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PRE-TAUT SUTURED MANIFOLDS AND ESSENTIAL LAMINATIONS

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1. Introduction

In 1989, D. Gabai and U. Oertel [8] introduced the concept of the essential lamination, which is a hybrid object lying between incompressible surfaces and taut foliations, and generalizing both. We say that a 3-manifold is *laminar* if it contains an essential lamination. An important result of [8] is that the universal covers of laminar manifolds are homeomorphic to $\boldsymbol{R}^3$.

This fact furnishes a strong method for studying the manifolds obtained by Dehn surgery along knots, especially concerning Property P Conjecture (nontrivial Dehn surgery on a nontrivial knot in S^3 never yields a simply-connected manifold) and Cabling Conjecture (Dehn surgery on a non-cable knot cannot yield a reducible manifold). For example, see [4] for non-torus alternating knots, [3], [12] for 2-bridge knots, [17] for most algebraic knots and [9] for knots with some kind of essential tangle decompositions. We note that by [8] a 3-manifold is laminar if and only if it contains an essential branched surface (for the definition see §2), and the above authors who followed [8] obtained their results by constructing essential branched surfaces. We note that sutured manifold theory was used in [14] and [18].

One of their approaches is to construct a closed essential branched surface B in the exterior $E(K)$ of a knot K and show that B remains essential after any nontrivial Dehn filling along $\partial E(K)$ (we call such B *persistently essential*). Then we see, by [8], that K has Property P in a strong form and that the cabling conjecture is true for K. (We say that a knot K has *strong Property P* if every manifold obtained by a nontrivial Dehn surgery along K has universal cover $\boldsymbol{R}^3$.) It is, however, an open question whether or not every knot with strong Property P admits a persistently essential lamination in its complement.

In [1], [2], M. Brittenham had a paradigm shift in proving strong Property P for knots. Instead of constructing a branched surface in the complement of a given knot, he first constructed a branched surface and then embedded a knot in its complement. More precisely, he first constructed a closed branched surface B in S^3 from any in-

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compressible Seifert surface for any knot and then associated with B a pair of 'canonical' compressing disks D_+ and D_- for the horizontal boundary of a fibered neighborhood $N(B)$ of B. He next showed that if a knot K in $S^3 - N(B)$ transversely intersects D_+ and D_- each in a point, then B is persistently essential with respect to K.

In this paper we develop Brittenham's approach by using Gabai's sutured manifold theory. One of the consequences of our result is as follows:

Corollary 1.1. *Let B be a branched surface obtained by Brittenham's construction as in* [2] *from any minimal genus Seifert surface for any knot in S^3, and let D_+ and D_- be as above. Then for a knot K in $S^3 - N(B)$, B is persistently essential with respect to K if and only if K is not cabled and not ambient isotopic in $S^3 - N(B)$ to a knot disjoint form D_+ or D_-.*

This corollary is proved in Section 5 as Example 5.2.

The contents of this paper are as follows. In Section 2, we recall basic concepts such as sutured manifold, (persistently essential) branched surface, Dehn filling and Dehn surgery, and we introduce the notion of *pre-taut* sutured manifold equipped with the canonical disk pair. We also state our results in Section 2. Proposition 2.2 assures that a pre-taut sutured manifold becomes taut when we remove a knot in it, if and only if the knot 'inevitably meets each component of the canonical disk pair'. Theorem 2.3 gives a necessary and sufficient condition for a knot K in a pre-taut sutured manifold to yield via Dehn surgery a taut sutured manifold for any nontrivial slope. Theorem 2.3 is applied to prove strong Property P for some knots (Corollary 2.5). In Sections 3 and 4, we prove the above result. Finally in Section 5, we give examples of branched surfaces satisfying the condition of Corollary 2.5, constructed from minimal genus Seifert surfaces for knots and 2-component links. Then by using the branched surfaces, we give examples of knots with strong Property P.

2. Preliminaries and statement of results

In this paper, we work in the smooth category. All manifolds are oriented and all submanifolds are in general position unless otherwise specified. For a link K in a 3-manifold X, $N(K, X)$, or $N(K)$ denotes a regular neighborhood of K in X, and $E_X(K)$, or $E(K)$ denotes the exterior $\mathrm{cl}(X - N(K))$. Following the usual convention as in [10], we regard a 2-sphere bounding a 3-ball as compressible.

DEFINITION ([16]). For a compact surface F, $\chi_-(F)$ is defined by $\chi_-(F) = \Sigma_i\ |\chi(F_i)|$ where the sum is taken over the components F_i of F with $\chi(F_i) \leq 0$. If F has no component F_i with $\chi(F_i) \leq 0$, then $\chi_-(F) = 0$

Let N be a subsurface of ∂M for a compact oriented 3-manifold M.

DEFINITION ([16]). For an integral lattice homology class $a \in H_2(M, N; \boldsymbol{R})$, the (*Thurston*) *norm* of the class a is defined by
$x(a) = \min\{\chi_-(F) \mid [F] = a, F$ is an oriented surface properly embedded in $(M, N)\}$.

Let S be an oriented surface properly embedded in M with $\partial S \subset N$.

DEFINITION. We say that S is *norm-minimizing* in $H_2(M, N; \boldsymbol{R})$ if S is incompressible in M and $\chi_-(S) = x([S])$ for $[S] \in H_2(M, N; \boldsymbol{R})$.

DEFINITION. A *sutured manifold* is a manifold pair (M, γ) such that;
(1) M is a compact oriented 3-manifold and $\gamma \subset \partial M$ is a (possibly empty) union of mutually disjoint annuli $A(\gamma)$ and tori $T(\gamma)$,
(2) the interior of each component of $A(\gamma)$ contains a *suture*, i.e., an oriented simple loop which is homologically nontrivial in $A(\gamma)$, and
(3) $R(\gamma) = \mathrm{cl}(\partial M - \gamma)$ is oriented so that each component of $\partial R(\gamma)$ with the boundary orientation is homologous in γ to a suture.

