$

From the Riemann-Hurwitz formula, we have

$$\sum_{c \in C_f} (d(c) - 1) = 2d - 2. \tag{B}$$

Therefore, if $d \geq 2$, then $\# C_f \geq 2$ and $\# P_f \geq 2$.

The case $\# A = 2$ is almost trivial.

Proposition 2.3. *Let (f, A) be a furnished branched covering of degree $d \geq 2$. If $\# A = 2$, then (f, A) is equivalent to either $(z^d, \{0, \infty\})$ or $(z^{-d}, \{0, \infty\})$.*

Proof. By Proposition 2.2, $A = P_f$. We write $P_f = \{a, b\}$. If $\#C_f > 2$, then $\sum_{c \in C_f} d(c) = 2d - 2 + \#C_f \geq 2d + 1$. Since $f(C_f) \subset P_f$, this contradicts (A). Thus $\#C_f = 2$; we write $C_f = \{y, z\}$. By (A) and (B), $d(y) = d(z) = d$. From (A), we have $f(y) \neq f(z)$. We can assume that $f(y) = a$ and $f(z) = b$. Then $f^{-1}(a) = \{y\}$ and $f^{-1}(b) = \{z\}$. Either $f(a) = a$ or $f(a) = b$. If $f(a) = a$, then $a = y$; if $f(a) = b$, then $b = y$. Therefore $C_f = P_f$, and we have two possibilities: (1) $a \mapsto a \quad b \mapsto b$ and (2) $a \mapsto b \mapsto a$.

Let l be a simple path joining a and b. Then $f^{-1}(l)$ is the union of simple paths $l_1, l_2, \ldots, l_d$ joining a and b, where we take l_i's such that l_i and l_{i+1} are neighboring. By E_i, we denote the simply connected domain bounded by $l_i \cup l_{i+1}$. We take a homeomorphism $\phi_1 : S^2 \to \hat{\mathbb{C}}$ such that $\phi_1(a) = 0$, $\phi_1(b) = \infty$ and $\phi_1(l) = \{0 \leq x \leq \infty\} \subset \hat{\mathbb{C}}$. Since $f : E_i \to S^2 - l$ is homeomorphic, we can define a homeomorphism

$$\phi_{2,i} : E_i \to \left\{ r \exp\left(\sqrt{-1}\theta\right) \;\middle|\; 0 \leq r \leq \infty, \frac{2\pi(i-1)}{d} < \theta < \frac{2\pi i}{d} \right\}$$

as $g \circ \phi_2(x) = \phi_1 \circ f(x)$, where $g(z) = z^d$ in the case (1) and $g(z) = z^{-d}$ in the case (2). Then we obtain the homeomorphism $\phi_2 : S^2 \to \hat{\mathbb{C}}$ by $\phi_2 | E_i = \phi_{2,i}$, which satisfies $g \circ \phi_2 = \phi_1 \circ f$. Since l_i is isotopic to l with the endpoints fixed, ϕ_2 is isotopic to ϕ_1 relative to A. $\square$

3. Branch groups

For a homeomorphism $\phi : S^2 - A \to S^2 - A$, the induced homomorphism $\phi_* : \pi_1(S^2 - A, x) \to \pi_1(S^2 - A, x)$ is a 'representation' of ϕ, provided x is a fixed point of ϕ. Indeed, we can reconstruct the homeomorphism ϕ from the homomorphism ϕ_* up to isotopy. However, if a furnished branched covering (f, A) is of degree more than one, it is hard to imagine the original mapping f from the induced homomorphism $f_* : \pi_1(S^2 - f^{-1}(A), x) \to \pi_1(S^2 - A, x)$. Therefore we introduce the branch groups, which are closely related to the branched covering. Roughly speaking, the induced homomorphism $f_\dagger$ on the branch group is something like the 'inverse' of $f_* : \pi_1(S^2 - f^{-1}(A), x) \to \pi_1(S^2 - A, x)$.

Let (f, A) be a furnished branched covering of degree d. By $\rho : U \to S^2 - A$, we denote the universal covering. Then there exist mappings $q_1, q_2, \ldots, q_d : U \to U$ such that

$$\begin{array}{ccc} U - \rho^{-1} f^{-1}(A) & \xleftarrow{\;q_i\;} & U \\ \rho \big\downarrow & & \big\downarrow \rho \\ S^2 - f^{-1}(A) & \xrightarrow[f]{\qquad} & S^2 - A \end{array}$$

commutes and $f^{-1}(\rho(x)) = \{\rho(q_1(x)), \rho(q_2(x)), \ldots, \rho(q_d(x))\}$ for any $x \in U$. Indeed, let us take $x \in U$ and $x_1, x_2, \ldots, x_d \in U$ such that $f(\rho(x_i)) = \rho(x)$. Since $f : S^2 - f^{-1}(A) \to S^2 - A$ is a covering, so is $f \circ \rho : U - \rho^{-1}(f^{-1}(A)) \to S^2 - A$. Therefore there exists the covering $q_i : U \to U - \rho^{-1}(f^{-1}(A))$ that satisfies $q_i(x) = x_i$. Namely, q_i is defined as follows. Let γ be a path between x and y. There exists a path $\tilde{\gamma}$ such that $f \circ \rho(\tilde{\gamma}) = \rho(\gamma)$ and $\tilde{\gamma}$ has an endpoint x_i. We define $q_i(y)$ as the other endpoint of $\tilde{\gamma}$. We call $(q_1, q_2, \dots, q_d)$ a *system of lifts* of f^{-1}. The idea of the branch group is founded on the existence of these mappings.

NOTATION. We denote the set of *words* of d symbols by

$$W_k = \{1, 2, \dots d\}^k = \{a_1 a_2 \dots a_k \mid a_i \in \{1, 2, \dots, d\}\} \text{ for } k = 1, 2, \dots, \text{ and } W_0 = \{\emptyset\}.$$

Let Λ_k denote the set of the bijections of W_k to itself. Then Λ_k is the symmetric group on d^k elements with the product $hh' = h \circ h'$. Remark that Λ_0 is a trivial group.

The space $U \times W_k$ is the disjoint union of d^k copies of U. Since W_0 consists of one point, $U \times W_0 = U$. A projection $\xi : U \times W_k \to U$ is naturally defined as $\xi(x, w) = x$. We consider the mapping $\rho_k = \rho \circ \xi : U \times W_k \to S^2 - A$. Although $U \times W_k$ is not connected, we may consider $\rho_k : U \times W_k \to S^2 - A$ as a covering. By $G_{d,k}(S^2 - A)$ (we write G_k for simplicity), we denote the group of covering transformations of ρ_k. In other words, G_k consists of homeomorphisms $g : U \times W_k \to U \times W_k$ satisfying $\rho_k \circ g = \rho_k$.

For a covering transformation $g \in G_k$, a covering transformation $p_1 g(w) \in G_0$ is defined by

$$x \mapsto \xi(g(x, w))$$

for each $w \in W_k$, and a permutation $p_2 g \in \Lambda_k$ is defined by

$$p_2 g(w) = w' \iff g(x, w) = (p_1 g(w)(x), w').$$

We consider $p_1 g$ as a mapping of W_k to G_0. Conversely, if $\tau \in G_0^{W_k}$ and $h \in \Lambda_k$ are given, a covering transformation $g \in G_k$ is determined by $g(x, w) = (\tau(w)(x), h(w))$. Note that $p_1 g = g$ and $p_2 g = \mathrm{id}$ if $g \in G_0$. Therefore, as a set, G_k is the direct product $G_0^{W_k} \times \Lambda_k$. In fact, the group G_k is a semi-direct product of $G_0^{W_k}$ and Λ_k. For $g \in G_k$, suppose $g(x', w') = (x, w)$. Then $p_2 g(w') = w$ and $p_2(g^{-1})(w) = w'$, so $(p_2 g)^{-1} = p_2(g^{-1})$. Since $p_1(g^{-1})(w)(x) = x'$ and $p_1 g(w')(x') = x$, we have $p_1(g^{-1})(w) = p_1 g(p_2 g^{-1}(w))$. For $g, g' \in G_k$, we have

$$\begin{aligned} gg'(x, w) &= g(p_1 g'(w)(x), p_2 g'(w))) \\ &= (p_1 g(p_2 g'(w)) p_1 g'(w)(x), p_2 g p_2 g'(w)). \end{aligned}$$

Therefore

$$\text{(C)} \qquad p_1(gg')(w) = p_1 g(p_2 g'(w)) p_1 g'(w), \quad p_2(gg') = p_2 g p_2 g'.$$

Proposition 3.1. *For $g \in G_0$ and $i \in \{1, 2, \dots, d\}$, there uniquely exist $g' = T_i(g) \in G_0$ and $j = e(i, g) \in \{1, 2, \dots, d\}$ such that $q_j \circ g = g' \circ q_i$.*

Proof. Take a point $x \in U$. There uniquely exists j such that $\rho \circ q_j \circ g(x) = \rho \circ q_i(x)$. Let g' denote the covering transformation such that $g'(q_i(x)) = q_j(g(x))$. Since $g' \circ q_i$ and $q_j \circ g$ are covering, we have $g' \circ q_i = q_j \circ g$. □

It is easily seen that $e(\cdot, g) : \{1, 2, \dots, d\} \to \{1, 2, \dots, d\}$ is a permutation. For $g, g' \in G_0$, suppose $j = e(i, g)$ and $j' = e(j, g')$. Then $q_j \circ g = T_i(g) \circ q_i$ and $q_{j'} \circ g' = T_j(g') \circ q_j$. Therefore $q_{j'} \circ g' \circ g = T_j(g') \circ q_j \circ g = T_j(g') \circ T_i(g) \circ q_i$. Consequently,

(D) $$e(i, g'g) = e(e(i, g), g'), \;\; T_i(g'g) = T_{e(i,g)}(g') \circ T_i(g).$$

The induced homomorphism $f_\dagger : G_k \to G_{k+1}$ is defined for $k = 0, 1, 2, \dots$. In this paper, however, we deal with only the case $k = 0$.

For $g \in G_0$, we define $g' \in G_1$ by $p_1 g'(i) = T_i(g)$ and $p_2 g'(i) = e(i, g)$. Then $g'(q_i(x), i) = (q_j \circ g(x), j)$, where $j = e(i, g)$. By Proposition 3.1, g' is unique. By (C) and (D), it is easily seen that the mapping $f_\dagger : g \mapsto g'$ is a homomorphism.

We say G_k is the k-th *d-branch group*. The homomorphism $f_\dagger$ is the *induced homomorphism*.

REMARK. The definition of $f_\dagger : G_k \to G_{k+1}$ for general k is as follows. A left action of G_k on

$$V_k = \{x \in U^{W_k} \mid f^k(\rho x(w)) = f^k(\rho x(w')), \rho x(w) \neq \rho x(w') \text{ for any } w \neq w' \in W_k\}$$

is defined by $(g \cdot x)(w) = p_1 g(p_2 g^{-1}(w))(x(p_2 g^{-1}(w)))$. A mapping $F : V_{k-1} \to V_k$ is defined by $F(x)(iw) = q_i(x(w))$. Then $f_\dagger(g)$ is characterized as the element that satisfies $f_\dagger(g) \cdot F(x) = F(g \cdot x)$.

Now the induced homomorphism depends on a system of lifts. Therefore we may write $f_\dagger = f_{r,\dagger}$ for a system of lifts $r = (q_1, q_2, \dots, q_d)$. For two systems of lifts $r = (q_1, q_2, \dots, q_d)$ and $r' = (q'_1, q'_2, \dots, q'_d)$, we define $a = a(r, r') \in G_1$ as follows:

$$p_2 a(i) = j \text{ if } \rho q_i = \rho q'_j; \quad p_1 a(i) q_i = q'_j.$$

Note that $a(r', r) = a(r, r')^{-1}$.

Proposition 3.2. *We have*

$$f_{r,\dagger}(g) = a(r, r')^{-1} f_{r',\dagger}(g) a(r, r')$$

for $g \in G_0$.

Proof. We write $e(\cdot,\cdot) = e_r(\cdot,\cdot)$ and $T_i(\cdot) = T_{r,i}(\cdot)$ for a system of lifts r.

Suppose $g \in G_0$. We set $j = p_2a(i)$, $j' = e_r(i,g)$ and $j'' = e_{r'}(j,g)$. Then $\rho q_i = \rho q'_j$, $\rho q_{j'}g = \rho q_i$ and $\rho q'_{j''}g = \rho q'_j$. Thus $\rho q_{j'}g = \rho q'_{j''}g$, and so $j'' = p_2a(j')$. Therefore $p_2(f_{r,\dagger}(g)) = p_2(a^{-1}f_{r'\dagger}(g)a)$.

We set $g' = T_{r,i}(g)$ and $g'' = T_{r',j}(g)$. Then $g'q_i = q_{j'}g$ and $g''q'_j = q'_{j''}g$. Since $p_1a(i)q_i = q'_j$ and $p_1a(j')^{-1}q'_{j''}g = q_{j'}g$, we have $p_1a(j')^{-1}g''p_1a(i)q_i = q_{j'}g$. Thus

$$p_1a(j')^{-1}g''p_1a(i) = g' \,[= T_{r,i}(g) = p_1(f_{r,\dagger}(g))(i)\,],$$

and so

$$\begin{aligned} p_1(f_{r,\dagger}(g))(i) &= p_1a(p_2a^{-1}\circ e_{r'}(\cdot,g)\circ p_2a(i))^{-1}T_{r',p_2a(i)}(g)p_1a(i) \\ &= p_1(a^{-1})(e_{r'}(\cdot,g)\circ p_2a(i))T_{r',p_2a(i)}(g)p_1a(i) \\ &= p_1(a^{-1}f_{r',\dagger}(g)a)(i). \end{aligned}$$

The proof is completed. □

Conversely, suppose $b \in G_1$. Then it is easily seen that there exists a system of lifts r' such that $f_{r,\dagger}(g) = b^{-1}f_{r',\dagger}(g)b$.

For a homeomorphism $\phi : (S^2, A) \to (S^2, A)$ we can similarly define $\phi_\dagger : G_k \to G_k$ ($k = 0, 1$). In fact, we choose $\psi : U \to U$ a lift of ϕ^{-1}. For $g \in G_0$, a covering transformation $\phi_{\psi,\dagger}(g)$ is defined such that $\phi_{\psi,\dagger}(g)\psi = \psi g$. Then $\phi_{\psi,\dagger} : G_1 \to G_1$ is defined by $p_1(\phi_{\psi,\dagger}(g))(i) = \phi_{\psi,\dagger}(p_1g(i))$ and $p_2(\phi_{\psi,\dagger}(g)) = p_2g$. For homeomorphisms ϕ, ϕ', we have $(\phi f \phi')_{r',\dagger} = \phi'_{\psi',\dagger}f_{r,\dagger}\phi_{\psi,\dagger}$ provided $r = (q_1, q_2, \ldots, q_d)$ and $r' = (\psi' q_1\psi, \psi' q_2\psi, \ldots, \psi' q_d\psi)$ where ψ, ψ' are lifts of ϕ, ϕ'. Indeed, for $g \in G_0$,

$$\begin{aligned} p_1(\phi'_{\psi',\dagger}f_{r,\dagger}\phi_{\psi,\dagger}(g))(i) &= \phi'_{\psi',\dagger}T_{r,i}\phi_{\psi,\dagger}(g) \\ p_2(\phi'_{\psi',\dagger}f_{r,\dagger}\phi_{\psi,\dagger}(g))(i) &= j, \end{aligned}$$

where j satisfies $q_j \circ \phi_{\psi,\dagger}(g) = T_{r,i}\phi_{\psi,\dagger}(g)\circ q_i$. When we write

$$g' = \phi'_{\psi',\dagger}T_{r,i}\phi_{\psi,\dagger}(g),$$

we have $g'\psi' q_i\psi = \psi' q_j\psi g$. Therefore

$$p_1((\phi f \phi')_{r',\dagger}(g))(i) = g' \quad \text{and} \quad p_2((\phi f \phi')_{r',\dagger}(g))(i) = j.$$

Proposition 3.3. *If ϕ, ϕ' are homotopic to the identity relative to A, then there exists r'' a system of lifts such that $\phi'_{\psi',\dagger}f_{r,\dagger}\phi_{\psi,\dagger} = f_{r'',\dagger}$.*

Proof. Let $h(\cdot,\cdot)$ be a homotopy between the identity and ϕ. We take a continuous map $\psi(\cdot,\cdot) : U \times [0,1] \to U$ such that $\psi(\cdot,1) = \psi$ and $\psi(\cdot,t)$ is a lift of $h(\cdot,t)^{-1}$. Since $b = \psi(\cdot,0)$ is a lift of the identity, b is a member of G_0. For each $x \in U$, the path $\gamma_x = \psi(x,\cdot)$ is the lift of the path $h(\rho(x),\cdot)$ with endpoints $b(x)$ and $\psi(x)$. For each $g \in G_0$, the path $\gamma_{g^{-1}(x)}$ has endpoints $bg^{-1}(x)$ and $\psi(g^{-1}(x))$. Since $\phi_{\psi,\dagger}(g) = \psi g \psi^{-1}$, we have $\phi_{\psi,\dagger}(g)(\psi(g^{-1}(x))) = \psi(x)$, and hence $\phi_{\psi,\dagger}(g)(bg^{-1}(x)) = b(x)$. Therefore $\phi_{\psi,\dagger}(g) = bgb^{-1}$ for $g \in G_0$. Similarly there exists $b' \in G_0$ such that ${\phi'}_{\psi',\dagger}(g) = b'gb'^{-1}$ for $g \in G_0$. Define b'_1 by $b'_1(x,i) = (b'x,i)$. Then $\phi'_{\psi',\dagger}(g) = b'_1 g {b'_1}^{-1}$ for $g \in G_1$.