We denote the union of sutures by $s(\gamma)$, and denote by $R_+(\gamma)$ (resp. $R_-(\gamma)$) the union of those components of $R(\gamma)$ whose positive normal vectors point out of (resp. into) M.

DEFINITION. A sutured manifold (M, γ) is *taut*, if M is irreducible and $R(\gamma)$ is norm-minimizing in $H_2(M, \gamma; \boldsymbol{R})$.

DEFINITION. A sutured manifold (M, γ) is *pre-taut*, if $\chi_-(R_+(\gamma)) = \chi_-(R_-(\gamma))$ and M is obtained from a (possibly disconnected) taut sutured manifold $(\tilde{M}, \tilde{\gamma})$ by attaching two 1-handles, one on $R_+(\tilde{\gamma})$ and the other on $R_-(\tilde{\gamma})$. We denote by D_+ (resp. D_-) the co-core of the 1-handle attached on $R_+(\tilde{\gamma})$ (resp. $R_-(\tilde{\gamma})$). We call $D_+ \cup D_-$ the *canonical disk pair* of (M, γ).

REMARK 2.1. For a given pre-taut sutured manifold (M, γ), it is elementarily observed as in [11, Lemma 4.4] that the canonical disk pair of (M, γ) is unique up to isotopies, i.e., if (M, γ) is obtained from another taut sutured manifold by attaching two 1-handles as above and $D'_+ \cup D'_-$ is the canonical disk pair obtained from it, then $(D'_+ \cup D'_-, \partial(D'_+ \cup D'_-))$ is properly isotopic to $(D_+ \cup D_-, \partial(D_+ \cup D_-))$ in $(M, R(\gamma))$.

Note that this does not mean that compressing disks for $R_+(\gamma)$ are unique up to isotopies.

Let (M, γ) be a sutured manifold, and K a knot in $\mathrm{int} M$. Then the manifold pair $(E_M(K), \gamma \cup \partial N(K))$ naturally inherits a sutured manifold structure from (M, γ). We prove the following in Section 3.

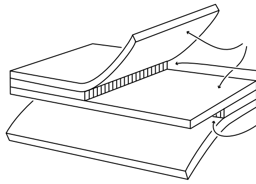


Fig. 2.1.

Proposition 2.2. *Let (M, γ) be a pre-taut sutured manifold, and K a knot in* int$M$. *Then the sutured manifold $(E_M(K), \gamma \cup \partial N(K))$ is taut if and only if K is not ambient isotopic in M to a knot disjoint from a component of the canonical disk pair of (M, γ).*

For the definitions of *branched surface* and *essential lamination* and terminologies concerning them (e.g., *fibered neighborhood, horizontal* and *vertical boundary, disk of contact, Reeb branched surface, branched surface fully carrying a lamination* etc.), see [8]. In this paper, we assume branched surfaces are closed unless otherwise specified. For a branched surface B, $N(B)$ denotes its fibered neighborhood. The boundary $\partial N(B)$ is the union of the *horizontal boundary* (denoted by $\partial_h N(B)$) and the *vertical boundary* (denoted by $\partial_v N(B)$). See Fig. 2.1.

For a branched surface B in a 3-manifold X, we denote by $E_X(B)$ the exterior cl$(X - N(B))$ of B in X.

DEFINITION. A closed branched surface B in a 3-manifold X with empty or incompressible boundary is *essential* if it satisfies the following conditions.
1. B has no disks of contact.
2. $\partial_h N(B)$ is incompressible in $E_X(B)$.
3. There are no monogons in $E_X(B)$.
4. No component of $\partial_h N(B)$ is a sphere.
5. $E_X(B)$ is irreducible.
6. B contains no Reeb branched surface.
7. B fully carries a lamination.

For the definition of the term 'to contain no Reeb branched surface', see [8, p. 46, ll. 9–13]. In an orientable, irreducible 3-manifold, a branched surface satisfying the conditions 1, 2, 3 and 4 is called *incompressible*.

DEFINITION. A *transversely oriented branched surface* is a branched surface B with a global orientation on the 1-foliation of $N(B)$ whose leaves are the fibers of $N(B)$.

DEFINITION. Let B be a closed transversely oriented branched surface embedded in a closed 3-manifold X. Then the manifold pair $(E_X(B), \partial_v N(B))$ naturally has a sutured manifold structure and we call this the *complementary sutured manifold* for B.

DEFINITION. Let X be a 3-manifold with a toral boundary component T, and α (the isotopy class of) an essential simple loop in T. Then $X(\alpha)$ denotes the 3-manifold obtained from X and a solid torus V by identifying ∂V with T by a homeomorphism which takes a meridian loop of V to α. We say that $X(\alpha)$ is obtained from X by a *Dehn filling* along T with *slope* α.

DEFINITION. Let K be a knot in a 3-manifold X, and let T be the component of $\partial E_X(K)$ corresponding to $\partial N(K)$ and let α be (the isotopy class of) an essential simple loop in T. Then $X(K, \alpha)$ denotes the manifold obtained from $E_X(K)$ by a Dehn filling along T with slope α. We say that $X(K, \alpha)$ is obtained from X by a *Dehn surgery* along K with slope α. We say that α is a *trivial slope* if α is a meridian loop on $N(K)$. If α is not a trivial slope, we say that $X(K, \alpha)$ is obtained by a *nontrivial surgery*.

DEFINITION. A knot K in a 3-manifold X is a *cable knot*, or called *cabled* if there exists a solid torus V in X with $K \subset \partial V$ such that K is not isotopic in V to a core circle of V and does not bound a disk in V.

Let (M, γ) be a sutured manifold and K a knot in int$M$. Then the manifold pair $(M(K, \alpha), \gamma)$ naturally inherits a sutured manifold structure from (M, γ). We prove the following in Section 4:

Theorem 2.3. *Let (M, γ) be a pre-taut sutured manifold, and K a knot in* int$M$. *Then the sutured manifold $(M(K, \alpha), \gamma)$ is taut (and hence $M(K, \alpha)$ is irreducible and $R_+(\gamma) \cup R_-(\gamma)$ is incompressible in $M(K, \alpha)$) for any nontrivial slope α, if and only if K is not cabled and not ambient isotopic in M to a knot disjoint from a component of the canonical disk pair of (M, γ).*

DEFINITION. Let B be a closed branched surface embedded in a closed 3-manifold X, and K a knot in $E_X(B)$. We say that B is *persistently essential* with respect to K, if B is essential in $E_X(K)$ and remains essential in $X(K, \alpha)$ for any nontrivial slope α. A *persistently incompressible* branched surface is also defined analogously.

As an immediate corollary of Theorem 2.3, we have the following:

Corollary 2.4. *Let B be a closed transversely oriented branched surface embedded in a 3-manifold X such that the complementary sutured manifold for B is pre-taut,*

and let K be a knot in $E_X(B)$. Then the following two conditions are equivalent:
(1) *B is persistently incompressible with respect to K and the surgered manifold $E_X(B)(K, \alpha)$ is irreducible for any nontrivial slope α.*
(2) *K is not cabled in $E_X(B)$ and not ambient isotopic in $E_X(B)$ to a knot disjoint from a component of the canonical disk pair of the complementary sutured manifold for B.*

Proof of Corollary 2.4. By Theorem 2.3, the condition (1) follows from (2). The condition (2) obviously follows from (1). In fact, if K can be isotoped to be disjoint from D_+ or D_-, then B is not incompressible in $E(K)$ and hence in $E_X(B)(K, \alpha)$ for any nontrivial slope α. If K is cabled, then some Dehn surgery along K yields a manifold with a lens space summand. □

Then consequently we have:

Corollary 2.5. *Let B be a closed transversely oriented branched surface embedded in a 3-manifold X such that the complementary sutured manifold for B is pre-taut. Suppose that B satisfies the conditions* 1, 4, 6 *and* 7 *of the definition of the essential branched surface. Let K be a knot in $E_X(B)$ which is not cabled in $E_X(B)$ and not ambient isotopic in $E_X(B)$ to a knot disjoint from a component of the canonical disk pair of the complementary sutured manifold for B. Then B is persistently essential with respect to K. In particular, K has strong Property P.*

Proof of Corollary 2.5. We show that B persistently satisfies the conditions 1–7 of the definition of the essential branched surface. The conditions 2 and 5 are already assured by Corollary 2.4. The conditions 1, 4, 6 and 7 are not affected by surgeries. Since B is transversely oriented, B persistently satisfies the condition 3. Now we have proved Corollary 2.5. □

3. Proof of Proposition 2.2

The 'only if' part of Proposition 2.2 is obvious. Hence we give a proof of the 'if' part. Since (M, γ) is pre-taut, there exists a taut sutured manifold $(\tilde{M}, \tilde{\gamma})$ from which (M, γ) is obtained by attaching two 1-handles. Let $D_+ \cup D_-$ be the canonical disk pair of (M, γ). Since we consider knots which inevitably meet each component of the canonical disk pair, we may assume, without loss of generality, that M is connected and hence that $\tilde{M}$ has at most three connected components. Let (N, δ) denote the sutured manifold $(E_M(K), \gamma \cup \partial N(K))$. We first show:

CLAIM 1. N is irreducible.

Proof of Claim 1. Suppose there exists an essential 2-sphere S in N. Regard S as a sphere in M. Then by using standard innermost disk argument together with the irreducibility of $M - (D_+ \cup D_-)$, we can move S by an ambient isotopy of M so that $S \cap (D_+ \cup D_-) = \emptyset$. Since $M - (D_+ \cup D_-)$ is irreducible, S bounds a 3-ball B in $M - (D_+ \cup D_-)$, and hence in M. Since S is essential, the image of K by the above ambient isotopy is contained in B. Hence it misses $D_+ \cup D_-$, a contradiction. This establishes Claim 1. □

Next we show:

CLAIM 2. $\chi_-(R_+(\delta)) = x([R_+(\delta)]) = \chi_-(R_-(\delta)) = x([R_-(\delta)])$.

Proof of Claim 2. By the definition of the pre-taut sutured manifold, $\chi_-(R_+(\delta)) = \chi_-(R_-(\delta))$. We only prove $\chi_-(R_+(\delta)) = x([R_+(\delta)])$, for we can analogously prove $\chi_-(R_-(\delta)) = x([R_-(\delta)])$. Suppose, for a contradiction, that there exists a surface F in (N, δ) such that $[F] = [R_+(\delta)]$ in $H_2(N, \delta)$ and $\chi_-(F) < \chi_-(R_+(\delta))$. Since $[F] = [R_+(\delta)]$, we may assume that $\partial F = s(\delta)$ and that $F \cap \partial N(K) = \emptyset$, by capping off, if necessary, pairs of boundary components by annuli.

SUBCLAIM 1. $F \cap (D_+ \cup D_-) \neq \emptyset$.