Thus

$$\begin{aligned} \phi'_{\psi',\dagger} f_{r,\dagger} \phi_{\psi,\dagger}(g) &= {\phi'}_{\psi',\dagger} f_{r,\dagger}(bgb^{-1}) \\ &= {\phi'}_{\psi',\dagger}(f_{r,\dagger}(b) f_{r,\dagger}(g) f_{r,\dagger}(b^{-1})) \\ &= b'_1 f_{r,\dagger}(b) f_{r,\dagger}(g) f_{r,\dagger}(b)^{-1} {b'_1}^{-1} \end{aligned}$$

The proposition follows from the remark just after the proof of Proposition 3.2. □

Fix a basepoint $x \in S^2 - A$ and its lift $\tilde{x} \in \rho^{-1}(x)$. The induced homomorphism gives us the information of the behavior of loops in $S^2 - A$. The 0-th branch group G_0 is isomorphic to the fundamental group $\pi_1(S^2 - A, x)$. Let $\gamma : [0,1] \to S^2 - A$ be a closed curve such that $\gamma(0) = \gamma(1) = x$. By $\tilde{\gamma}$, we denote the lift of γ by $\rho : U \to S^2 - A$ such that $\tilde{\gamma}(0) = \tilde{x}$, which uniquely determines the covering transformation $g_\gamma \in G_0$ by $g_\gamma(\tilde{\gamma}(1)) = \tilde{\gamma}(0)$. For $g \in G_0$, a path between $\tilde{x}$ and $g(\tilde{x})$ is uniquely determined up to homotopy. Thus we obtain a homomorphism $\pi_1(S^2 - A, x) \ni \gamma \to g_\gamma \in G_0$.

DEFINITION. Consider the graph in the plane

$$Q_d = \left\{ te^{\theta\sqrt{-1}} \in \mathbb{C} \;\middle|\; 0 \le t \le 1, \theta = \frac{2\pi}{d}, 2\cdot\frac{2\pi}{d}, \dots, (d-1)\cdot\frac{2\pi}{d}, d\cdot\frac{2\pi}{d} \right\}.$$

A *radial* of f is a continuous map $r : Q_d \to S^2 - A$ such that

$$f^{-1}(r(0)) = \left\{ r\left(e^{k\cdot 2\pi\sqrt{-1}/d}\right) \mid k = 1, 2, \dots, d \right\}.$$

We say $r(0)$ is the *basepoint* of r and a point of $r\left(e^{k\cdot 2\pi\sqrt{-1}/d}\right)$ is a *radial points* of r. The arc $l_k : [0,1] \ni t \mapsto r\left(te^{k\cdot 2\pi\sqrt{-1}/d}\right) \in S^2 - A$ is called the k-th *spoke* of r. Two radials r, r' are said to be homotopic if there exists a homotopy $h : Q_d \times I \to S^2 - A$ such that $h(\cdot,0) = r, h(\cdot,1) = r'$ and $h(\cdot,t)$ is a radial of f for $0 \le t \le 1$.

There exists a one-to-one correspondence between the radials of f with basepoint x up to homotopy and the systems of lifts of f^{-1}. Indeed, for a radial r we take the lift $\tilde{r}$ by ρ such that $\tilde{r}(0) = \tilde{x}$. Then q_k is determined by $q_k(\tilde{x}) = \tilde{r}\left(e^{k\cdot 2\pi\sqrt{-1}/d}\right) = \tilde{x}_k$.

Let $\gamma : [0,1] \to S^2 - A$ be a curve with $\gamma(0) = \gamma(1) = x$. Suppose $\gamma_1, \gamma_2, \dots, \gamma_d$ are the lift of γ by $f : S^2 - f^{-1}(A) \to S^2 - A$ with $\gamma_i(0) = \rho(\tilde{x}_i)$. Then $p_2(f_\dagger(g_\gamma))(i) =$

$j \iff \gamma_i(1) = \gamma_j(0)$. Therefore $\alpha = l_i \gamma_i l_j^{-1}$ is a closed curve, where l_i is the spoke of r. We have $p_1(f_\dagger(g_\gamma))(i) = g_\alpha$. For a permutation $h \in \Lambda_k$, we say $(a_1, a_2, \dots, a_n = a_0)$ is an *orbit* of h if $h(a_{i-1}) = a_i$ for $i = 1, 2, \dots, n$. Consequently,

Proposition 3.4. *Let γ be a closed curve in $S^2 - A$ with $\gamma(0) = \gamma(1) = x$. If there exists $N = (a_1, a_2 \dots, a_l)$ an orbits of $p_2((f_\dagger)(g_\gamma)) \in \Lambda_j$, then there exists closed curve γ' such that $f : \gamma' \to \gamma$ is of degree l and $g_{\gamma'} = p_1(f_\dagger(g_\gamma))(a_l) \dots p_1(f_\dagger(g_\gamma))(a_2) p_1(f_\dagger(g_\dagger))(a_1)$.*

In particular, if γ is a simple closed curve, then the number of the orbits of $p_2((f_\dagger)(g_\gamma))$ is equal to the number of the component of $f^{-1}(\gamma)$.

As for a homeomorphism ϕ, a radial is a path l between x and $\phi^{-1}(x)$. The path l determines the isomorphism $l_* : \pi_1(S^2 - A, x) \to \pi_1(S^2 - A, \phi^{-1}(x))$ by $\gamma \mapsto l\gamma l^{-1}$. Write $\phi_{l,\dagger}$ instead of $\phi_{\psi,\dagger}$, where ψ is the lift of ϕ^{-1} by ρ such that $\psi(\tilde{x}) = \tilde{l}(1)$, and $\tilde{l}$ is the lift of l by ρ with $\tilde{l}(0) = \tilde{x}$. Then $\phi_{l,\dagger}$ is identified with

$$\pi_1(S^2 - A, x) \xrightarrow{\phi_*^{-1}} \pi_1(S^2 - A, \phi^{-1}(x)) \xrightarrow{l_*^{-1}} \pi_1(S^2 - A, x).$$

From now on we identify G_0 and $\pi_1(S^2 - A, x)$ for simplicity. An element of G_k is written in the form

$$g = \sum_{w \in W_k} \gamma_w \cdot (w, h(w)),$$

where γ_w is the element of $\pi_1(S^2 - A, x)$ such that $g_{\gamma_w} = p_1 g(h(w))$, $h(w)$ is the element of W_k such that $p_2 g(h(w)) = w$ (i.e. $h = p_2 g^{-1}$). Remark that the summation is formal. For two elements $g = \sum_{w \in W_k} \gamma_w \cdot (w, h(w))$ and $g' = \sum_{w \in W_k} \gamma'_w \cdot (w, h'(w))$, the composition is

$$gg' = \sum_{w \in W_k} \gamma_w \gamma'_{h(w)} \cdot (w, h'(h(w))).$$

DEFINITION. Let (f, A) be a furnished branched covering. Fix a radial r with basepoint x. We set $A = \{a_1, a_2 \dots, a_n\}$, that is, we choose a mapping $a : \{1, 2, \dots, n\} \to A$. Let us take simple closed curves $C_1, C_2, \dots, C_n : [0, 1] \to S^2 - A$ that satisfy the following: $C_i(0) = C_i(1) = x$, C_i's are disjoint except at x, each C_i bounds a simply connected domain D_i anticlockwise such that $D_i \cap A = \{a_i\}$ and the product $C_1 C_2 \dots C_n$ is null-homotopic in $S^2 - A$. Considering $C_1, \dots, C_n$ as elements of $\pi_1(S^2 - A, x)$, we obtain a generator set — the set $\{C_1, \dots, C_{n-1}\}$ generates $\pi_1(S^2 - A, x)$ freely. We say $(C_1, \dots, C_n)$ is a *generator chain* of $S^2 - A$. Each element $a \in \pi_1(S^2 - A, x)$ can be expressed in the form $a = C_{i(1)}^{\epsilon(1)} C_{i(2)}^{\epsilon(2)} \dots C_{i(m)}^{\epsilon(m)}$ with m minimal, where $i(j) \in \{1, 2, \dots, n-1\}$ and $\epsilon(j) = \pm 1$. This expression is said to be the *minimal expression* of a. We say $|a| = m$ is the *length* of a. For another generator chain $(C_1', C_2', \dots, C_n')$, there exists a homeomorphism $\phi : (S^2, A) \to (S^2, A)$ that pointwise fixes A such that $\phi_*(C_i) = C_i'$.

The homomorphism $f_\dagger : G_0 \to G_1$ is determined by the following diagram:

$$\begin{cases} C_1 & \mapsto Z_{1,1} \cdot (1, h_1(1)) + Z_{1,2} \cdot (2, h_1(2)) + \cdots + Z_{1,d} \cdot (d, h_1(d)) \\ C_2 & \mapsto Z_{2,1} \cdot (1, h_2(1)) + Z_{2,2} \cdot (2, h_2(2)) + \cdots + Z_{2,d} \cdot (d, h_2(d)) \\ & \vdots \\ C_{n-1} & \mapsto Z_{n-1,1} \cdot (1, h_{n-1}(1)) + Z_{n-1,2} \cdot (2, h_{n-1}(2)) + \cdots + Z_{n-1,d} \cdot (d, h_{n-1}(d)) \end{cases}$$

where $Z_{k,i} = p_1(f_\dagger(C_k))(h_k(i))$ and $h_k = p_2(f_\dagger(C_k))^{-1}$. This diagram is said to be the *fundamental system* of $f_\dagger$ with respect to the generator chain $(C_1, \dots, C_n)$.

EXAMPLE.
(1) Consider $f(z) = z^d$ with $A = P_f$. Let us set $x = 1$ as the basepoint. We take a radial r such that the k-th spoke is $l_k(t) = \exp\big(2\pi\sqrt{-1}(k-1)t/d\big)$ $(k = 1, 2, \dots, d)$, and take a generator chain (C_1, C_2) such that C_1 is homotopic to $\{|z| = 1\}$. Then the fundamental system of $f_{r,\dagger}$ is

$$C_1 \mapsto C_1 \cdot (1, d) + (2, 1) + \cdots + (d-1, d-2) + (d, d-1).$$

See Fig. 1 and 2.

Even if we take another radial r with spokes $l_k(t) = \exp\big(2\pi\sqrt{-1}kt/d\big)$, the fundamental system is unchanged; because r and r' are homotopic. If we take a radial r'' with spokes $l_k'' = l_k$ $(k = 1, 2, \dots, d-1)$ and $l_d''(t) = \exp\big(-2\pi\sqrt{-1}t/d\big)$, then the fundamental system of $f_{r'',\dagger}$ is

$$C_1 \mapsto (1, d) + (2, 1) + \cdots + (d-1, d-2) + C_1 \cdot (d, d-1).$$

(2) Consider $f(z) = z^2 + \sqrt{-1}$ with $A = P_f = \{\sqrt{-1}, -1 + \sqrt{-1}, -\sqrt{-1}, \infty\}$. We take a radial r and a generator chain (C_1, C_2, C_3, C_4) as in Fig. 3. Then the fundamental system of $f_{r,\dagger}$ is

$$\begin{cases} C_1 \mapsto C_2^{-1}C_1^{-1} \cdot (1, 2) + C_1C_2 \cdot (2, 1) \\ C_2 \mapsto C_3 \cdot (1, 1) + C_1C_2C_1C_2^{-1}C_1^{-1} \cdot (2, 2) \\ C_3 \mapsto (1, 1) + C_1C_2C_1^{-1} \cdot (2, 2) \end{cases}.$$

See Fig. 4. If we take a radial r' as in Fig. 5, then the fundamental system of $f_{r',\dagger}$ is

$$\begin{cases} C_1 \mapsto (1, 2) + (2, 1) \\ C_2 \mapsto C_3 \cdot (1, 1) + C_1 \cdot (2, 2) \\ C_3 \mapsto (1, 1) + C_2 \cdot (2, 2) \end{cases},$$

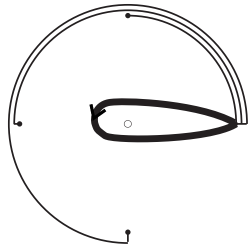

Fig. 1. The branched covering $f(z) = z^4$. The thick arrow is C_1. The three thin curves between 1 and $\exp(2\pi ik/4)$ $(k = 1, 2, 3)$ and the constant curve $t \mapsto 1$ form the radial r.

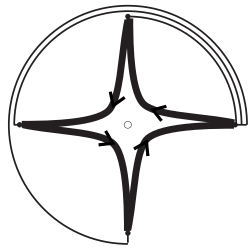

Fig. 2. The thick arrows are $f^{-1}(C_1)$.

since $f_{r',\dagger}(\gamma) = ((1, 1) + C_2^{-1}C_1^{-1} \cdot (2, 1))f_{r,\dagger}(\gamma)((1, 1) + C_1C_2 \cdot (2, 2))$.

We take another generator chain (C_1', C_2', C_3', C_4') such that C_1' is homotopic to $C_1C_2C_1C_2^{-1}C_1^{-1}$, C_2' is homotopic to $C_1C_2C_1^{-1}$ and $C_3' = C_3, C_4' = C_4$. Remark that C_1 is homotopic to $C_2'^{-1}C_1'C_2'$ and C_2 is homotopic to $C_2'^{-1}C_1'^{-1}C_2'C_1'C_2'$. Then the fundamental system of $f_{r',\dagger}$ is

$$\begin{cases} C_1' \mapsto C_2'^{-1}C_1'C_2'C_3'^{-1} \cdot (1, 2) + C_3'C_2^{-1}C_1'^{-1}C_2' \cdot (2, 1) \\ C_2' \mapsto C_2'^{-1}C_1'C_2' \cdot (1, 1) + C_3' \cdot (2, 2) \\ C_3' \mapsto (1, 1) + C_2'^{-1}C_1'^{-1}C_2'C_1'C_2' \cdot (2, 2) \end{cases},$$

since $f_{r',\dagger}(C_1') = C_1C_3^{-1} \cdot (1, 2) + C_3C_1^{-1} \cdot (2, 1)$, $f_{r',\dagger}(C_2') = C_1 \cdot (1, 1) + C_3$ and $f_{r',\dagger}(C_3') = (1, 1) + C_2 \cdot (2, 2)$.

Lemma 3.5. *Suppose a homeomorphism $\phi : (S^2, A) \to (S^2, A)$ satisfies $\phi_{l,\dagger} = \mathrm{id}$ for some l. Then $\phi : S^2 - A \to S^2 - A$ is isotopic to the identity in $S^2 - A$.*

Proof. Let $H : (S^2 - A) \times I \to S^2 - A$ be a homotopy such that $H(\cdot, 0) = \mathrm{id}$ and $H(x, \cdot) = l$, where x is the basepoint. Then $h = H(\cdot, 1)$ is homotopic to the identity, and the induced homomorphisms $\phi_*^{-1}, h_* : \pi_1(S^2 - A, \phi(x)) \to \pi_1(S^2 - A, x)$ coincides. Therefore ϕ is homotopic to the identity (for example see [7], Chapter VI, Exercise F),

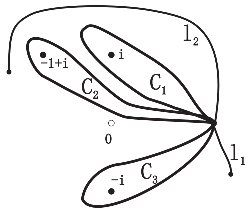

Fig. 3. The branched covering $f(z) = z^2 + i$. The closed curves C_1, C_2, C_3 together with a closed curve homotopic to $C_3^{-1}C_2^{-1}C_1^{-1}$ form a generator chain. The curves l_1 and l_2 form the radial r.