Proof. Suppose $F \cap (D_+ \cup D_-) = \emptyset$ and regard F as a surface in (M, γ) such that $\partial F = s(\gamma)$ and $F \cap K = \emptyset$. Since $[F] = [R_+(\delta)]$, we see that $[F] = [R_+(\gamma)]$ in $H_2(M, \gamma)$. Hence F separates M into two submanifolds M_+ and M_- such that $D_+ \subset M_+$ and $D_- \subset M_-$. Since $K \cap F = \emptyset$, $K \subset M_+$ or M_-, say M_+. However this shows $K \cap D_- = \emptyset$, a contradiction. □

Since $M - (D_+ \cup D_-)$ is irreducible, standard innermost disk argument allows us to assume that by isotopies each component of $F \cap (D_+ \cup D_-)$ is essential in F. Let F_1 be the surface obtained by compressing F ($\subset M$) along an innermost disk in $D_+ \cup D_-$ bounded by a component of $F \cap (D_+ \cup D_-)$. If $F_1 \cap (D_+ \cup D_-) \neq \emptyset$, we apply the above procedure to F_1 to obtain F_2 (i.e., remove inessential simple loops in F_1 by isotopy, and then compress along a compressing disk contained in $D_+ \cup D_-$). After a finite number of applications of the procedures, we obtain a surface F_n such that $F_n \cap (D_+ \cup D_-) = \emptyset$. Let $F' = F_n$. We regard F' as a surface in $(\tilde{M}, \tilde{\gamma})$ such that $\partial F' = s(\tilde{\gamma})$. Since $[F] = [R_+(\delta)]$, it is easy to see that $[F'] = [R_+(\tilde{\gamma})]$ in $H_2(\tilde{M}, \tilde{\gamma})$. We have the following three cases.

CASE 1. $\chi_-(F') \leq \chi_-(F) - 2$.
In this case, by the assumption and the definitions of (M, γ) and (N, δ), we have;

$\chi_-(F') \leq \chi_-(F) - 2 < \chi_-(R_+(\delta)) - 2 = \chi_-(R_+(\gamma)) - 2 \leq \chi_-(R_+(\tilde{\gamma}))$. In particular we have $\chi_-(F') < \chi_-(R_+(\tilde{\gamma}))$, which contradicts the tautness of $(\tilde{M}, \tilde{\gamma})$.

CASE 2. $\chi_-(F') = \chi_-(F) - 1$.

In this case, we first show the following claim:

SUBCLAIM 2. There exists a disk component, say D, of F'.

Proof. There exists $1 \leq i < n$ such that $\chi_-(F_{i+1}) = \chi_-(F_i) - 1$. Let $E \subset D_+ \cup D_-$ be the compressing disk used for obtaining F_{i+1} from F_i and let F_i^* be the component of F_i that contains ∂E.

CASE A. ∂E is non-separating in F_i^*. In this case, it is easy to see that F_i^* is a once-punctured torus, for $\chi_-(F_{i+1}) = \chi_-(F_i) - 1$. By compressing the once-punctured torus F_i^*, we obtain a disk component of F_{i+1}. Clearly this disk survives in F' to give desired D.

CASE B. ∂E is separating in F_i^*. In this case, it is easy to see that ∂E is parallel to a component of ∂F_i^*, say ∂_1. Obviously the component of F_{i+1} containing ∂_1 is a disk, which survives in F' to give desired D. □

Next we show the following subclaim:

SUBCLAIM 3. Let (A, α) be the component of $(\tilde{M}, \tilde{\gamma})$ containing the disk D in Subclaim 2. Then (A, α) has the product structure of $(D \times I, \partial D \times I)$.

Proof. Let α_1 be the component of α that contains ∂D. Let R_+ be the connected component of $R_+(\alpha)$ such that $R_+ \cap \alpha_1 \neq \emptyset$. Define R_- analogously. The existence of D implies that the boundary components $R_+ \cap \alpha_1$ of R_+ and $R_- \cap \alpha_1$ of R_- are contractible in A. Since R_+ and R_- are incompressible by the definition of norm-minimizing surfaces, this shows that they are disks. Since A is irreducible, the 2-sphere $R_+ \cup \alpha_1 \cup R_-$ bounds a 3-ball A, which shows that (A, α) has the product structure above. □

If $\tilde{M}$ is connected, we see, by Subclaims 2 and 3, that M is a genus two handlebody (see Fig. 3.1 (a)). In this case we have $\chi_-(R_+(\gamma)) = 1$. Hence $\chi_-(R_+(\gamma)) = \chi_-(R_+(\delta)) > \chi_-(F) = 0$. However, this contradicts the assumption that $\chi_-(F') = \chi_-(F) - 1$, for $\chi_-(*)$ is always non-negative.

Suppose $\tilde{M}$ is not connected (i.e., $\tilde{M}$ consists of two or three connected components). Then M appears as in one of Figs 3.1 (b), (c), (d), (e) and (f). Let (B, β) be the sutured manifold consisting of the components of $(\tilde{M}, \tilde{\gamma})$ other than (A, α) of Subclaim 3. Since M is connected, we may suppose, without loss of generality, that the 1-handle attached to $R_+(\tilde{\gamma})$ joins A and B.

In cases (b) and (f), and in cases (c) and (e) with one component of (B, β) being

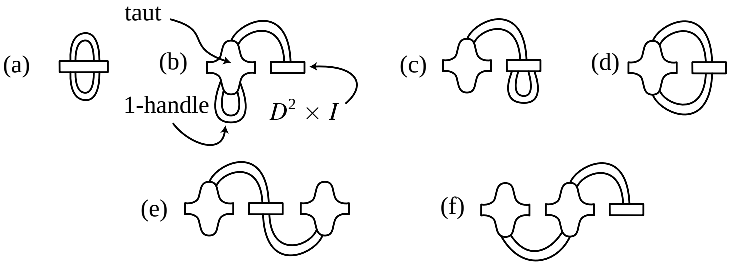


Fig. 3.1.

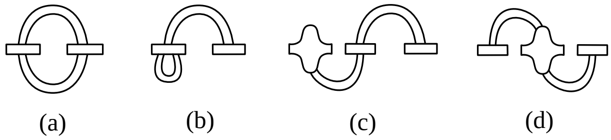


Fig. 3.2.

of the form $(D \times I, \partial D \times I)$, it is obvious that K can be isotoped to miss one component of the canonical disk pair, a contradiction. In case (d) with (B, β) being of the form $(D \times I, \partial D \times I)$, we have $\chi_-(R_+(\delta)) = 0$, which contradicts the assumption that $\chi_-(F) < \chi_-(R_+(\delta))$. In other cases (i.e., cases (c), (d) and (e) with no component of (B, β) being of the form $(D \times I, \partial D \times I)$), we have $\chi_-(R_+(\gamma)) - 1 = \chi_-(R_+(\tilde{\gamma}))$. By assumption, we have $\chi_-(F') = \chi_-(F) - 1 < \chi_-(R_+(\delta)) - 1 = \chi_-(R_+(\gamma)) - 1$. Hence we have $\chi_-(F') < \chi_-(R_+(\tilde{\gamma}))$, a contradiction.

CASE 3. $\chi_-(F') = \chi_-(F)$.

SUBCLAIM 4. Two components of F' are disks.

Proof. Let $E \subset D_+ \cup D_-$ be the compressing disk used for obtaining F_1 from F and let F^* be the component of F that contains ∂E.

CASE *A*. ∂E is non-separating in F^*.

In this case, F^* is a torus, for $\chi_-(F_1) = \chi_-(F') = \chi_-(F)$. Since M is irreducible, we see that F^* bounds a solid torus in M or is contained in a 3-ball in M. Hence F^* represents the trivial element in $H_2(M, \gamma)$ and we can eliminate, without loss of generality, such a component F^* from F.

CASE *B*. ∂E is separating in F^*.

In this case, it is easy to see that F^* is an annulus and hence the components of F_1 obtained from F^* are disks, which survive in F' to give desired two disk components of F'. □

Since F' has two disks, two components of $\tilde{M}$ are homeomorphic to $D^2 \times I$ as proved in Subclaim 3. Since we have assumed $\tilde{M}$ has at most three connected components, (M, γ) appears as in one of Fig. 3.2 (a), (b), (c) or (d). In (b), (c) and (d), it is obvious that K can be isotoped to miss one component of the canonical disk pair, a contradiction. In (a), $\chi_-(R_+(\delta)) = 0$, which contradicts the assumption that $\chi_-(F) < \chi_-(R_+(\delta))$. Claim 2 is proved. □

We note that the cases of Fig. 3.1 (b) and Fig. 3.2 (b) can be eliminated by the definition of the pre-taut sutured manifold because in these cases, we have $\chi_-(R_+(\gamma)) \neq \chi_-(R_-(\gamma))$.

Finally we show:

CLAIM 3. $R_+(\delta)$ and $R_-(\delta)$ are incompressible.

Proof of Claim 3. Suppose $R_+(\delta)$ has a compressing disk E and let R^* be the component of $R_+(\delta)$ that contains ∂E. Compress $R_+(\delta)$ along E and let R be a surface obtained by pushing the interior of the resulting surface into (N, δ). If $\chi_-(R^*) \geq 1$, then $\chi_-(R) < \chi_-(R_+(\delta))$, which contradicts Claim 2. Hence $\chi_-(R^*) = 0$. Since R^* is compressible, this shows R^* is either a torus or an annulus. If R^* is a torus, then $R_+(\tilde{\gamma})$ has a sphere component, contradicting the tautness of $(\tilde{M}, \tilde{\gamma})$. Therefore R^* is an annulus and R is a union of two disks, which are parallel to D_+ by Remark 2.1, and hence K can be isotoped off D_+, a contradiction. Claim 3 is proved. □

Claims 1, 2, and 3 establish Proposition 2.2. □

4. Proof of Theorem 2.3

For the proof of Theorem 2.3, we prepare two propositions, which are immediate consequences of a result of D. Gabai's in [6] and a result of M. Scharlemann's in [15].

Proposition 4.1. *Let X be a Haken manifold whose boundary is a non-empty union of tori. Let S be a norm-minimizing surface properly embedded in $(X, \partial X)$, and P a component of ∂X with $P \cap S = \emptyset$. Then we have the following;*
(1) *For any slope α in P but at most one exception, S remains norm-minimizing (and in particular incompressible) in $(X(\alpha), \partial X(\alpha))$, where $X(\alpha)$ is obtained by a Dehn filling along P with slope α, and moreover,*
(2) *if there is no essential torus T in $X - S$ that separates P and S, then $X(\alpha)$ is irreducible but for at most the exceptional α in* (1).

Note that a surface T is essential in $M - S$ if T is incompressible in $M - S$, and not parallel to a subsurface of $\partial M - S$ in $M - S$.

Proof. By [6, Corollary 2.4], Proposition 4.1 (1) holds. If X is S_P-atoroidal, then again by [6, Corollary 2.4], Proposition 4.1 (2) holds. Hence it is enough to show that X is S_P-atoroidal. (See [6], for the definitions of the terms *S_P-atoroidal* and *I-cobordism*.) Let V be an I-cobordism in $X - S$ between P and a surface, say P'. By [6, Lemma 1.5], P' is a torus. Now we use the fact that P' is incompressible in $X - S$, which is proved as follows; Suppose there exists a compressing disk $D \subset X - S$ for P'. Since V is an I-cobordism, $D \subset \mathrm{cl}(X - V)$. By compressing P' along D, we obtain a 2-sphere H in $X - S$. Since X is irreducible, H bounds a 3-ball B in X. Since P' separates P and S, we have either $P \subset B$ or $S \subset B$, which are both impossible. Therefore, P' is incompressible in $X - S$. If there is no essential torus T in $M - S$ that separates P and S, we see that V is a trivial cobordism $P \times I$. This shows that X is S_P-atoroidal. □

DEFINITION ([7]). Let $T \subset D^2 \times S^1$ be a torus bounding a solid torus $W = (1/2)D^2 \times S^1 \subset D^2 \times S^1$. We say that a knot K in $D^2 \times S^1$ with non-zero wrapping number is a *0-bridge braid* if K can be ambient isotoped to lie in T. K is a *1-bridge braid* if K is isotopic to a knot of the form $\beta_1 \cup \beta_2$ where $\beta_1 (\subset T)$ and $\beta_2 (\subset W)$ are arcs such that β_1 is transverse to each $D^2 \times \{*\}$ and that $\beta_2 \subset W \cap (D^2 \times \{p_1\})$, for some $p_1 \in S^1$.