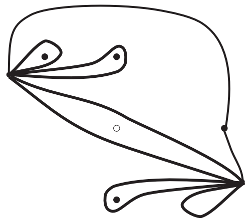

Fig. 4. The closed curves are $f^{-1}(C_k)$ $(k = 1, 2, 3)$

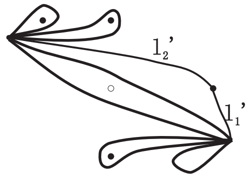

Fig. 5. Another radial r'.

and hence isotopic to the identity (see [5]). $\square$

Theorem 3.6. *Suppose two furnished branched coverings* (f_1, A) *and* (f_2, A) *satisfies* $(f_1)_\dagger = (f_2)_\dagger$. *Then there exist homeomorphisms* ϕ, ϕ' *of* $S^2 - A$ *isotopic to the identity such that* $f_1 = \phi f_2 \phi'$.

Proof. Let r and r' be the radials of f_1 and f_2 with radial points $x_1, x_2, \ldots, x_d$ and $x'_1, x'_2, \ldots, x'_d$. Let us take a generator chain $(C_1, C_2, \ldots, C_n)$ with basepoint x. By D_i, we denote the disc bounded by C_i as in the definition. By $\gamma_{i,k}$, we denote the component of $f_1^{-1}(C_i)$ that has the endpoint x_k and x_m, where $m = p_2(f_1)_\dagger(C_i)(k)$. We similarly define $\gamma'_{i,k}$ for f_2. Since $(f_1)_\dagger = (f_2)_\dagger$, we have $p_2(f_1)_\dagger(C_i)(k) = p_2(f_2)_\dagger(C_i)(k)$, and hence there exists a homeomorphism $\phi : f_1^{-1}\left(\bigcup_{i=1}^n C_i\right) \to f_2^{-1}\left(\bigcup_{i=1}^n C_i\right)$ satisfying $f_1 = f_2 \circ \phi$. Let $E_{i,1}, E_{i,2}, \ldots, E_{i,l}$ and $E'_{i,1}, E'_{i,2}, \ldots, E'_{i,l}$ be the components of $f_1^{-1}(D_i)$ and $f_2^{-1}(D_i)$ respectively, which are simply connected, for D_i contains at most one critical value. Remark that $N_{i,t} = \{k \mid \gamma_{i,k} \subset \partial E_{i,t}\}$ and $N'_{i,t} = \{k \mid \gamma'_{i,k} \subset \partial E_{i,t}\}$ are orbits of $p_2(f_1)_\dagger(C_i) = p_2(f_2)_\dagger(C_i)$, and l is the number of the orbits. We may assume that $E_{i,t}$ corresponds to $E'_{i,t}$ for each $t = 1, 2, \dots, l$, namely, $N_{i,t} = N'_{i,t}$. Therefore ϕ can be extended to a homeomorphism $\phi : \overline{\bigcup_{i,t} E_{i,t}} \to \overline{\bigcup_{i,t} E'_{i,t}}$ satisfying $\phi(E_{i,t}) = E'_{i,t}$ and $f_1 = f_2 \circ \phi$. Each of $E = f_1^{-1}\left(S^2 - \bigcup_i D_i\right)$ and $E' = f_2^{-1}\left(S^2 - \bigcup_i D_i\right)$ consists of d simply connected domains, on which f_1 and f_2 are one-to-one respectively. Thus ϕ can be extended to a homeomorphism $\phi : S^2 \to S^2$ satisfying $f_1 = f_2 \circ \phi$.

We show ϕ is isotopic to the identity. Beforehand we take the generator chain such that $x \in E'$. Even if we change the radial of f_2, we can take a radial of f_1 such that $(f_1)_\dagger = (f_2)_\dagger$. Since $f_1 = f_2 \circ \phi$, we can take a radial of ϕ such that $(f_1)_\dagger = \phi_\dagger (f_2)_\dagger$. Remark that $\overline{E'}$ is connected and that the radial points belong to the boundary of E'. Therefore we can take a radial r' of f_2 such that the image $r'(Q_d)$ is included in $\overline{E'}$ and r' is homotopic to an injective radial. Since each $E'_{i,t}$ contains at most one points of A and $\overline{E'}$ does not intersect A, we can define an injection $A \ni a \mapsto (i(a), t(a))$ by $a \in E'_{i(a),t(a)}$. Since the boundary of $E'_{i(a_k),t(a_k)}$ is homotopic to C_k, there exist $1 \le m \le d$, $s \in \mathbb{Z}$ and $Y_k \in \pi_1(S^2 - A, x)$ such that $p_1((f_2)_\dagger(C^s_{i(a_k)}))(m) = Y_k^{-1} C_k Y_k$. From $(f_1)_\dagger = \phi_\dagger (f_2)_\dagger$ we have $\phi_\dagger(Y_k^{-1} C_k Y_k) = Y_k^{-1} C_k Y_k$. By the choice of the radial r', there exist simple closed curves $C'_1, C'_2 \dots, C'_n$ disjoint except at the basepoint x such that C'_k is homotopic to $Y_k^{-1} C_k Y_k$. Thus $\{Y_k^{-1} C_k Y_k \mid k = 1, 2, \dots, n-1\}$ generates $\pi_1(S^2 - A, x)$. Therefore $\phi_\dagger = \mathrm{id}$, and by Lemma 3.5 ϕ is isotopic to the identity. $\square$

Corollary 3.7. *Let (f_1, A) and (f_2, A) be furnished branched coverings. If there exist a homeomorphism $\phi : (S^2, A) \to (S^2, A)$ and $g \in G_1$ such that $(f_1)_\dagger = g^{-1}(\phi_\dagger^{-1}(f_2)_\dagger \phi_\dagger) g$, then (f_1, A) and (f_2, A) are equivalent.*

Thus the fundamental system is the description of the furnished branched covering. We can consider that giving a fundamental system is equal to giving a furnished branched covering. Now, we restate our question: Let (f_1, A) and (f_2, A) be furnished branched coverings. When fundamental systems of $(f_1)_\dagger$ and $(f_2)_\dagger$ are given, can we know the existence of a homeomorphism $\phi : (S^2, A) \to (S^2, A)$ and $g \in G_1$ such that $(f_1)_\dagger = g^{-1}(\phi_\dagger^{-1}(f_2)_\dagger \phi_\dagger) g$?

In several cases, the Thurston equivalence can be directly checked by the description.

EXAMPLE. Let (f_1, A), (f_2, A) and (f_3, A) be furnished branched coverings as follows: the induced homomorphisms $(f_1)_\dagger$, $(f_2)_\dagger$ and $(f_3)_\dagger$ have the fundamental sys-

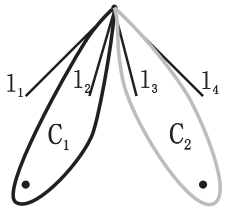

Fig. 6. The branched coverings f_1, f_2, f_3 of degree 4. $\#A = 3$. The closed curves C_1, C_2 together with a closed curve homotopic to $C_2^{-1}C_1^{-1}$ form a generator chain.

Fig. 7. The closed curves are $f_1^{-1}(C_k)$ $(k = 1, 2)$

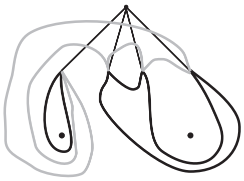

Fig. 8. The closed curves are $f_2^{-1}(C_k)$ $(k = 1, 2)$

Fig. 9. The closed curves are $f_3^{-1}(C_k)$ $(k = 1, 2)$

tems (see Fig. 6–9)

$$\begin{cases} C_1 \to C_1 \cdot (1,1) + (2,4) + (3,2) + (4,3) \\ C_2 \to C_1 \cdot (1,2) + (2,3) + C_2^{-1} \cdot (3,4) + C_2 C_1^{-1} \cdot (4,1), \end{cases}$$

$$\begin{cases} C_1 \to C_1 \cdot (1,1) + C_2 \cdot (2,4) + (3,2) + C_2^{-1} \cdot (4,3) \\ C_2 \to C_1^{-1} \cdot (1,2) + (2,3) + (3,4) + C_1 \cdot (4,1). \end{cases}$$

and

$$\begin{cases} C_1 \to C_1 \cdot (1,1) + (2,3) + (3,4) + (4,2) \\ C_2 \to (1,2) + C_2^{-1} \cdot (2,4) + C_2 \cdot (3,1) + (4,3). \end{cases}$$

They have the transition matrix $\left(\begin{array}{ccc|cc} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right)$. We try to find an element $g \in G_1$ such that $(f_1)_\dagger = g^{-1}(f_e)_\dagger g$ $(e = 2, 3)$.

(1) $e = 2$. Suppose there exists $g = X_1 \cdot (1,1) + X_2 \cdot (2,2) + X_3 \cdot (3,3) + X_4 \cdot (4,4)$ such that $(f_1)_\dagger = g^{-1}(f_2)_\dagger g$. Then

$$\begin{array}{llll} C_1 = X_1^{-1} C_1 X_1, & 1 = X_2^{-1} C_2 X_4, & 1 = X_3^{-1} X_2, & 1 = X_4^{-1} C_2^{-1} X_3, \\ C_1 = X_1^{-1} C_1^{-1} X_2, & 1 = X_2^{-1} X_3, & C_2^{-1} = X_3^{-1} X_4, & C_2 C_1^{-1} = X_4^{-1} C_1 X_1. \end{array}$$

Therefore

$$\begin{array}{lll} X_1 = C_1^l, & X_4 = C_2^{-1} X_2, & X_3 = X_2, \\ X_2 = C_1 X_1 C_1, & X_4 = C_1 X_1 C_1 C_2^{-1}, & \end{array}$$

where $l \in \mathbb{Z}$. Consequently,

$$X_1 = C_1^l,\ X_2 = X_3 = C_1^{l+2},\ X_4 = C_2^{-1} C_1^{l+2} = C_1^{l+2} C_2^{-1},$$

Thus $l = -2$ and $X_1 = C_1^{-2}, X_2 = X_3 = 1, X_4 = C_2^{-1}$. Conversely, $g = C_1^{-2} \cdot (1,1) + (2,2) + (3,3) + C_2 \cdot (4,4)$ satisfies $(f_1)_\dagger = g^{-1}(f_2)_\dagger g$. Hence there exist homeomorphisms ϕ, ϕ' isotopic to the identity such that $f_1 = \phi f_2 \phi'$.

(2) $e = 3$. We set $b = (1,1) + (2,2) + (3,4) + (4,3)$. Then the fundamental system of $b^{-1}(f_3)_\dagger b$ is

$$\begin{cases} C_1 \to C_1 \cdot (1,1) + (2,4) + (3,2) + (4,3) \\ C_2 \to (1,2) + C_2^{-1} \cdot (2,3) + (3,4) + C_2 \cdot (4,1). \end{cases}$$

Suppose there exists $g = X_1 \cdot (1,1) + X_2 \cdot (2,2) + X_3 \cdot (3,3) + X_4 \cdot (4,4)$ such that $(f_1)_\dagger = g^{-1} b^{-1} (f_3)_\dagger b g$. Then we have $1 = X_4^{-1} X_3$ and $C_2^{-1} = X_3^{-1} X_4$. This is a contradiction. Similarly, a contradiction follows from any other b. Thus there exist no homeomorphisms ϕ, ϕ' isotopic to the identity such that $f_1 = \phi f_3 \phi'$. Moreover we will see that f_1 and f_3 are not weakly equivalent (see §5.1).

EXAMPLE. Let (f_1, A) and (f_2, A) be furnished branched coverings as follows: the induced homomorphism $(f_1)_\dagger$ and $(f_2)_\dagger$ have the fundamental systems (see Fig. 10–12)

$$\begin{cases} C_1 \to C_1 \cdot (1,1) + C_3 \cdot (2,2) \\ C_2 \to C_2 \cdot (1,2) + C_2^{-1} \cdot (2,1) \\ C_3 \to C_2 \cdot (1,1) + (2,2), \end{cases}$$

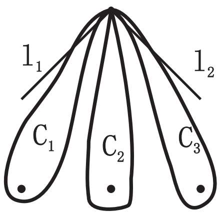

Fig. 10. The branched coverings f_1, f_2 of degree 2. $\#A = 4$. The closed curves C_1, C_2, C_3 together with a closed curve homotopic to $C_3^{-1}C_2^{-1}C_1^{-1}$ form a generator chain.

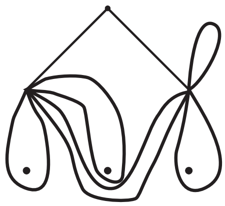

Fig. 11. The closed curves are $f_1^{-1}(C_k)$ $(k = 1, 2, 3)$

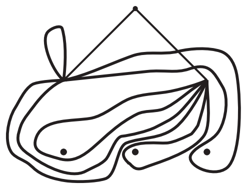

Fig. 12. The closed curves are $f_2^{-1}(C_k)$ $(k = 1, 2, 3)$

and

$$\begin{cases} C_1 \to C_3 \cdot (1,1) + C_1 \cdot (2,2) \\ C_2 \to C_1 \cdot (1,2) + C_1^{-1} \cdot (2,1) \\ C_3 \to (1,1) + C_2 \cdot (2,2). \end{cases}$$

They have the transition matrix $\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right)$.

It is easily seen that there is no $g \in G_1$ such that $(f_1)_\dagger = g^{-1}(f_2)_\dagger g$. From this, however, we cannot conclude that f_1 and f_2 are not equivalent.

Suppose there exist a homeomorphism ϕ and $g \in G_1$ such that

$$(f_1)_\dagger = g^{-1}(\phi_\dagger^{-1}(f_2)_\dagger\phi_\dagger)g. \tag{E}$$

We show that there exists $X, Y \in \pi_1(S^2 - A, x)$ such that

$$(f_2)_\dagger(X^{-1}C_1X) = Y^{-1}X^{-1}C_1XY \cdot (i,i) + \cdots \cdot (j,j) \tag{F}$$

$$(f_2)_\dagger((C_1C_2C_3)^2) = Y^{-1}C_1C_2C_3Y \cdot (i,i) + \cdots \cdot (j,j), \tag{G}$$

where

$$\begin{cases} (i,i) = (1,1), \\ (j,j) = (2,2) \end{cases} \text{or} \begin{cases} (i,i) = (2,2), \\ (j,j) = (1,1). \end{cases}$$

We have $\phi|_A = \mathrm{id}_A$, because, otherwise, the transition matrix of $\phi^{-1}f_2\phi$ differs from the original one. For this reason, we can set $\phi_\dagger(C_1) = X_1^{-1}C_1X_1$, $\phi_\dagger(C_1C_2C_3) = X_4^{-1}C_1C_2C_3X_4$. Set $g = Y_1 \cdot (i',1) + Y_2 \cdot (j',2)$. By (E),

$$\begin{aligned} (f_2)_\dagger\phi_\dagger(C_1) &= \phi_\dagger(g(f_1)_\dagger(C_1)g^{-1}), \\ (f_2)_\dagger\phi_\dagger((C_1C_2C_3)^2) &= \phi_\dagger(g(f_1)_\dagger((C_1C_2C_3)^2)g^{-1}). \end{aligned}$$

Consequently,

$$\begin{aligned} (f_2)_\dagger(X_1^{-1}C_1X_1) &= \phi_\dagger(g(C_1 \cdot (1,1) + C_3 \cdot (2,2))g^{-1}) \\ &= \phi_\dagger(Y_1C_1Y_1^{-1} \cdot (i',i') + Y_2C_3Y_2^{-1} \cdot (j',j')) \\ &= \phi_\dagger(Y_1)X_1^{-1}C_1X_1\phi_\dagger(Y_1)^{-1} \cdot (i',i') + \cdots \cdot (j',j'), \\ (f_2)_\dagger(X_4^{-1}(C_1C_2C_3)^2X_4) &= \phi_\dagger(g(C_1C_2C_3 \cdot (1,1) + C_3C_1C_2 \cdot (2,2))g^{-1}) \\ &= \phi_\dagger(Y_1)X_4^{-1}C_1C_2C_3X_4\phi_\dagger(Y_1)^{-1} \cdot (i',i') + \cdots \cdot (j',j'). \end{aligned}$$

When we write $(f_2)_\dagger(X_4) = X_{4,1} \cdot (i,i') + X_{4,2} \cdot (j,j')$, we have

$$\begin{aligned} &(f_2)_\dagger(X_4X_1^{-1}C_1X_1X_4^{-1}) \\ &\qquad = X_{4,1}\phi_\dagger(Y_1)X_1^{-1}C_1X_1\phi_\dagger(Y_1)^{-1}X_{4,1}^{-1} \cdot (i,i) + \cdots \cdot (j,j), \\ &(f_2)_\dagger((C_1C_2C_3)^2) \\ &\qquad = X_{4,1}\phi_\dagger(Y_1)X_4^{-1}C_1C_2C_3X_4\phi_\dagger(Y_1)^{-1}X_{4,1}^{-1} \cdot (i',i') + \cdots \cdot (j',j'). \end{aligned}$$

Thus (F) and (G) are satisfied for $X = X_1X_4^{-1}$ and $Y = X_4\phi_\dagger(Y_1)^{-1}X_{4,1}^{-1}$. From

$$(f_2)_\dagger((C_1C_2C_3)^2) = C_3C_1C_2 \cdot (1,1) + C_3C_1C_2 \cdot (2,2)$$

and (G), we have $Y = (C_1C_2C_3)^lC_3^{-1}$ ($l \in \mathbb{Z}$). If (F) and (G) are satisfied for $X = X'$ and $Y = (C_1C_2C_3)^lC_3^{-1}$, then it is easily seen that (F) and (G) are satisfied for $X = X'(C_1C_2C_3)^{2l}$ and $Y = C_3^{-1}$. Therefore we can assume $Y = C_3^{-1}$.