Proposition 4.2. *Let V be a solid torus and K a knot in V with $V - K$ irreducible. Let V' be a manifold obtained from V by a nontrivial Dehn surgery along K. Then either one of the following holds.*
(1) *V' is a solid torus and K is 0 or 1-bridge braid.*
(2) *K is cabled (and the slope of the surgery is that of the cabling annulus).*
(3) *V' is irreducible and $\partial V'$ is incompressible.*

We obtain Proposition 4.2 by applying [15, Theorem] to a solid torus. The conclusions (1), (2) and (3) above respectively correspond to the conclusions (a), (c) and (d) of [15, Theorem].

Proof of Theorem 2.3. The 'only if' part is easily verified. Actually, some Dehn surgery along a cable knot yields a lens space summand, and if K is disjoint from D_+ or D_-, then B is not essential in $E(K)$. Now we prove the 'if' part. Let $(\overline{N_1}, \overline{\delta_1}) = (M(K, \alpha), \gamma)$ be a sutured manifold obtained from (M, γ) by a Dehn surgery along K with slope α. We first show:

CLAIM 1. If α is nontrivial, then $R(\overline{\delta_1})$ is norm-minimizing.

Proof. Let $(N_1, \delta_1) = (E_M(K), \gamma \cup \partial N(K))$, (N_2, δ_2) a copy of (N_1, δ_1) and P_i the component of ∂N_i each corresponding to $\partial N(K)$. Identify $R(\delta_1)$ with $R(\delta_2)$ by an orientation reversing homeomorphism and let Q be the union of tori obtained from $\delta_1 \cup \delta_2$, and S the image of $R(\delta_1)$ (= that of $R(\delta_2)$). Let $N^* = (N_1 \cup N_2)/\!\sim$, where $\sim$ denotes the above identification. Note that each component of ∂N^* (= $P_1 \cup P_2 \cup Q$) is a torus and that $P_1 \cap S = \emptyset$. By Proposition 2.2, we see that (N_i, δ_i) is taut. Hence by [5, Corollary 5.3], (N_i, δ_i) has a taut foliation (in particular tangent to $R(\delta_i)$), and hence so does $(N^*, \partial N^*)$ with S a leaf. We note that [5, Corollary 5.3] involves an argument of sutured manifold hierarchy. We understand that the term "sutured manifold decomposition" in (a) of Corollary 5.3 in [5] should be read "sutured manifold hierarchy". Then by [5, Theorem 2.5] (by Thurston [16]), S is norm-minimizing. By Proposition 4.1, S remains norm-minimizing in the manifold $N^*(\alpha)$ obtained from N^* by a Dehn filling along P_1 with any nontrivial slope α, because the trivial filling along P_1 makes S compressible. Note that S cuts $(N^*(\alpha), \partial N^*)$ into $(M(K, \alpha), \gamma) = (\overline{N}_1, \overline{\delta}_1)$ and (N_2, δ_2), with $R(\overline{\delta}_1)$ corresponding to S. Since S remains norm-minimizing in $(N^*(\alpha), \partial N^*)$, $R(\overline{\delta}_1)$ is norm-minimizing in $(\overline{N}_1, \overline{\delta}_1)$. Claim 1 is proved. □

Next we show:

CLAIM 2. If α is nontrivial, then $\overline{N}_1 = M(K, \alpha)$ is irreducible.

Proof. According to Proposition 4.1, we divide the proof into the following two cases.

CASE 1. There does not exist an essential torus in N_1 which separates $P_1 = \partial N(K)$ from $R(\delta_1)$.

In this case, it immediately follows that there does not exist an essential torus T in N^* which separates P_1 and S. By the last half of Proposition 4.1, we see that, for any nontrivial slope $\alpha \subset P_1$, $N^*(\alpha)$ is irreducible. Since S remains incompressible in $N^*(\alpha)$, standard innermost disk argument shows that the manifold $\overline{N}_1 \cup N_2$ obtained by cutting $N^*(\alpha)$ apart along S is irreducible. This shows that $\overline{N}_1 = M(K, \alpha)$ is irreducible for any nontrivial slope α.

CASE 2. There exists an essential torus T in N_1 which separates $P_1 = \partial N(K)$ from $R(\delta_1)$.

Regard N_1 as embedded in M, and let the same symbol T denote the image of T in M. Let M_1 and M_2 be the closures of the components of $M - T$ such that $M_1 \supset K$.

SUBCLAIM 1. M_1 is a solid torus.

Proof. We consider the intersection $T \cap (D_+ \cup D_-)$. Since $K \cap (D_+ \cup D_-) \neq \emptyset$ and T separates K from $R(\gamma) = R(\delta_1)$, we see $T \cap (D_+ \cup D_-) \neq \emptyset$, and by standard innermost disk argument, we may assume that every component of $T \cap (D_+ \cup D_-)$ is an essential loop in T. Let E be the closure of an open disk component of $(D_+ \cup D_-) - T$. Then E is a compressing disk for T in M. Since M is irreducible, this shows that T bounds either a solid torus in M or a knot exterior contained in a 3-ball in M. Since T separates K and $R(\gamma) = R(\delta_1)$, the second situation implies that K is contained in the 3-ball and hence K is ambient isotopic to a knot disjoint form $D_+ \cup D_-$, a contradiction. Hence T bounds a solid torus in M containing K. This establishes Subclaim 1. □

SUBCLAIM 2. M_2 is irreducible and (the image of) T is incompressible in M_2.