By the form of the fundamental system of $(f_2)_\dagger$, we obtain $|Z| \geq |p_1(f_2)_\dagger(Z)(i)|$ for any $Z \in G_0$. In particular, $|X^{-1}C_1X| \geq |Y^{-1}X^{-1}C_1XY|$. Therefore X has the minimal presentation $X = \ldots C_3$, and

$$|X^{-1}C_1X| - |Y^{-1}X^{-1}C_1XY| = 2.$$

Hence there is no cancellation in $p_1(f_2)_\dagger(X^{-1}C_1X)(i)$. Namely, if $X^{-1}C_1X$ has the minimal presentation $C_{k(1)}C_{k(2)}\cdots C_{k(m)}$, then $p_1(f_2)_\dagger(C_{k(l)})(i_l)p_1(f_2)_\dagger(C_{k(l+1)})(i_{l+1}) \neq 1$ for any l, where $i_l = p_2(f_2)_\dagger(C_{k(1)}\cdots C_{k(l-1)})(i)$.

Suppose $i = 2$. Then

$$\begin{aligned}(f_2)_\dagger(X^{-1}C_1X) &= (f_2)_\dagger(C_3^{-1}\cdots C_1\cdots C_3)\\ &= C_2^{-1}\cdots C_1\cdots C_2\cdot(2,2) + \cdots\cdot(1,1)\end{aligned}$$

Consequently, X has the minimal presentation $\ldots C_2C_3$. Similarly, we see that X has the minimal presentation $\ldots C_3C_1C_2C_3$. It is easily seen that $X = C_3C_1C_2C_3$ implies a contradiction. If X has the minimal presentation

$$X = \ldots C_k^{\epsilon}C_3C_1C_2C_3 \ (\epsilon = 1 \text{ or } -1),$$

then

$$(f_2)_\dagger(X^{-1}C_1X) = (f_2)_\dagger(C_3^{-1}C_2^{-1}C_1^{-1}C_3^{-1}C_k^{-\epsilon}\cdots C_1\cdots C_k^{\epsilon}C_3C_1C_2C_3)$$

This is impossible because there is no C_k such that $(f_2)_\dagger(C_k^{\epsilon}) = C_k^{\epsilon}\cdot(\tilde{i},1) + \cdots\cdot(\tilde{j},2)$. We can similarly show the impossibility in the case $i = 1$. Thus f_1 and f_2 are not equivalent.

4. Mapping class groups

Let A be a finite subset of S^2. Consider the set

$$\widetilde{B}_A = \{f \mid f \text{ is an orientation-preserving branched covering, } P_f \subset A,\ f(A) \subset A\}.$$

If (f, A') is a furnished branched covering with $\#A = \#A'$, then there exists $f' \in \widetilde{B}_A$ such that (f, A') and (f', A) are equivalent. Remark that $\widetilde{B}_A$ contains all orientation-preserving homeomorphisms that map A to itself.

It is clear that $\widetilde{B}_A$ is closed under the operation of the composition $(f, g) \mapsto f \circ g$. Therefore we can consider $\widetilde{B}_A$ as a semigroup. By identifying 'isotopic' branched coverings, we obtain the *mapping class semigroup*

$$B_A = \frac{\widetilde{B}_A}{\{\phi \mid \phi \text{ is isotopic to the identity relative to } A\}}.$$

In other words, we identify f and g if there exist homeomorphisms ϕ_1, ϕ_2 isotopic to the identity relative to A such that $g \circ \phi_1 = \phi_2 \circ f$. If f and f' are identified and if g and g' are identified, then so are $f \circ g$ and $f' \circ g'$ by virtue of Proposition 2.1. Therefore the semigroup structure of B_A is well-defined. When we think of a mapping class $f \in B_A$, we denote, by the same symbol f, the representative of f. This will not cause confusion. As for the composition of f and g, we use the notation fg as the member of B_A, and $f \circ g$ as the member of $\widetilde{B}_A$.

We consider a homeomorphism as a branched covering of degree one. Hence the mapping class semigroup includes the *mapping class group*:

$$M(A) = \frac{\{\phi \mid \text{ a homeomorphism, } \phi(A) = A\}}{\{\phi \mid \text{ a homeomorphism isotopic to the identity relative to } A\}} \subset B_A.$$

By $1 \in M(A)$, we denote the unit element of $M(A)$, or the mapping class of the identity. The subgroup

$$M^0(A) = \frac{\{\phi \mid \text{ a homeomorphism, } \phi|A = id\}}{\{\phi \mid \text{ a homeomorphism isotopic to the identity relative to } A\}} \subset M(A)$$

is called the *pure mapping class group*.

REMARK. The transition matrix of $f \in B_A$ is denoted by T_f. If $\#A = n$ and $\#(C_f \cup A) = m$, then T_f is an $n \times m$ matrix. By $\mathrm{Mat}(A)_d$, we denote the set of $n \times m$ matrices with $n \leq m \leq n + 2d - 2$. In other words, if S is a member of $\mathrm{Mat}(A)_d$, then there exists a finite set D_S with $\#D_S = m - n$, and S is a mapping of $A \times (A \sqcup D_S)$ to $\{0\} \cup \mathbb{N}$. For $S \in \mathrm{Mat}(A)_d$ and $S' \in \mathrm{Mat}(A)_{d'}$, we define the product $SS' : A \times (A \sqcup \bigsqcup_{d'} D_S \sqcup D_{S'}) \to \{0\} \cup \mathbb{N}$ by

$$\begin{array}{ll} SS'(x, y) = \sum_{z \in A} S(x, z) S'(z, y) & \text{if} \quad y \in A \cup D_{S'} \\ SS'(x, y) = S(x, y) & \text{if} \quad y \in D_S, \end{array}$$

where $\bigsqcup_{d'} D_S$ is the disjoint union of d' copies of D_S. Then $SS' \in \mathrm{Mat}(A)_{dd'}$. Thus $\bigsqcup_{d \geq 1} \mathrm{Mat}(A)_d$ is a semigroup with respect to the product. We consider the mapping $f \mapsto T_f$ from B_A to $\bigsqcup \mathrm{Mat}(A)_d$. It is easily seen that $T_{fg} = T_f T_g$, and hence the mapping is a 'linear representation'.

Proposition 4.1. *For $f \in B_A$ and $\phi \in M^0(A)$, if $f = f\phi$, then $\phi = 1$.*

Proof. We consider that f and ϕ denote also representatives of f and ϕ, that is, we think of f as a branched covering, and ϕ as a homeomorphism. Then $f = f\phi$ means $\phi_1 \circ f = f \circ \phi \circ \phi_2$, where ϕ_1, ϕ_2 are some homeomorphisms isotopic to the identity relative to A. By Proposition 2.1, we can assume that ϕ_1 is the identity. Therefore it is sufficient to show that ϕ is isotopic to the identity relative to A whenever $f = f \circ \phi$.

Suppose that ϕ satisfies $f = f \circ \phi$. Since the case where f is a homeomorphism is trivial, we assume that f is of degree $d \geq 2$. From Proposition 2.2, A consists of more than one point. Let $\gamma : [0,1] \to S^2$ be a simple path in $S^2 - A$ with the endpoints $\gamma(0) \neq \gamma(1)$ in A. Then $f^{-1}(\gamma)$ consists of d simple paths $\gamma_1, \gamma_2, \ldots, \gamma_d$, which are disjoint except the endpoints. From $f = f \circ \phi$, we see that ϕ induces a permutation of $\{\gamma_1, \gamma_2, \ldots, \gamma_d\}$. Suppose that the permutation has a fixed point, say $\phi(\gamma_1) = \gamma_1$. Then for any path γ homotopic to γ_1 with the endpoints fixed, we have $\phi(\gamma) = \gamma$, and hence $\phi = \mathrm{id}$. Next we suppose that $\phi(\gamma_i) = \gamma_{i+1}$ for $i = 1, 2, \ldots, m-1$ and $\phi(\gamma_m) = \gamma_1$. Then $\gamma_1, \gamma_2, \ldots, \gamma_m$ have the common endpoints, say y and z. We can define the cyclic order of γ_i's around y. If γ_1 is next to γ_{j+1}, then γ_k is next to γ_{j+k} for $k = 1, 2, \ldots, m$, where indices are considered modulo m. By E_k, we denote the simply connected domain bounded by $\gamma_k \cup \gamma_{k+j}$ that includes no γ_i $(1 \leq i \leq m)$. Then $\phi|_{E_k} : E_k \to E_{k+1}$ is bijective. Since $m > 1$, $S^2 - \bigcup_{i=1}^m \gamma_i = \bigcup_{i=1}^m E_k$ contains no points of A, that is, $A = \{y, z\}$. By Proposition 2.3, we can assume that (f, A) is either $\left(z^d, \{0, \infty\}\right)$ or $\left(z^{-d}, \{0, \infty\}\right)$. Hence we have $\phi(z) = \exp\left(2\pi\sqrt{-1}k/d\right)z$ $(k = 0, 1, \ldots, d-1)$, and ϕ is isotopic to the identity. $\square$

DEFINITION. We say f and g in B_A are *p-weakly equivalent* if there exist $\phi_1, \phi_2 \in M^0(A)$ such that $g\phi_1 = \phi_2 f$. In case $\phi_1 = \phi_2$, two mapping classes f and g are said to be *p-equivalent*. We write $f \sim g$ if f and g are p-equivalent.

By Proposition 4.1, for $\phi \in M^0(A)$, if there exists $\phi' \in M^0(A)$ such that $\phi f = f\phi'$, then ϕ' is unique. For $f \in B_A$, we set

$$M_f(A) = \{\phi \in M^0(A) \,|\, \text{there exists } \phi' \in M^0(A) \text{ such that } \phi f = f\phi'\}.$$

Then $M_f(A)$ is a subgroup of $M^0(A)$. From the uniqueness of ϕ', we obtain a homomorphism

$$\lambda_f : M_f(A) \to M^0(A)$$

by

$$\phi f = f\lambda_f(\phi).$$

We define an equivalence relation $\sim_f$ on $M^0(A)$: we say $\phi_1 \sim_f \phi_2$ if there exists $\phi \in M_f(A)$ such that

$$\phi_2 = \lambda_f(\phi^{-1})\phi_1\phi.$$

Proposition 4.2.

$$f\phi_1 \sim f\phi_2 \iff \phi_1 \sim_f \phi_2.$$

Proof. The equivalence $f\phi_1 \sim f\phi_2$ means that there exists $\phi \in M^0(A)$ such that

$$f\phi_1\phi = \phi f\phi_2.$$

Therefore $\phi f = f\phi_1\phi\phi_2^{-1}$, and hence ϕ is contained in $M_f(A)$ and $\lambda_f(\phi) = \phi_1\phi\phi_2^{-1}$. $\square$

From this proposition, classifying $M^0(A)$ by $\sim_f$ is equal to classifying $\{f\phi \mid \phi \in M^0(A)\}$ by the p-equivalence. Moreover, if $g \in B_A$ is p-weakly equivalent to f, then there exists $\phi \in M^0(A)$ such that $g \sim f\phi$. Indeed, g can be expressed as $\phi_1 f\phi_2$; therefore $g = \phi_1 f\phi_2 \sim f\phi_2\phi_1^{-1}$. We write

$$\hat{\Omega}_f = \{\phi f\phi' \mid \phi, \phi' \in M^0(A)\},$$

the p-weak equivalence class including f, and

$$\Omega_f = \frac{\hat{\Omega}_f}{\sim},$$

the set of p-equivalence classes in $\hat{\Omega}_f$. Consequently,

Proposition 4.3. *We have a one-to-one correspondence*

$$\Omega_f \longleftrightarrow \frac{M^0(A)}{\sim_f}.$$

We consider

$$\phi_1 \to \lambda_f(\phi)^{-1}\phi_1\phi$$

as a right action of $M_f(A)$ on $M^0(A)$. Then the equivalence classes of $\sim_f$ are the orbits of the action. Let $\mu : M^0(A) \to \mathrm{GL}(L)$ be a representation of $M^0(A)$ in a linear space L. By

$$\mu(\phi_1) \to \mu(\lambda_f(\phi))^{-1}\mu(\phi_1)\mu(\phi),$$

we define a linear right action ρ of $M_f(A)$ on $\mu(M^0(A))$. Clearly,

Corollary 4.4. *If $f\phi_1 \sim f\phi_2$, then $\mu(\phi_1)$ and $\mu(\phi_2)$ lie in the same orbit of ρ.*

5. Some applications

5.1. #A $\leq$ 3 The pure mapping class group $M^0(A)$ is trivial if $\#A = 2, 3$. This is easily proved from the fact that $M(A)$ is isomorphic to the symmetric group

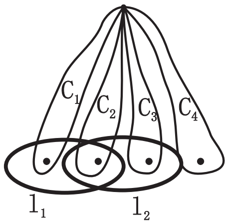

Fig. 13. The generator chain (C_1, C_2, C_3, C_4) and the closed curves l_1, l_2.

on A under this assumption. Therefore, for $f \in B_A$, each p-weak equivalence class $\{\phi f \phi' \,|\phi, \phi' \in M^0(A)\}$ consists of one member f. Consequently,

Proposition 5.1. *Let (f_1, A_1) and (f_2, A_2) be furnished branched coverings with $\#A_1 = \#A_2 \leq 3$. Then (f_1, A_1) and (f_2, A_2) are equivalent if and only if (f_1, A_1) and (f_2, A_2) are weakly equivalent.*

The local type is not a complete invariant, but so is the fundamental system up to conjugation.

Theorem 5.2. *Suppose $\#A \leq 3$. Mapping classes f_1 and f_2 in B_A are p-equivalent if and only if there exist radials r and r' such that $(f_1)_{r,\dagger} = (f_2)_{r',\dagger}$.*

5.2. #A = 4 In the case $\#A \geq 4$, the group $M^0(A)$ is an infinite group. In particular, it is a free group generated by two elements if $\#A = 4$. This section is devoted to the case $\#A = 4$.

We start with the structure of the mapping class group $M(A)$. Refer to [1] for the details of the mapping class groups. We set $A = \{a_1, a_2, a_3, a_4\}$ and take a generator chain $\{C_1, C_2, C_3, C_4\}$. We take simple closed curves l_1 and l_2 such that l_1 is homotopic to C_1C_2 and l_2 is homotopic to C_2C_3 (see Fig. 13). Then l_1 separates $\{a_1, a_2\}$ and $\{a_3, a_4\}$ (i.e. l_1 divide S^2 into two simply connected domains $D_{1,1}$ which contains a_1, a_2, and $D_{1,2}$ which contains a_3, a_4), l_2 separates $\{a_2, a_3\}$ and $\{a_4, a_1\}$. Let σ_1, σ_2 and σ_3 denote 'half Dehn twists' along l_1, l_2 and $-l_1$ respectively. Namely, for example, σ_1 is the homeomorphism that is identity on $D_{1,2}$ and interchanges a_1 and a_2 as shown in Fig. 14. A Dehn twist along a simple closed curve l is defined as a homeomorphism which is the identity outside an annular neighborhood of l and which 'twists' as Fig. 15 inside the neighborhood (see [11]). Remark that the Dehn twist is unique up to isotopy. Then σ_1^2 and σ_3^2 are isotopic to a Dehn twist along l_1 and σ_2^2 is isotopic to a Dehn twist along l_2. The mapping class group $M(A)$ has a finite presen-

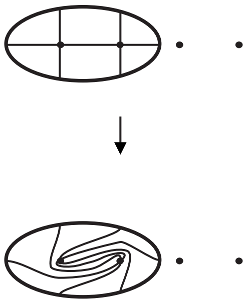

Fig. 14. The ‘harf Dehn twist’ σ_1.

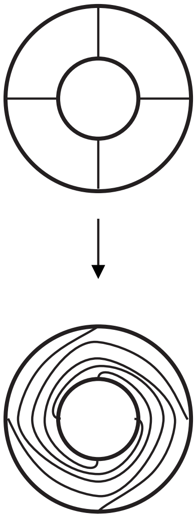

Fig. 15. A Dehn twist.

tation $\langle \sigma_1, \sigma_2, \sigma_3 \mid R_1, R_2, R_3, R_4, R_5 \rangle$, where

$$\begin{aligned} R_1 &= \sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}, \\ R_2 &= \sigma_2\sigma_3\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}, \\ R_3 &= \sigma_1\sigma_3\sigma_1^{-1}\sigma_3^{-1}, \\ R_4 &= \sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1, \\ R_5 &= (\sigma_1\sigma_2\sigma_3)^4. \end{aligned}$$

(Note that $\sigma_i\sigma_j^2\sigma_i = \sigma_j^{-2}$ if $|i - j| = 1$.) By group theoretical calculation, we conclude that the pure mapping class group $M^0(A)$ is the subgroup $\langle \sigma_1^2, \sigma_2^2 \rangle \subset M(A)$. A homomorphism $\mu : M(A) \to PSL(2, \mathbb{Z})$ is defined by

$$\mu(\sigma_1) = \mu(\sigma_3) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mu(\sigma_2) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Then we have

$$\mu(M^0(A)) = \left\langle \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \right\rangle = \Gamma(2),$$

which is the principal congruence subgroup and is known to be a free group. Therefore $M^0(A)$ is a free group generated by two Dehn twists $s_1 = \sigma_1^2, s_2 = \sigma_2^2$.

If $\#A \leq 4$ and f is of degree 2, the structure of Ω_f is completely understood as we will state in the next subsections. The following proposition holds true for any $n = \#A$.

Lemma 5.3. *Let (f_1, A_1) and (f_2, A_2) be furnished branched coverings of degree 2. If they are locally equivalent, then they are weakly equivalent.*

Proof. We write $A_1 = \{a_1, a_2, \dots, a_n\}$ and $A_2 = \{b_1, b_2, \dots, b_n\}$ such that a_i corresponds to b_i ($i = 1, 2, \dots, n$). Suppose a_1, a_n are the critical values of f_1 and b_1, b_n are the critical values of f_2. Let l be a simple path joining a_1 and a_n that touches $a_2, a_3, \dots, a_{n-1}$ in order, and similarly take a simple path l' joining b_1 and b_n. Then $f_1^{-1}(l)$ and $f_2^{-1}(l')$ are simple closed curves. Cyclic orders on A_1 and A_2 can be defined by the closed curves. If the cyclic orders agree, then there exists a homeomorphism $\phi_1, \phi_2 : S^2 \to S^2$ such that $\phi_1(f_1^{-1}(l)) = f_2^{-1}(l')$, $\phi_2(l) = l'$, $\phi_k(a_i) = b_i$ ($k = 1, 2$) and $\phi_2 \circ f_1 = f_2 \circ \phi_1$. Therefore f_1 and f_2 are weakly equivalent. Although the cyclic orders do not agree, we can retake l' so that they agree. Indeed, let us take a closed curve γ as follows: $D \cap A_2 = \{b_1, b_i\}$, where D is one of the domain bounded by γ, $\#(\gamma \cap l') = 1$ if $i = 2$ and $\#(\gamma \cap l') = 3$ if $i = 3, 4, \dots, n-1$. By σ, we denote the Dehn twist along γ. Let us compare the cyclic order on A_2 defined by $f_2^{-1}(l')$ with that defined by $f_2^{-1}(\sigma(l'))$. We can see that the two inverse image of b_i are exchanged (see Fig. 16 and 17. b_1' and b_5' are the critical points such that $f_2(b_1') = b_1$, $f_2(b_5') = b_5$.) From this we conclude that l' can be deformed by finite Dehn twists such that the cyclic order agrees with that defined by $f_1^{-1}(l)$. □

There are thirty local types of furnished branched coverings of degree 2 if $\#A = 4$:
(I) $f(A \cup C_f) \neq A$. 5 types:

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{etc.}$$

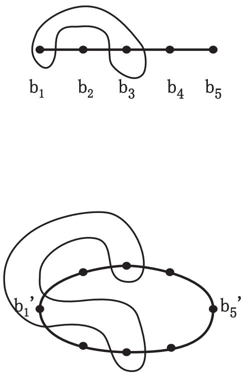

Fig. 16. The closed curve γ and its inverse image $f_2^{-1}(\gamma)$.

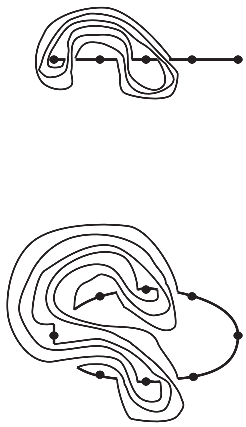

Fig. 17. The path $\sigma(l')$ and its inverse image $f_2^{-1}(\sigma(l'))$.

(II) $f(A \cup C_f) = A, C_f \subset A$. 10 types:

$$\begin{pmatrix} 2&0&0&0\\ 0&0&1&0\\ 0&0&0&2\\ 0&1&0&0 \end{pmatrix}, \quad \begin{pmatrix} 2&0&0&0\\ 0&0&1&0\\ 0&2&0&0\\ 0&0&0&1 \end{pmatrix}, \quad \text{etc.}$$

(III) $f(A \cup C_f) = A, \#(C_f \setminus A) = 1$. 12 types:

$$\left(\begin{array}{cccc|c} 2&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&2\\ 0&1&0&1&0 \end{array}\right), \quad \left(\begin{array}{cccc|c} 2&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&2\\ 0&0&1&1&0 \end{array}\right), \quad \text{etc.}$$

(IV) $f(A \cup C_f) = A, \#(C_f \setminus A) = 2$. 3 types:

$$\left(\begin{array}{cccc|cc} 0&0&0&0&2&0\\ 1&1&0&0&0&0\\ 0&0&0&0&0&2\\ 0&0&1&1&0&0 \end{array}\right), \quad \left(\begin{array}{cccc|cc} 0&0&0&0&2&0\\ 0&1&0&1&0&0\\ 0&0&0&0&0&2\\ 1&0&1&0&0&0 \end{array}\right), \quad \left(\begin{array}{cccc|cc} 0&0&0&0&2&0\\ 1&0&0&1&0&0\\ 0&0&0&0&0&2\\ 0&1&1&0&0&0 \end{array}\right).$$

REMARK. The *type of the orbifold* of a furnished branched covering (f, A) is the smallest function $\nu : A \to \mathbb{N} \cup \{\infty\}$ such that $\nu(x)$ is a multiple of $\nu(y)\deg_y(f)$ for any $y \in f^{-1}(x)$, where $\deg_y(f)$ is the local degree of f at y, considering $\nu(x) = 1$ for $x \notin A$. In the cases (IV), the types of the orbifolds are $(2, 2, 2, 2)$.

We take $\{a_1, a_2, a_3, a_4\}$ such that a_1 and a_3 are the critical values. Then

Proposition 5.4. *In the cases* (II), (III) *and* (IV), *$M_f(A)$ is generated by $\{s_1^2, s_2^2, s_1s_2\}$. In other words, $M_f(A)$ is the kernel of the homomorphism $h : M^0(A) \to \mathbb{Z}/(2)$ defined by $h(s_1) = h(s_2) = 1$.*

DEFINITION. A simple closed curve γ in $S^2 - A$ is called *peripheral* if a disc bounded by γ contains at most one point of A.

Proof. Let l be a non-peripheral simple closed curve, and let ϕ be the Dehn twist along l. Suppose $f^{-1}(l)$ has two components γ_1, γ_2. Then $f : \gamma_i \to l$ is of degree one. By ϕ', we denote the composition of the Dehn twists along γ_1 and γ_2. Then $\phi f = f\phi'$, and so $\phi \in M_f(A)$. From $s_1s_2 = \sigma_1^2\sigma_2^2 = \sigma_1^2\sigma_2^2\sigma_1\sigma_1^{-1} = \sigma_1\sigma_2^{-2}\sigma_1^{-1} = \sigma_1 s_2^{-1}\sigma_1^{-1}$, we see that $(s_1s_2)^{-1}$ is the Dehn twist along l_0, which is homotopic to C_1C_3. Since the inverse image of l_0 has two components, $s_1s_2 \in M_f(A)$.

In case $\gamma = f^{-1}(l)$ has only one component, $f : \gamma \to l$ is of degree two. By ϕ', we denote the Dehn twist along γ. Then $\phi^2 f = f\phi'$. The inverse image of l_i $(i = 1, 2)$ has one component, and hence $s_1^2, s_2^2 \in M_f(A)$. Thus $\mathrm{Ker}(h) \subset M_f(A)$.

To complete the proof, we show that $s_1 \notin M_f(A)$. Let D_1 and D_2 denote the discs bounded by $\gamma = f^{-1}(l_1)$, and let E_1 and E_2 denote the discs bounded by l_1. Since we are working with the cases (II), (III) and (IV), each of $f^{-1}(a_2) \cap A$ and $f^{-1}(a_4) \cap A$ consists of at least one points. Set $f^{-1}(a_2) = \{c_1, c_2\}$ and $f^{-1}(a_4) = \{c_3, c_4\}$. Let α be a simple path between a_2 and a_4, and let α_1, α_2 be the components of $f^{-1}(\alpha)$. If α_1 joins c_1 and c_3, then α_2 joins c_2 and c_4. The two components β_1, β_2 of $(s_1 f)^{-1}(\alpha)$ join c_1 and c_4, and c_2 and c_3 respectively. Thus by no homeomorphism in $M^0(A)$ the paths β_1, β_2 can be moved to α_1, α_2. $\square$

The types of (I) are reduced to the case $\#A = 3$. Indeed, $\phi f = f$ for each $\phi \in M_f(A)$, because each inverse image of l_i bounds a domain that contains at most one point of A.

5.2.1. Cases (II) and (III) We choose a couple of model examples and investigate them.

Let f_1 denote the mapping class in B_A with the induced homomorphism $(f_1)_\dagger$:

$$\begin{cases} C_1 \to & (1,2) + C_1 \cdot (2,1) \\ C_2 \to & C_3 \cdot (1,1) + \quad (2,2) \\ C_3 \to & C_3^{-1}C_2^{-1}C_1^{-1} \cdot (1,2) + \quad (2,1) \end{cases}$$

The transition matrix is $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

Theorem 5.5. *The p-weak equivalence class* (= *the local equivalence class*) *including f_1 consists of three p-equivalence classes*:

$$\Omega_{f_1} = \{[f_1], [f_1 s_1], [f_1 s_2]\},$$

where $[f]$ means the p-equivalence class including f.

Proof. Since $(f_1)_\dagger(C_1C_2) = (1,2) + C_1C_3 \cdot (2,1)$ and $(f_1)_\dagger(C_2C_3)_\dagger = C_2^{-1}C_1^{-1} \cdot (1,2) + (2,1)$, we conclude that $\lambda_{f_1}(s_1^2) = s_2^{-1}s_1^{-1}$ is the Dehn twist with respect to C_1C_3 and $\lambda_{f_1}(s_2^2) = s_1$ is the Dehn twist with respect to C_1C_2. Since $s_2s_1 = \sigma_2^2\sigma_1^2 = \sigma_2^2\sigma_1^2\sigma_2\sigma_2^{-1} = \sigma_2\sigma_1^{-2}\sigma_2^{-1} = \sigma_2 s_1^{-1}\sigma_2^{-1}$, s_2s_1 is the Dehn twist with respect to $C_3^{-1}C_2^{-1}C_1^{-1}C_2$. Since $(f_1)_\dagger(C_3^{-1}C_2^{-1}C_1^{-1}C_2) = (1,1) + C_1C_2C_1^{-1} \cdot (2,2)$, $\lambda_{f_1}(s_2s_1) = 1$. Therefore

$$\lambda_{f_1} : \begin{cases} s_1^2 & \to s_2^{-1}s_1^{-1} \\ s_2^2 & \to s_1 \\ s_2s_1 & \to 1 \\ s_2^{-1}s_1 & \to s_1^{-1} \\ s_2s_1^{-1} & \to s_1s_2 \\ s_2^{-1}s_1^{-1} & \to s_2 \end{cases}.$$

We prove

$$\frac{M^0(A)}{\sim_{f_1}} = \{[1], [s_1], [s_2]\}.$$

Since $M^0(A)$ is the free group generated by s_1, s_2, the length of an element is defined. For $\phi \in M_{f_1}(A)$, we have $|\lambda_{f_1}(\phi)| \le |\phi|$.

Lemma 5.6. *For $\phi \in M^0(A)$, there exists $\phi' \in M^0(A)$ such that $\phi \sim_{f_1} \phi'$ and $|\phi'| \le 1$.*

Proof. Let $\phi = s_{i(1)}^{\epsilon(1)} s_{i(2)}^{\epsilon(2)} \ldots s_{i(m)}^{\epsilon(m)}$ be the minimal presentation of ϕ. Then

$$\phi \sim_{f_1} \lambda_{f_1}\left(s_{i(m-1)}^{\epsilon(m-1)} s_{i(m)}^{\epsilon(m)}\right) s_{i(1)}^{\epsilon(1)} s_{i(2)}^{\epsilon(2)} \ldots s_{i(m-2)}^{\epsilon(m-2)}. \tag{H}$$

Suppose that there exists no $\phi' \in M^0(A)$ such that $\phi \sim_{f_1} \phi'$ and $|\phi'| \leq 1$. We can assume that there exists no $\phi' \in M^0(A)$ such that $\phi \sim_{f_1} \phi'$ and $|\phi'| < |\phi|$. By (H), $|\lambda_{f_1}\left(s_{i(m-2k-1)}^{\epsilon(m-2k-1)} s_{i(m-2k)}^{\epsilon(m-2k)}\right)| = 2$ for $k = 0, 1, \ldots, (m-2)/2$ (or $(m-3)/2$). Therefore $s_{i(m-2k-1)}^{\epsilon(m-2k-1)} s_{i(m-2k)}^{\epsilon(m-2k)} = s_1^2,\ s_1^{-2},\ s_2 s_1^{-1}$ or $s_1 s_2^{-1}$. Moreover, $\lambda_{f_1}\left(s_{i(1)}^{\epsilon(1)} s_{i(2)}^{\epsilon(2)} \ldots s_{i(m)}^{\epsilon(m)}\right) = (s_1 s_2)^{\pm m/2}$ if m is even, $\lambda_{f_1}\left(s_{i(2)}^{\epsilon(2)} s_{i(3)}^{\epsilon(3)} \ldots s_{i(m)}^{\epsilon(m)}\right) = (s_1 s_2)^{\pm(m-1)/2}$ if m is odd. We write $\phi_1 = (s_1 s_2)^{\pm m/2}$ if m is even, $\phi_1 = (s_1 s_2)^{\pm(m-1)/2} s_{i(1)}^{\epsilon(1)}$ if m is odd. If m is even,

$$\phi \sim_{f_1} \phi_1 \sim_{f_1} \lambda_{f_1}\left((s_1 s_2)^{\pm m/2}\right) = s_2^{\mp m/2}.$$

This is a contradiction. If m is odd,

$$\phi \sim_{f_1} \phi_1 \sim_{f_1} \lambda_{f_1}\left((s_2 s_1)^{\pm(m-2\mp 1)/2} s_2 s_{i(1)}^{\epsilon(1)}\right) s_1 = \lambda_{f_1}\left(s_2 s_{i(1)}^{\epsilon(1)}\right) s_1,$$

which leads to a contradiction. □

Since

$$s_1^{-1} \sim_{f_1} \lambda_{f_1}(s_1 s_2^{-1})^{-1} s_1^{-1} s_1 s_2^{-1} = s_1 \text{ and } s_2^{-1} \sim_{f_1} \lambda_{f_1}(s_2 s_1)^{-1} s_2^{-1} s_2 s_1 = s_1,$$

we remain to show that $1, s_1, s_2$ are not equivalent to one another. Assume $1 \sim_{f_1} s_1$. Then there exists $\phi \in M_{f_1}(A)$ such that $1 = \lambda_{f_1}(\phi^{-1}) s_1 \phi$. Since $|\lambda_{f_1}(\phi)| \leq |\phi|$, we have $|\lambda_{f_1}(\phi)| = |\phi| - 1$. Therefore the minimal presentation of ϕ consists of some of $s_1^{\pm 2}$, $(s_2 s_1^{-1})^{\pm 1}$ and only one of $s_2^{\pm 2}$, $(s_1 s_2)^{\pm 1}$. Moreover we have $|\lambda_{f_1}(\phi^{-1}) s_1| = |\phi|$. Thus the minimal presentation of ϕ is $s_1^{-1} \cdots$, and the minimal presentation of $\lambda_{f_1}(\phi)$ is not $s_1 \cdots$. This is a contradiction. We can similarly show the other inequalities. This completes the proof of the theorem. □

Set $f_2 = f_1 \sigma_1$. The transition matrix of f_2 is

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The fundamental system is

$$\begin{cases} C_1 \to & (1,2) + C_2 \cdot (2,1) \\ C_2 \to & C_3 \cdot (1,1) + (2,2) \\ C_3 \to & C_3^{-1} C_2^{-1} C_1^{-1} \cdot (1,2) + (2,1) \end{cases}.$$

Theorem 5.7. *The p-weak equivalence class* (*=the local equivalence class*) *including f_2 consists of infinite p-equivalence classes*:

$$\Omega_{f_2} = \{[f_2], [f_2 s_1]\} \cup \{[f_2 s_2 (s_1 s_2)^n] \mid n \in \mathbb{Z}\}.$$

Proof. Since $\lambda_{f_2}(\phi) = \sigma_1^{-1}\lambda_{f_1}(\phi)\sigma_1$, we have

$$\lambda_{f_2} : \begin{cases} s_1^2 & \to \sigma_1^{-1}s_2^{-1}s_1^{-1}\sigma_1 = s_2 \\ s_2^2 & \to \sigma_1^{-1}s_1\sigma_1 = s_1 \\ s_2 s_1 & \to \sigma_1^{-1}\sigma_1 = 1 \\ s_2^{-1}s_1 & \to \sigma_1^{-1}s_1^{-1}\sigma_1 = s_1^{-1} \\ s_2 s_1^{-1} & \to \sigma_1^{-1}s_1 s_2\sigma_1 = s_2^{-1} \\ s_2^{-1}s_1^{-1} & \to \sigma_1^{-1}s_2\sigma_1 = s_1^{-1}s_2^{-1} \end{cases}.$$

Let ϕ be an element of $M^0(A)$. If there exists no $\phi' \in M^0(A)$ such that $\phi \sim_{f_2} \phi'$ and $|\phi'| < |\phi|$, then either $|\phi| = 1$ or $\phi = s_2(s_1s_2)^n$. We can show that $1, s_1, s_2$ and s_1^{-1} are not equivalent to one another in a fashion similar to the previous theorem. In order to complete the proof, it is sufficient to show the following.