Proof. Since T is an essential torus in N_1, we immediately see that T is incompressible in M_2. Assume that there exists an essential 2-sphere S^2 in M_2. Regard S^2 as embedded in M. Since M is irreducible, S^2 bounds a 3-ball B in M, where $M_1 \subset B$. This shows that K is contained in a 3-ball in M, a contradiction. Hence we have Subclaim 2. □

Now assume further that T is farthest from P_1, i.e., any other essential torus in M_2 satisfying the condition of Case 2 is parallel to T. Now we consider the manifold $\bar{M}_1 = M_1(K, \alpha)$. By Subclaim 1, we have one of the conclusions of Proposition 4.2 by regarding $M_1 = V$ and $\bar{M}_1 = V'$. Since K is not cabled, we do not have the conclusion (2), or the conclusion (1) with K a 0-bridge braid. If $\bar{M}_1$ satisfies the conclusion (3), then it is easy to show that $\bar{N}_1$ is irreducible by Subclaim 2. Suppose that $\bar{M}_1$ satisfies the conclusion (1) with K a 1-bridge braid. Let K^* be the core circle of M_1. Since K is a 1-bridge braid in M_1, K is not representing a trivial element of $H_1(M_1)$. Hence by an easy homological calculation, we see that $M(K, \alpha) = M(K^*, \alpha^*)$, where α^* is nontrivial. Since T is farthest from P_1, we can apply the argument of Case 1 to show that $M(K^*, \alpha^*)$ is irreducible, and hence that $M(K, \alpha)$ is irreducible. Claim 2 is proved. □

Claims 1 and 2 establish Theorem 2.3. □

5. Examples

Finally in this section, we construct branched surfaces satisfying the condition of Corollary 2.5 from minimal genus Seifert surfaces for knots and 2-component links in S^3. Then by using the branched surfaces, we give examples of knots with strong Property P.

DEFINITION. For an oriented Seifert surface S for a link L, we denote $S \cap E(L)$ by the same symbol S. Then the manifold pair $(N_S, \delta_S) = (\text{cl}(E(L) - N(S, E(L))), \text{cl}(\partial E(L) - N(\partial S, \partial E(L))))$ naturally has a sutured manifold structure and we call (N_S, δ_S) the *complementary sutured manifold* for S.

REMARK 5.1. It is elementary that if S is of minimal genus and N_S is irreducible, then (N_S, δ_S) is taut.

EXAMPLE 5.2. (*Brittenham's branched surfaces*) and proof of Corollary 1.1.

We first recall Brittenham's construction of branched surfaces [2]. Let S be an incompressible Seifert surface for any knot $\tilde{K}$. First we perform a tubing operation to S in $N(\tilde{K})$ along a half of $\tilde{K}$, i.e., remove two disks from S and cap off by a thin tube 'parallel' to a half of $\tilde{K}$. Let l be a simple loop on tubed S running parallel to the other half of $\tilde{K}$ and otherwise running on the tube. Then curl $N(\partial S, S)$ to l and glue ∂S $(= \tilde{K})$ to l so that we obtain a transversely oriented closed branched surface B_S with one locus l. (See Fig. 5.1 (a), and for the detail [2]. The shaded two disks define the canonical disk pair.) In [2, §2], it is shown that the complementary sutured manifold for B_S is of the form $(N_S, \delta_S) \cup$ (two 1-handles), where one 1-handle is attached on $R_+(\delta_S)$ and the other on $R_-(\delta_S)$. Hence if S is of minimal genus, then by Remark 5.1, the complementary sutured manifold for B_S is pre-taut. Moreover, it is shown in [1] that B_S satisfies the conditions 1, 4, 6 and 7 of the definition of the essential branched surface. Therefore, by Corollary 2.5, B_S is persistently essential with respect to any non-cable knot in the exterior of B_S inevitably meeting each component of the canonical disk pair. The above argument gives a proof to the 'if' part of Corollary 1.1 and the 'only if' part is easily verified as in the proof of Theorem 2.3.

Fig. 5.1 (b) ([1], [8], [13]) depicts the simplest case of Example 5.2, where S is the disk. The complementary sutured manifold for B is a pre-taut sutured manifold of the form $(D^2 \times I, \partial D^2 \times I) \cup$(two 1-handles). This fact was already pointed out in [1].

EXAMPLE 5.3. (*Iterated tubing operations*)

Let S be a minimal genus Seifert surface for a knot K. We concentrate our attention to a regular neighborhood of ∂S as in [2, §2]. Let B_0 be a branched surface obtained from S as in Example 5.2, which looks as in Fig. 5.1 (a) in the regular neighborhood of ∂S. Let α_1 be the arc as in Fig. 5.1 (a) and B_1 the branched surface obtained by tubing B_0 along α_1. By tubing B_0 successively as in Fig. 5.1 (c), we obtain a sequence of branched surfaces $B_0, B_1, B_2, \ldots$. Then by using the argument as in [11, §7], we see that the complementary sutured manifold (N_n, δ_n) for B_n is homeomorphic to $(N_S \cup_{\delta_S = \partial_1 \Sigma_n \times I} (\Sigma_n \times I) \cup \text{(two 1-handles)}, \partial_2 \Sigma_n \times I)$, where Σ_n is a twice punctured orientable surface of genus n with boundary components $\partial_1 \Sigma_n$ and $\partial_2 \Sigma_n$, and 1-handles are attached to each component of $\partial(N_S \cup (\Sigma_n \times I)) - (\partial_2 \Sigma_n \times I)$. (The cocores of the 1-handles are indicated in Fig. 5.1 (c) as $D_i^{\pm}$.) It is elementary to show that $(N_S \cup_{\delta_S = \partial_1 \Sigma_n \times I} (\Sigma_n \times I), \partial_2 \Sigma_n \times I)$ is taut, and hence that (N_n, δ_n) is pre-taut. By the argument in [2, §2], we see that B_n $(n \geq 0)$ satisfies the conditions 1, 4, 6 and 7 of the definition of the essential branched surface. Therefore, by Corollary 2.5, B_n is persistently essential with respect to any non-cable knot in the exterior of B_n inevitably meeting each component of the canonical disk pair.