Lemma 5.8. *Suppose that* $\phi' \sim_{f_2} s_2(s_1s_2)^n$ *and* $|\phi'| \leq |2n+1|$. *Then* $\phi' = s_2(s_1s_2)^n$.

Proof. If $n = 0$ or -1, the statement is true. Suppose $\phi = s_2(s_1s_2)^n$ with $n > 0$. Assume that there exists $\alpha \in M_{f_2}(A)$ such that $\phi' = \lambda_{f_2}(\alpha)^{-1}\phi\alpha$ and $|\phi'| \leq 2n + 1$. We can assume $\alpha \neq 1$. Remark that $|\lambda_{f_2}(\alpha)| \leq |\alpha|$. Let $s_{j(1)}^{\delta(1)}\ldots$ be the minimal presentation of α. Suppose $s_{j(1)}^{\delta(1)} \neq s_2^{-1}$. Then

$$\begin{array}{ll} |\phi'| \geq |\lambda_{f_2}(\alpha)^{-1}\phi| + |\alpha| & \text{if } |\lambda_{f_2}(\alpha)^{-1}| \leq |\phi|, \\ |\phi'| \geq |\alpha| - \big(|\lambda_{f_2}(\alpha)^{-1}| - |\phi|\big) & \text{if } |\lambda_{f_2}(\alpha)^{-1}| > |\phi|. \end{array}$$

Therefore

$$|\phi'| \geq |\phi| - |\lambda_{f_2}(\alpha)| + |\alpha| \geq |\phi| = 2n + 1.$$

Thus $|\phi'| = 2n+1$, and hence $|\lambda_{f_2}(\alpha)| = |\alpha|$ and the minimal presentation of $\lambda_{f_2}(\alpha)$ is $s_2s_1s_2s_1\cdots$. Consequently, $\alpha = (s_1s_2)^m$. Then

$$\phi' = \lambda_{f_2}(\alpha)^{-1}\phi\alpha = (s_2s_1)^{-m}s_2(s_1s_2)^n(s_1s_2)^m = s_2(s_1s_2)^n.$$

If $s_{j(1)}^{\delta(1)} = s_2^{-1}$, then $\lambda_{f_2}\big(s_{j(1)}^{\delta(1)}s_{j(2)}^{\delta(2)}\big)^{-1}\phi s_{j(1)}^{\delta(1)}s_{j(2)}^{\delta(2)}$ has three possibilities:

(i) $$\lambda_{f_2}(s_2^{-1}s_1)^{-1}\phi s_2^{-1}s_1 = s_1s_2(s_1s_2)^{n-1}s_1^2$$

(ii) $$\lambda_{f_2}(s_2^{-1}s_1^{-1})^{-1}\phi s_2^{-1}s_1^{-1} = s_2(s_1s_2)^n$$

(iii) $$\lambda_{f_2}(s_2^{-2})^{-1}\phi s_2^{-2} = s_1 s_2 (s_1 s_2)^{n-1} s_1 s_2^{-1}.$$

Since $s_{j(2)}^{\delta(2)} \neq s_{j(3)}^{-\delta(3)}$, we can similarly prove that $|\lambda_{f_2}\left(s_{j(1)}^{\delta(1)} s_{j(2)}^{\delta(2)}\right)^{-1}\phi s_{j(1)}^{\delta(1)} s_{j(2)}^{\delta(2)}| \leq |\phi'|$. Consequently, (i) and (iii) are impossible, and hence $\alpha = (s_2^{-1}s_1^{-1})^k$. Therefore $\phi' = s_2(s_1s_2)^n$. The proof of the case $n < -1$ is similar. $\square$

REMARK. In the p-weak equivalence class $\{\phi f_2 \phi' \mid \phi, \phi' \in M^0(A)\}$, the homeomorphism $s_1 s_2$ has a special meaning. This is the Dehn twist along the curve l_0, which has the following property: there exists a component $l' \subset (f_2 \circ s_2)^{-1}(l_0)$ isotopic to l_0 and $f_2 : l' \to l_0$ is of degree one. As to the p-weak equivalence class including f_1, there is no curve satisfying this property. In general, the p-equivalence classes of (II) and (III) are divided into two categories by the property. According to the category, the p-weak equivalence class consists of infinite p-equivalence classes or consists of finite p-equivalence classes.

The following conjecture would be natural: Let (f, A) be a furnished branched covering. Suppose there exists a non-peripheral simple closed curve $l \subset S^2 - A$ satisfying that there exists only one component $l' \subset f^{-1}(l)$ isotopic to l such that $f : l' \to l$ is of degree one. By σ, we denote the Dehn twist along l. Then $f \not\sim f\sigma^n$ for any integer $n \neq 0$.

By proofs similar to the previous theorems, we recognize that this conjecture is true for all types of (II) and (III). Note that generally the conjecture is not true when we do not assume the uniqueness of l'. Indeed, if $f^{-1}(l)$ has two components l_1, l_2 isotopic to l such that $f : l_1 \to l$ and $f : l_2 \to l$ are of degree one, and if the other components are peripheral, then $\sigma f = f\sigma^2$. Therefore $f \sim \sigma^k f \sigma^{-k} = f\sigma^k$ for any k.

5.2.2. Case (IV) In order to study the case (IV), we need some different approaches. While we cannot explicitly describe the p-equivalence classes, we construct a complete invariant.

Let f_3 be the mapping class in B_A with the induced homomorphism $(f_3)_\dagger$:

$$\begin{cases} C_1 \to C_1 \cdot (1,2) + C_1^{-1} \cdot (2,1) \\ C_2 \to C_1 \cdot (1,1) + C_2 \cdot (2,2) \\ C_3 \to C_3^{-1} \cdot (1,2) + C_3 \cdot (2,1) \end{cases}.$$

We have

$$\lambda_{f_3} : \begin{cases} s_1^2 \to s_1 \\ s_2^2 \to s_2^{-1} s_1 s_2 \\ s_2 s_1 \to s_2^2 \end{cases}.$$

We write

$$R = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}, S_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix},$$

and set

$$\mathcal{L} = \{TRT' \mid T, T' \in \Gamma(2)\},$$

where

$$\Gamma(2) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in PSL(2,\mathbb{Z}) \,\middle|\, \begin{array}{ll} x, w = 1 & \pmod 2 \\ y, z = 0 & \pmod 2 \end{array} \right\}.$$

Note that we identify X and $-X$ in $\mathcal{L}$. It is easily seen that

$$\mathcal{L} = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \,\middle|\, \begin{array}{llll} x = 1 & \pmod 2, & y = 0 & \pmod 2 \\ z = 1 & \pmod 2, & w = 0 & \pmod 2 \end{array}, xw - yz = 2 \right\}.$$

An isomorphism $\mu : M^0(A) \to \Gamma(2)$ is defined by $\mu(s_1) = S_1$, $\mu(s_2) = S_2$. Then μ can be extended on $\hat{\Omega}_{f_3} = \{\phi f_3 \phi' \mid \phi, \phi' \in M^0(A)\}$ by $\mu(f_3) = R$. Indeed, by calculation, we have

$$S_1^2 R = RS_1, \;\; S_2^2 R = RS_2^{-1} S_1 S_2, \;\; S_2 S_1 R = RS_2^{-2}.$$

Lemma 5.9. *$\mu : \hat{\Omega}_{f_3} \to \mathcal{L}$ is bijective.*

Proof. Since $\mu : M^0(A) \to \Gamma(2)$ is isomorphic, μ is surjective. We set

$$\Gamma_R = \{X \in \Gamma(2) \mid \text{ there exists } X' \in \Gamma(2) \text{ such that } XR = RX'\}.$$

By calculation,

$$\begin{aligned} \Gamma_R &= \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \Gamma(2) \,\middle|\, x - y + z - w = 0 \pmod 4 \right\} \\ &= \langle S_1^2, S_2^2, S_1 S_2 \rangle. \end{aligned}$$

Suppose $\mu(\phi f_3 \phi') = \mu(f_3)$. Since $M_{f_3}(A) = \langle s_1^2, s_2^2, s_1 s_2 \rangle$, we can assume that $|\phi| \leq 1$. If $\phi = 1$, then $R\mu(\phi') = R$. Therefore $\phi' = 1$. In case $|\phi| = 1$ we can assume $\phi = s_1$. Then $S_1 R = R\mu(\phi')^{-1}$. This implies $S_1 \in \Gamma_R$, and a contradiction. Thus μ is injective. $\square$

Theorem 5.10. *For $f, f' \in \Omega_{f_3}$, $f \sim f'$ if and only if there exist $S \in PSL(2,\mathbb{Z})$ such that $\mu(f) = S^{-1}\mu(f')S$.*

Proof. It is sufficient to show that S is a member of $\Gamma(2)$ provided $Z \in \mathcal{L}$, $X \in PSL(2,\mathbb{Z})$ and $S^{-1}ZX \in \mathcal{L}$. We can check this by calculation. $\square$

EXAMPLE.
(1) $f_3 s_1^n$ ($n \in \mathbb{Z}$) are p-equivalent to one another. Indeed,

$$\mu(f_3 s_1^n) = RS_1^n = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 4n-1 & 2 \end{pmatrix}.$$

For $X_n = \left(\begin{smallmatrix} 1 & 0 \\ -4n & 1 \end{smallmatrix}\right)$, we have

$$\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = X_n^{-1} \begin{pmatrix} 1 & 0 \\ 4n-1 & 2 \end{pmatrix} X_n.$$

(2) It is easily seen that $f \not\sim f'$ if $|\operatorname{trace}(\mu(f))| \neq |\operatorname{trace}(\mu(f'))|$. For example, $f_3 s_1^n s_2$ ($n \in \mathbb{Z}$) are not p-equivalent to one another since $\operatorname{trace}(\mu(f_3 s_1^n s_2)) = 8n + 5$. But the trace is not a complete invariant.

The representation μ has a topological meaning. Let T^2 be the 2-torus, and let $h : (T^2, \tilde{A}) \to (S^2, A)$ be a 2-fold branched covering with branch points $\tilde{A}$. Then the branched covering $\phi f_3 \phi' : S^2 \to S^2$ can be lifted to a 2-fold covering $\tilde{f} : T^2 \to T^2$. It is easily seen that $\mu(\phi f_3 \phi')$ is a matrix representation of $\tilde{f}_* : H_1(T^2) \to H_1(T^2)$. This is generalized in the next subsection.

5.2.3. branched coverings with $(2, 2, 2, 2)$-orbifolds Let (f, A) be a furnished branched covering with $(2, 2, 2, 2)$-orbifolds without restriction on the degree of f. In this subsection, we construct a representation $\mu : \hat{\Omega}_f \to \{TRT' \mid T, T' \in \Gamma(2)\}$, where R is some 2×2 matrix. Using this representation, we can check the p-equivalence.

Fix a generator chain (C_1, C_2, C_3, C_4). We first show that $f^{-1}(A) = A \cup C_f$. It is clear that $A \cup C_f \subset f^{-1}(A)$ and that $A \cap C_f = \emptyset$. Since all critical points are of degree two, $\#C_f = 2d - 2$. Therefore $\# f^{-1}(A) = 4d - (2d - 2) = 2d + 2 = 4 + 2d - 2 = \#(A \cup C_f)$. Thus $f^{-1}(A) = A \cup C_f$.

Consider the induced homomorphism

$$f_{\gamma,*} : \pi_1(S^2 - f^{-1}(A), x) \xrightarrow{f_*} \pi_1(S^2 - A, f(x)) \xrightarrow{\gamma_*} \pi_1(S^2 - A, x),$$

where γ is a path between x and $f(x)$. Set $L_2 = \{a \in \pi_1(S^2 - A, x) : |a| \text{ is even}\}$, where $|a|$ is the length of a with respect to the generator chain (C_1, C_2, C_3, C_4). Remark that L_2 is independent of the choice of the generator chain. Using the inclusion $i : S^2 - f^{-1}(A) \to S^2 - A$, we define $L_2' = i_*^{-1}(L_2)$. Then $\pi_1(S^2 - f^{-1}(A), x)$ is generated by $\{C_1', C_2', C_3', u_1, u_2, \ldots, u_k\}$, where $i_*(C_l') = C_l$ ($l = 1, 2, 3, 4$), and u_j corresponds to a closed curve enclosing a point of C_f such that $C_1' C_2' C_3' C_4' u_1 u_2 \ldots u_k$ is trivial in $S^2 - f^{-1}(A)$. From the fact that all critical points are of degree two and $f^{-1}(A) = A \cup C_f$, we obtain $f_{\gamma,*}(L_2') \subset L_2$. By $ab(f_{\gamma,*}) : ab(L_2') \to ab(L_2)$, we denote the abelization of $f_{\gamma,*} : L_2' \to L_2$. We set

$$w_1 = C_1^2, w_2 = C_2^2, w_3 = C_3^2, w_4 = C_1 C_2, w_5 = C_2 C_3,$$

and

$$w'_1 = {C'_1}^2,\ w'_2 = {C'_2}^2,\ w'_3 = {C'_3}^2,\ w'_4 = C'_1C'_2,\ w'_5 = C'_2C'_3.$$

Since L_2 is the free group generated by $\{w_1, w_2, w_3, w_4, w_5\}$, $ab(L_2)$ is the free module generated by $\{w_1, w_2, w_3, w_4, w_5\}$. Similarly, $ab(L'_2)$ is the free module generated by

$$\{w'_1, w'_2, w'_3, w'_4, w'_5, u_1, v_1, u_2, v_2, \dots, u_k, v_k\},$$

where $v_j = {C'_1}^{-1}u_jC'_1$. Then

$$\bar{L}_2 = \frac{ab(L_2)}{\langle w_1, w_2, w_3\rangle}$$

is the free module generated by w_4 and w_5, and

$$\bar{L}'_2 = \frac{ab(L'_2)}{\langle w'_1, w'_2, w'_3, u_1, v_1, u_2, v_2 \dots, u_k, v_k\rangle}$$

is the free module generated by w'_4 and w'_5. Since

$$ab(f_{\gamma,*})\big(\langle w'_1, w'_2, w'_3, u_1, v_1, u_2, v_2 \dots, u_k, v_k\rangle\big) \subset \langle w_1, w_2, w_3\rangle,$$

we can reduce $ab(f_{\gamma,*})$ to $\bar{f}_{\gamma,*} : \bar{L}'_2 \to \bar{L}_2$, which is independent of the choice of C'_l and u_j (depends on only the generator chain (C_1, C_2, C_3, C_4) and the path γ). By setting the basis w_4, w_5, we obtain $\mu(f)_\gamma$, the matrix representation of $\bar{f}_{\gamma,*}$. Namely, the matrix representation of $ab(f_{\gamma,*})$ is

$$\begin{array}{cc} & \begin{array}{cc} \bar{L}'_2 & \quad K' \end{array} \\ \begin{array}{c} \bar{L}_2 \\ K \end{array} & \begin{pmatrix} \mu(f)_\gamma & 0 \\ * & * \end{pmatrix} \end{array},$$

where $K = \langle w_1, w_2, w_3\rangle$ and $K' = \langle w'_1, w'_2, w'_3, u_1, v_1, u_2, v_2 \dots, u_k, v_k\rangle$. The matrix $\mu(f)_\gamma$ is a member of $\mathrm{Mat}(2, \mathbb{Z})$, the set of 2×2 matrices with integer components. When γ is replaced by $\gamma' = C_1\gamma$, we have $\mu(f)_{\gamma'} = -\mu(f)_\gamma$, since $C_1w_4C_1^{-1} = C_1^2C_2C_1^{-1} = w_1w_2^{-1}w_4^{-1}$ and $C_1w_5C_1^{-1} = C_1C_2C_3C_1^{-1} = w_4w_3w_5^{-1}w_2w_4^{-1}$. In case that $\gamma' = \alpha\gamma$ with $\alpha \in L_2$, it is clear that $\mu(f)_{\gamma'} = \mu(f)_\gamma$. Thus the matrix representation depends on only (C_1, C_2, C_3, C_4) up to ± 1, and is independent of γ. We consider the matrix as a member of $\mathrm{Mat}(2, \mathbb{Z})/\pm 1$ and denote by $\mu(f)$.

EXAMPLE. Let (f, A) be a furnished branched covering with induced homomorphism

$$f_{\gamma,*}\begin{cases} C_1' \to C_1C_2C_1^{-1} \\ C_2' \to C_2 \\ C_3' \to C_3 \\ u_1 \to C_1C_2(C_3^{-1}C_2^{-1}C_1^{-1})^2C_2^{-1}C_1^{-1} \\ u_2 \to C_1C_2C_1^2C_2^{-1}C_1^{-1} \end{cases}.$$

Then

$$f_{\gamma,*}:\begin{cases} w_1' \to C_1C_2^2C_1^{-1} & = w_4w_2w_4^{-1} \\ w_2' \to C_2^2 & = w_2 \\ w_3' \to C_3^2 & = w_3 \\ w_4' \to C_1C_2C_1^{-1}C_2 & = w_4w_1^{-1}w_4 \\ w_5' \to C_2C_3 & = w_5 \\ u_1 \to C_1C_2(C_3^{-1}C_2^{-1}C_1^{-1})^2C_2^{-1}C_1^{-1} & = w_4w_5^{-1}w_1^{-1}w_4w_2^{-1}w_5w_3^{-1}w_4^{-2} \\ v_1 \to C_1(C_3^{-1}C_2^{-1}C_1^{-1})^2C_1^{-1} & = w_4w_2^{-1}w_5w_3^{-1}w_4^{-1}w_5^{-1}w_1^{-1} \\ u_2 \to C_1C_2C_1^2C_2^{-1}C_1^{-1} & = w_4w_1w_4^{-1} \\ v_2 \to C_1^2 & = w_1 \end{cases}.$$

Thus the matrix representation of $ab(f_{\gamma,*})$ is

$$\begin{array}{c} \\ w_4 \\ w_5 \\ w_1 \\ w_2 \\ w_3 \end{array}\begin{array}{c} \begin{array}{ccccccccc} w_4' & w_5' & w_1' & w_2' & w_3' & u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \end{pmatrix} \end{array},$$

and we obtain

$$\mu(f) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

For a homeomorphism $\phi : (S^2, A) \to (S^2, A)$, we similarly define the homomorphism $\overline{\phi}_{l,*} : \overline{L}_2 \to \overline{L}_2$ and the matrix representation $\mu(\phi)$. Clearly, $\mu(\phi^{-1}) = \mu(\phi)^{-1}$. For a homeomorphism $\phi' : (S^2, f^{-1}(A)) \to (S^2, f^{-1}(A))$ such that $\phi'(A) = A$, we define the homomorphism $\overline{\phi}'_{l',*} : \overline{L}_2' \to \overline{L}_2'$ and the matrix representation $\mu(\phi')$. Therefore if $g = \phi f \phi'$, we have $\mu(g) = \mu(\phi)\mu(f)\mu(\phi')$. When we extend ϕ' to $\tilde{\phi}' : (S^2, A) \to (S^2, A)$, we have $\mu(\phi') = \mu(\tilde{\phi}')$. It is clear that μ is a representation of the subsemigroup

$$B_A(2, 2, 2, 2) = \{f \in B_A \mid f \text{ has } (2, 2, 2, 2)\text{-orbifold}\} \cup M(A).$$

If f and g in $B_A(2,2,2,2)$ are p-equivalent, then there exists $\phi \in M^0(A)$ such that $g = \phi^{-1} f \phi$. Therefore $\mu(g) = \mu(\phi)^{-1}\mu(f)\mu(\phi)$. We will show the converse, that is, if $\mu(g) = \mu(\phi)^{-1}\mu(f)\mu(\phi)$ with $T_f = T_g$, then f and g are p-equivalent.

Let us take simple closed curves β_1 and β_2 homotopic to C_1C_2 and C_2C_3 respectively. Let $\alpha_1, \alpha_1' \subset S^2 - \beta_1$ be simple paths joining a_1 and a_2, and a_3 and a_4 respectively; let $\alpha_2, \alpha_2' \subset S^2 - \beta_2$ be simple paths joining a_2 and a_3, and a_4 and a_1 respectively. We can assume that $\alpha_1, \alpha_1', \alpha_2$ and α_2' are disjoint except at the endpoints. Cutting S^2 along α_1 and α_1', we obtain an annulus N with boundary $\tilde{\alpha}_1 \cup \tilde{\alpha}_1{}'$. We take N^+, a copy of N. Identifying the boundaries of N and N^+ (gluing $\tilde{\alpha}_1$ to $\tilde{\alpha}_1'^{+}$, and $\tilde{\alpha}_1'$ to $\tilde{\alpha}_1^+$), we obtain a 2-torus T^2 and a branched covering $h : T^2 \to S^2$ such that $h \circ j = h$, where j is defined by $j(x) = x^+$, $j(x^+) = x$ for $x \in N$. By $\tilde{A} = \{\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4\}$, we denote the branch point of h, namely $\tilde{A} = h^{-1}(A)$. Then $h_*(\pi_1(T^2 - \tilde{A}, \tilde{x})) = L_2$ and $h_*(\pi_1(T^2 - (\tilde{A} \cup h^{-1}(C_f)), \tilde{x})) = L_2'$, where $h(\tilde{x}) = x$. Therefore there exists a covering $\tilde{f} : T^2 - (\tilde{A} \cup h^{-1}(C_f)) \to T^2 - \tilde{A}$ such that $h\tilde{f} = fh$. It is easily seen that $\tilde{f}$ can be extended to a covering $\tilde{f} : T^2 \to T^2$. Then the induced homomorphism $\tilde{f}_* : H_1(T^2; \mathbb{Z}) \to H_1(T^2; \mathbb{Z})$ is identified with $\overline{f}_{\gamma,*} : \overline{L}_2' \to \overline{L}_2$ if a lift of γ joins $\tilde{x}$ and $\tilde{f}(\tilde{x})$.

If $\phi : (S^2, A) \to (S^2, A)$ is a homeomorphism, the lift $\tilde{\phi} : T^2 \to T^2$ is a homeomorphism. The matrix $\mu(\phi)$ is determined by the following. For some γ, we have

$$(\sigma_1)_{\gamma,*} : \begin{cases} C_1 \to C_1C_2C_1^{-1} \\ C_2 \to C_1 \\ C_3 \to C_3 \end{cases}, (\sigma_2)_{\gamma,*} : \begin{cases} C_1 \to C_1 \\ C_2 \to C_2C_3C_2^{-1} \\ C_3 \to C_2 \end{cases},$$

$$(\sigma_3)_{\gamma,*} : \begin{cases} C_1 \to C_1 \\ C_2 \to C_2 \\ C_3 \to C_2^{-1}C_1^{-1}C_3^{-1} \end{cases}.$$

Therefore

$$(\sigma_1)_{\gamma,*} : \begin{cases} w_1 \to w_4w_2w_4^{-1} \\ w_2 \to w_1 \\ w_3 \to w_3 \\ w_4 \to w_4 \\ w_5 \to w_4w_2^{-1}w_5 \end{cases}, (\sigma_2)_{\gamma,*} : \begin{cases} w_1 \to w_1 \\ w_2 \to w_3 \\ w_3 \to w_5w_3w_5^{-1} \\ w_4 \to w_4w_3w_5^{-1} \\ w_5 \to w_5 \end{cases},$$

$$(\sigma_3)_{\gamma,*} : \begin{cases} w_1 \to w_1 \\ w_2 \to w_2 \\ w_3 \to w_4^{-1}w_5^{-1}w_1^{-1}w_4w_2^{-1}w_5w_3^{-1} \\ w_4 \to w_4 \\ w_5 \to w_1^{-1}w_4w_2^{-1}w_5w_3^{-1} \end{cases}.$$

So,

$$\overline{(\sigma_1)}_{\gamma,*}:\begin{cases} w_4 \to w_4 \\ w_5 \to w_4+w_5 \end{cases}, \ \overline{(\sigma_2)}_{\gamma,*}\begin{cases} w_4 \to w_4-w_5 \\ w_5 \to w_5 \end{cases},$$
$$\overline{(\sigma_3)}_{\gamma,*}:\begin{cases} w_4 \to w_4 \\ w_5 \to w_4+w_5 \end{cases},$$

and

$$\mu(\sigma_1)=\mu(\sigma_3)=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mu(\sigma_2)=\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

The matrix $\mu(f)$ can be computed from the fundamental system of f.

Lemma 5.11. *Let (f,A) be a furnished branched covering of degree d with $(2,2,2,2)$-orbifold. Suppose $C\subset S^2-A$ is a non-peripheral simple closed curve (i.e. C is a simple closed curve corresponding to a member of L_2). Then all components of $f^{-1}(C)$ are non-peripheral and isotopic to one another. Moreover, there exists d' such that $f:C'\to C$ is of degree d' for each components $C'\in f^{-1}(C)$.*

Proof. Let γ_i be a simple path in D_i joining two points of A. Each component of $f^{-1}(\gamma_i)$ is either a simple closed curve containing no points of A or a simple path with both endpoints in A (recall that $f^{-1}(A)=A\cup C_f$). This implies the first assertion. Each component of $f^{-1}(D_i)$ is either an annulus or a disc. It is easily seen that f has common degree on the two boundaries of the annulus. □

Let us take the minimal $k_1,k_2>0$ satisfying

$$p_2f_\dagger\big((C_1C_2)^{k_1}\big)(1)=1, \ \ p_2f_\dagger\big((C_2C_3)^{k_2}\big)(1)=1.$$

By the lemma, $g_1=p_1f_\dagger\big((C_1C_2)^{k_1}\big)(1)$ and $g_2=p_1f_\dagger\big((C_2C_3)^{k_2}\big)(1)$ belong to L_2. Therefore there exist $g_1'\in i_*^{-1}(g_1), g_2'\in i_*^{-1}(g_2)$ such that $f_{\gamma,*}(g_1')=w_4^{k_1}$, $f_{\gamma,*}(g_2')=w_5^{k_2}$, where γ is the first spoke of the radial. Suppose g_1 and g_2 are carried to $c_1w_4+c_2w_5$ and $d_1w_4+d_2w_5$ by the projection $L_2\to\overline{L}_2$. Then $\overline{f}_{\gamma,*}(c_1w_4+c_2w_5)=k_1w_4$ and $\overline{f}_{\gamma,*}(d_1w_4+d_2w_5)=k_2w_5$. Since $c_1w_4+c_2w_5$ and $d_1w_4+d_2w_5$ are linearly independent, we obtain

$$\mu(f)=\begin{pmatrix} c_1/k_1 & c_2/k_1 \\ d_1/k_2 & d_2/k_2 \end{pmatrix}^{-1}.$$

For example, $\mu(f_3)$ in §5.2.2 is computed as follows. Since $(f_3)_\dagger(C_1C_2)=C_1C_2\cdot(1,2)+(2,1)$ and $(f_3)_\dagger(C_2C_3)=C_1C_3^{-1}\cdot(1,2)+C_2C_3\cdot(2,1)$, we have

$$\overline{(f_3)}_{\gamma,*}:\begin{cases} w_4 & \to 2w_4 \\ w_4+2w_5 & \to 2w_5 \end{cases}.$$

Therefore

$$\mu(f_3) = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}.$$

Lemma 5.12. *Let $\beta : S^1 \to S^2 - A$ be a non-peripheral simple closed curve. Then $h^{-1}(\beta)$ consists of two non-trivial simple closed curve $\tilde{\beta}, \tilde{\beta}_* \subset T^2$, which are homotopic to each other in T^2, but which are not homotopic in $T^2 - \tilde{A}$. Let β' be another non-peripheral simple closed curve. If lifts $\tilde{\beta}, \tilde{\beta}' \in T^2$ are homotopic in T^2, then β and β' are isotopic in $S^2 - A$.*

Proof. If β is a simple closed curve β_0 satisfying $\beta_0 \cap \alpha_1 = \beta_0 \cap \alpha_1' = \emptyset$, then the assertion is true. In general, there exists a homeomorphism $\phi : (S^2, A) \to (S^2, A)$ such that $\phi(\beta) = \beta_0$. Since ϕ is lifted to a homeomorphism $\tilde{\phi} : (T^2, \tilde{A}) \to (T^2, \tilde{A})$, the theorem is true. $\square$

REMARK. From this lemma, there exists an injection from the set of isotopy classes of non-peripheral simple closed curves in $S^2 - A$ to the set of isotopy classes of non-trivial simple closed curves in T^2. The class of a non-trivial simple closed curve in T^2 is determined by a pair of relatively prime integers $\binom{a}{b}$. Let β be a non-peripheral simple closed curve in $S^2 - A$, and $\tilde{\beta}$ be a component of $h^{-1}(\beta)$. By $c(\beta)$, we denote the class $\binom{a}{b}$ of $\tilde{\beta}$. If β' is a component of $f^{-1}(\beta)$, then $\mu(f)c(\beta') = c(\beta)$.

Theorem 5.13. *Let (f, A) and (f', A) be furnished branched coverings with $(2, 2, 2, 2)$-orbifolds. If there exists $a_k \in A$ such that $f(a_k) = f'(a_k)$ and if $\mu(f) = \mu(f')$, then f and f' are equal in B_A.*

Proof. We can assume that $a_k = a_2$. Consider a universal covering $\tau : \mathbb{R}^2 \to T^2$ such that

$$h \circ \tau\left(0, \frac{1}{2}\right) = a_1, h \circ \tau(0, 0) = a_2, h \circ \tau\left(\frac{1}{2}, 0\right) = a_3, h \circ \tau\left(\frac{1}{2}, \frac{1}{2}\right) = a_4,$$

and

$$h \circ \tau\left(0 \times \left[0, \frac{1}{2}\right]\right) = \alpha_1, h \circ \tau\left(\left[0, \frac{1}{2}\right] \times 0\right) = \alpha_2,$$
$$h \circ \tau\left(\frac{1}{2} \times \left[0, \frac{1}{2}\right]\right) = \alpha_1', h \circ \tau\left(\left[0, \frac{1}{2}\right] \times \frac{1}{2}\right) = \alpha_2'.$$

Then for $x_1, x_2 \in \mathbb{R}^2$,

$$\tau(x_1) = \tau(x_2) \Leftrightarrow x_1 - x_2 \in \mathbb{Z}^2,$$

$$h \circ \tau(x_1) = h \circ \tau(x_2) \Leftrightarrow x_1 - x_2 \in \mathbb{Z}^2 \text{ or } x_1 + x_2 \in \mathbb{Z}^2.$$

We set $b = f(a_2)$. Let $\hat{b}$ be the point in $\{(x, y) \mid 0 \le x < 1, 0 \le y < 1\}$ such that $h \circ \tau(\hat{b}) = b$. Then

$$\hat{F} : x \mapsto \mu(f)(x) + \hat{b}$$

is considered as a mapping of $\mathbb{R}^2$ to itself, and two (branched) coverings $\tilde{F} : T^2 \to T^2$, $F : S^2 \to S^2$ are induced by $\tilde{F}$. On the other hand, f is lifted to $\tilde{f} : T^2 \to T^2$, and further to $\hat{f} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\hat{f}(0, 0) = \hat{b}$. If $x_1 - x_2 \in \mathbb{Z}^2$, then

$$\hat{f}(x_1) - \hat{f}(x_2) = \mu(f)(x_1 - x_2)$$

since $\mu(f)$ is considered as $\tilde{f}_* : \pi_1(T^2, \tilde{x}) \to \pi_1(T^2, \tilde{f}(\tilde{x}))$. In particular, for a *lattice point* x_1 (i.e. a point in $\mathbb{Z}^2$), we have $\hat{f}(x_1) = \hat{F}(x_1)$. A $1/2$-*lattice point* is a point of $1/2\ \mathbb{Z}^2 = \{(x/2, y/2) \mid x, y \in \mathbb{Z}\}$. Let x_1 be a $1/2$-lattice point. For any $x \in \mathbb{R}^2$, we have $h \circ \tau(x_1 + x) = h \circ \tau(x_1 - x)$. Therefore $h \circ \tau \circ \hat{f}(x_1 + x) = h \circ \tau \circ \hat{f}(x_1 - x)$. Consequently, $\hat{f}(x_1 + x) + \hat{f}(x_1 - x) \in \mathbb{Z}^2$, and that is a constant function with respect to x. Considering $x = 0$ and $x = x_1$, we have $2\hat{f}(x_1) = \hat{f}(2x_1) + \hat{b} = \hat{f}(x_1 + x) + \hat{f}(x_1 - x)$. In other words, if $y_1 + y_2 = y_3 \in \mathbb{Z}^2$, then $\hat{f}(y_1) + \hat{f}(y_2) = \hat{f}(y_3) + \hat{b} = \hat{F}(y_1) + \hat{F}(y_2)$. In particular, we have $\hat{F} = \hat{f}$ on the $1/2$-lattice points. Therefore F and f are locally equivalent.

The homeomorphism $\hat{\phi} = \hat{f}^{-1} \circ \hat{F}$ satisfies $\hat{F} = \hat{f} \circ \hat{\phi}$ and $\hat{\phi}(x_1) = x_1$ for $x_1 \in 1/2\ \mathbb{Z}^2$. If $x_1 \pm x_2 = x_3 \in \mathbb{Z}^2$, then $\hat{\phi}(x_1) \pm \hat{\phi}(x_2) = x_3$. Therefore $\hat{\phi}$ induces the homeomorphisms $\tilde{\phi} : T^2 \to T^2$ and $\phi : S^2 \to S^2$ such that $\tilde{F} = \tilde{f} \circ \tilde{\phi}$ and $F = f \circ \phi$.

From the above remark, for a non-peripheral simple closed curve β in $S^2 - A$, a component of $f^{-1}(\beta)$ is isotopic to a component of $F^{-1}(\beta)$. Therefore $\phi(\alpha_1) \cup \phi(\alpha_2) \cup \phi(\alpha'_1) \cup \phi(\alpha'_2)$ is isotopic to $\alpha_1 \cup \alpha_2 \cup \alpha'_1 \cup \alpha'_2$ with A kept fixed. Consequently, ϕ is isotopic to the identity relative to A. This completes the proof. □

Corollary 5.14.
(1) *Suppose that* $f, f' \in \tilde{B}_A$ *are branched coverings with* $(2, 2, 2, 2)$*-orbifold. Then* f *and* f' *are p-equivalent if and only if* $T_f = T_{f'}$ *and there exists* $S \in \Gamma(2)$ *such that* $\mu(f) = S^{-1}\mu(f')S$.
(2) *Suppose that* $f, f' \in \tilde{B}_A$ *are branched coverings with* $(2, 2, 2, 2)$*-orbifold and* f *has a fixed point in* A*. Then* (f, A) *and* (f', A) *are equivalent if and only if they are locally equivalent and there exists* $S \in PSL(2, \mathbb{Z})$ *such that* $\mu(f) = S^{-1}\mu(f')S$.

Proof. The first half is an immediate consequence of the previous theorem. The last half is proved as follows.

Suppose (f, A) and (f', A) are locally equivalent and there $\mu(f) = S^{-1}\mu(f')S$ for some $S \in PSL(2, \mathbb{Z})$. Without loss of generality, we may assume $f(a_k) = f'(a_k)$ for $k = 1, 2, 3$ and $f(a_4) = f'(a_4) = a_4$. Recall that $\sigma_1(a_4) = \sigma_2(a_4) = a_4$. Since $PSL(2, \mathbb{Z})$

is generated by $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, we have $\mu(f) = \mu(s^{-1}f's)$ for some $s \in \langle\sigma_1, \sigma_2\rangle$. Applying the previous theorem, we see that f and f' are equivalent. $\square$

REMARK. Unlike Theorem 5.10, in general, it is untrue that a matrix $S \in PSL(2,\mathbb{Z})$ with $\mu(f) = S^{-1}\mu(f')S$ belongs to $\Gamma(2)$. For example, we set

$$X = \begin{pmatrix}1&0\\0&3\end{pmatrix} \text{ and } X' = \begin{pmatrix}1&0\\0&3\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix} = \begin{pmatrix}1&2\\0&3\end{pmatrix}.$$

Then X and X' induce two branched coverings $f, f' \in \tilde{B}_A$ such that $f(a_k) = f'(a_k) = a_k$ for $k = 1,2,3,4$. Since

$$\begin{pmatrix}1&1\\0&1\end{pmatrix} X \begin{pmatrix}1&-1\\0&1\end{pmatrix} = X',$$

we conclude (f, A) and (f', A) are equivalent. However, there exists no $S \in \Gamma(2)$ such that $X = S^{-1}X'S$, and hence f and f' are not p-equivalent in $\bar{B}_A$.

We turn back to the example f_3 in §5.2.2. We say two matrices X and Y in $\mathcal{L}$ are *equivalent* if there exists $S \in PSL(2,\mathbb{Z})$ such that $X = S^{-1}YS$.

For two given matrices X and Y in $\mathcal{L}$, we have an algorithm to check whether X and Y are equivalent. If trace $X = k$, we can write $X = \left(\begin{smallmatrix}2x+k & y\\ z & -2x\end{smallmatrix}\right)$, where x is an integer, z an odd integer, y an even integer and $-2x(2x+k) - yz = 2$. The eigenvalues are $\alpha = \left(k+\sqrt{m}\right)/2, \overline{\alpha} = \left(k-\sqrt{m}\right)/2$, where $m = k^2-8$. Let $\left(\begin{smallmatrix}a_1\\a_2\end{smallmatrix}\right)$ be an eigenvector with eigenvalue α, and $\left(\begin{smallmatrix}a_1'\\a_2'\end{smallmatrix}\right)$ an eigenvector with eigenvalue $\overline{\alpha}$. We have $\xi = a_1/a_2 = \left(4x + k + \sqrt{m}\right)/2z, \overline{\xi} = a_1'/a_2' = \left(4x + k - \sqrt{m}\right)/2z$. We say ξ is the *base* of X. Remark that if $z, 4x + k$ and y are relatively prime and m is not a square, the minimal polynomial of $\xi, \overline{\xi}$ is $zt^2 - (4x+k)t - y$. Then the *discriminant* $D(\xi) = (4x+k)^2 + 4zy = m$. If b is the greatest common divisor of $z, 4x+k, y$, then $D(\xi) = m/b^2$.

Suppose there exists $S = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ such that $X = S^{-1}YS$. Then X and Y have the common eigenvalues $\alpha, \overline{\alpha}$. By $\left(\begin{smallmatrix}b_1\\b_2\end{smallmatrix}\right), \left(\begin{smallmatrix}b_1'\\b_2'\end{smallmatrix}\right)$, we denote eigenvectors of Y corresponding to $\alpha,\overline{\alpha}$. We write $\eta = b_1/b_2$ and $\overline{\eta} = b_1'/b_2'$. Since $S\left(\begin{smallmatrix}a_1\\a_2\end{smallmatrix}\right)$ is an eigenvector of Y with eigenvalue α, we have

$$\eta = \frac{a\xi + b}{c\xi + d}. \tag{I}$$

We say two algebraic numbers ξ and η are *modularly equivalent* if they have the relation (I) with $ad - bc = 1$. Conversely, suppose X and Y have the same eigenvalues. If ξ and η are modularly equivalent, then X and Y are equivalent.

Thus our problem is concerned with the arithmetic of quadratic number fields. We consult a textbook of number theory, for example, Section 2.7 of [2].

Consider the case $m < 0$. Since k is odd, $k = \pm 1$. This case has a special significance: the condition trace $\mu(f) = \pm 1$ is necessary and sufficient for (f, A) to be equivalent to a rational map [4].

Proposition 5.15. *If $f \in \Omega_{f_3}$ satisfies* $\mathrm{trace}(\mu(f)) = \pm 1$, *then f is equivalent to either $f_3 s_2^{-1}$ or $s_1 f_3 s_2$.*

Proof. We first see that

$$\mu(f_3 s_2^{-1}) = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}, \mu(s_1 f_3 s_2) = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}.$$

The bases are $\left(-1 - \sqrt{-7}\right)/2$, $\left(1 + \sqrt{-7}\right)/2$.

Recall the fundamental domain $P = \{x + \sqrt{-1}y \mid y > 0, -1/2 < x \leq 1/2, x^2 + y^2 \geq 1\ (x^2 + y^2 > 1 \text{ if } -1/2 < x < 0)\}$ of the modular group $PSL(2, \mathbb{Z})$. By calculation, we have only one quadratic number $\theta = (1 + \sqrt{-7})/2 \in P$ such that $D(\theta) = -7$. Set $\mu(f) = X = \left(\begin{smallmatrix} 2x+1 & y \\ z & -2x \end{smallmatrix}\right)$. Since $-m = 7$ is a prime, $D(\xi)$ the discriminant of the base ξ is $m = -7$. Remark that $D(\xi) = D(\eta)$ if ξ and η are modularly equivalent. Therefore, if $z > 0$, there exists $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2, \mathbb{Z})$ such that $\theta = (a\xi + b)/(c\xi + d)$. If $z < 0$, then ξ is modularly equivalent to the complex conjugate of θ. According to the sign of z, f is equivalent to $s_1 f_3 s_2$ or $f_3 s_2^{-1}$. $\square$

Proposition 5.16. *Suppose $m = k^2 - 8 > 0$ is square-free. By h, we denote the number of ideal classes of the quadratic number field $\mathbb{Q}\left(\sqrt{m}\right)$, and by h', the number of ideal classes in the narrow sense. Then*

$$\frac{h'}{h} \leq \# \frac{\{f \in \Omega_{f_3} \mid \mathrm{trace}\, \mu(f) = \pm k\}}{\sim} \leq h'.$$

In particular, in the case $h = 1$,

$$\# \frac{\{f \in \Omega_{f_3} \mid \mathrm{trace}\, \mu(f) = \pm k\}}{\sim} = h'.$$

Proof. We have h quadratic numbers $\theta_1, \theta_2, \ldots, \theta_h$ such that for any quadratic number ξ with $D(\xi) = m$, there exist i and integers a, b, c, d with $\theta_i = (a\xi + b)/(c\xi + d)$ and $ad - bc = \pm 1$. Since m is square-free, the discriminant of the base of $\mu(f)$ is equal to m. Therefore the base of $\mu(f)$ is modularly equivalent to one of $\theta_1^{\pm 1}, \theta_2^{\pm 1}, \ldots, \theta_h^{\pm 1}$. If $h = h'$, then θ_i and θ_i^{-1} are modularly equivalent. Consequently,

$$\# \frac{\{f \in \Omega_{f_3} \mid \mathrm{trace}\, \mu(f) = \pm k\}}{\sim} \leq h'.$$

If $h' = 2h$, then $\left(\begin{smallmatrix} k & 2 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} k & -2 \\ 1 & 0 \end{smallmatrix}\right)$ are not modularly equivalent, and hence

$$\#\frac{\{f \in \Omega_{f_3} \mid \operatorname{trace}\mu(f) = \pm k\}}{\sim} \geq 2. \qquad \square$$

In the case $m > 0$, the modular equivalence can be checked by the continued fraction expansions of ξ and η.

EXAMPLE.

(1) $k = 5$, $m = 17$. The bases of

$$\mu(f_3 s_2^{-1} s_1) = \begin{pmatrix} 5 & 2 \\ -1 & 0 \end{pmatrix} = X, \quad \mu(s_1 f_3 s_1^{-1} s_2^{-1} s_1^2) = \begin{pmatrix} 9 & 2 \\ -19 & -4 \end{pmatrix} = Y$$

are $\xi = \left(-5 - \sqrt{17}\right)/2$, $\eta = \left(-13 - \sqrt{17}\right)/38$. The continued fraction expansions of ξ and η:

$$\text{(J)} \qquad \xi = -5 + \cfrac{1}{2 + \cfrac{1}{\theta}}, \quad \eta = -1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \theta}}},$$

where

$$\text{(K)} \qquad \theta = \frac{3 + \sqrt{17}}{2} = \left[\ \overline{3, 1, 1}\ \right] = 3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\theta}}}.$$

By (J) and (K),

$$\xi = \frac{-9\theta - 5}{2\theta + 1}, \quad \eta = \frac{-\theta - 1}{2\theta + 3}, \quad \theta = \frac{7\theta + 4}{2\theta + 1}.$$

Therefore a matrix $S \in GL(2, \mathbb{Z})$ satisfying $X = S^{-1} X S$ is written in the form $S = \left(\begin{smallmatrix} 9 & 4 \\ -2 & -1 \end{smallmatrix}\right)^n$. On the other hand,

$$Y = \begin{pmatrix} 17 & 4 \\ -4 & -1 \end{pmatrix}^{-1} X \begin{pmatrix} 17 & 4 \\ -4 & -1 \end{pmatrix}.$$

So, we finally obtain

$$Y = \begin{pmatrix} 17 & 4 \\ -4 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 9 & 4 \\ -2 & -1 \end{pmatrix}^{-n} X \begin{pmatrix} 9 & 4 \\ -2 & -1 \end{pmatrix}^{n} \begin{pmatrix} 17 & 4 \\ -4 & -1 \end{pmatrix}.$$

Since the determinant of $\begin{pmatrix} 9 & 4 \\ -2 & -1 \end{pmatrix}^n \begin{pmatrix} 17 & 4 \\ -4 & -1 \end{pmatrix}$ is 1 for odd n, we have $f_3 s_2^{-1} s_1 \sim s_1 f_3 s_1^{-1} s_2^{-1} s_1^2$.

In general, all matrices $X \in \mathcal{L}$ satisfying trace $X = 5$ are modularly equivalent, for $h' = 1$. Indeed, for X and Y, there exist $S, T \in GL(2, \mathbb{Z})$ such that $Y = T^{-1} X T$, $X = S^{-1} X S$ and $|S| = -1$. Thus either $|T| = 1$ or $|ST| = 1$.

(2) $k = 13, m = 161$. The bases of

$$\mu\left(f_3 s_2^{-1} s_1^3\right) = \begin{pmatrix} 13 & 2 \\ -1 & 0 \end{pmatrix} = X, \quad \mu\left(s_1 f_3 s_2 s_1^{-3}\right) = \begin{pmatrix} 13 & -2 \\ 1 & 0 \end{pmatrix} = Y$$

are $\xi = \left(-13 - \sqrt{161}\right)/2$, $\eta = \left(13 + \sqrt{161}\right)/2$. The continued fraction expansions of ξ and η:

$$\xi = -13 + \frac{1}{1 + \theta}, \quad \eta = 12 + \cfrac{1}{1 + \cfrac{1}{\theta}},$$

where

$$\theta = \frac{\left(9 + \sqrt{161}\right)}{4} = \left[\ \overline{5, 2, 2, 1, 2, 2, 5, 1, 11, 1}\ \right].$$

By a calculation similar to the previous example, we conclude that a matrix $S \in GL(2, \mathbb{Z})$ satisfying $Y = S^{-1} X S$ is written in the form $S = \begin{pmatrix} 23839 & 3712 \\ -1856 & -289 \end{pmatrix}^n \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Since $|S| = -1$, we have $f_3 s_2^{-1} s_1^3 \not\sim s_1 f_3 s_2 s_1^{-3}$.

References

[1] J.S. Birman: Braids, links, and mapping class groups, Princeton University Press, Princeton, 1975.

[2] Z.I. Borevich and I.R. Shafarevich: Number theory, Academic Press, New York, 1966.

[3] E. Brezin, R. Byrne, J. Levy, K. Pilgrim and K. Plummer: *A census of rational maps*, Conform. Geom. Dynam. **4** (2000), 35–74.

[4] A. Douady and J.H. Hubbard: *A proof of Thurston's topological characterization of rational functions*, Acta. Math. **171** (1993), 263–297.

[5] D.B.A. Epstein: *Curves on 2-manifolds and isotopies*, Acta Math. **115** (1966), 83–107.

[6] V.L. Hansen: Braids and coverings, London Mathematical Society student texts 18, Cambridge University Press, 1989.

[7] S.T. Hu: Homotopy theory, Academic Press, London, 1959.

[8] A. Kameyama: *On Julia sets of postcritically finite branched coverings*, preprint.

[9] W. de Melo and S. van Strien: One-dimensional dynamics, Springer-Verlag, Berlin, 1993

[10] K.M. Pilgrim: *Combination, decomposition, and structure theory for branched coverings of the two-sphere to itself*, preprint, 2001.

[11] D. Rolfsen: Knots and links, Publish or Perish, Berkeley, 1976.

[12] W.P. Thurston: *The combinatorics of iterated rational maps*, preprint, Princeton.

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