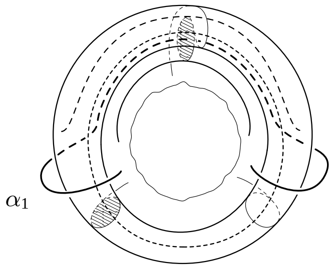


Fig. 5.1 (a).

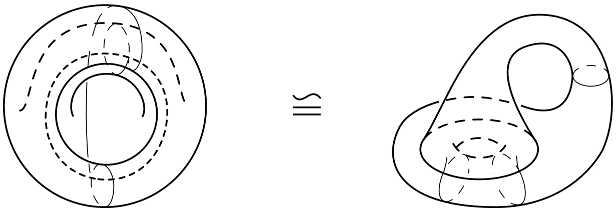


Fig. 5.1 (b).

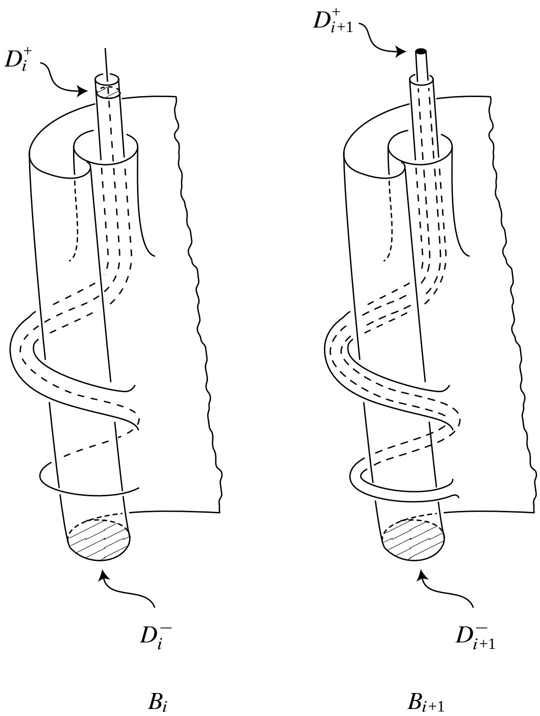
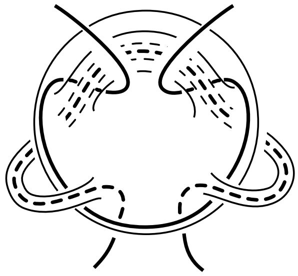
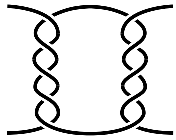


Fig. 5.1 (c).



B_1 is obtained by curling the edge of the above surface.



$(-5, 5)$-tangle

Fig. 5.1 (d).

In [1], Brittenham introduced the concept of *persistently laminar tangles*. A tangle T is called *persistently laminar*, if every knot obtained by summing another tangle with T has a persistently essential lamination in its complement. In particular, by using B_n with S a disk, we see that the $(-2n-3, 2n+3)$-pretzel tangle (Fig. 5.1 (d)) is persistently laminar. The case $n = 0$ in the above was shown in [1].

EXAMPLE 5.4. (*Branched surfaces constructed from minimal genus Seifert surfaces for 2-component links*)

Let $L = K_1 \cup K_2$ be an oriented 2-component link in S^3 which has a connected minimal genus Seifert surface S. Let α be an arc properly embedded in S connecting K_1 and K_2. We perform a tubing operation on S in a neighborhood of α as in Fig. 5.2 (a) to obtain a compressible surface S'. Let l_1, l_2 be simple loops in S' such that l_i is parallel to K_i except in a neighborhood of α, where l_i appears as in Fig. 5.2 (a). Then curl $N(K_i, S')$ and glue K_i to l_i to obtain a transversely oriented closed branched surface B_S. By using an argument similar to that in [2, §2], it is directly observed that the complementary sutured manifold for B_S is of the form $(N_S \cup (\text{two 1-handles}), \delta_S)$, where D_+ and D_- in Fig.5.2 (b) are disks corresponding to the co-cores of the 1-handles, i.e., the complementary sutured manifold for B_S is pre-taut with canonical disk pair $D_+ \cup D_-$. Let K be a non-cable knot in the complement of B_S inevitably meeting each component of the canonical disk pair. Now we show that B_S is persistently essential. By Corollary 2.5, it is enough to show that B_S satisfies the conditions 1, 4, 6 and 7 of the definition of the essential branched surface. By using the weight argument as in [1, §1], we see that B_S satisfies the conditions 1, 4 and 6. Finally we show that B_S satisfies the condition 7, i.e., construct a lamination $\mathcal{L}$ fully carried by B_S as follows.

We take a product $S' \times I$ in S^3, where $I = [-1, 1]$. Let A_1 and A_2 be annuli in $S' \times I$ such that $A_1 = l_1 \times [1/2, 1]$ and $A_2 = l_2 \times [-1, -1/2]$. We take the product lamination $\mathcal{L}' = S' \times C$, where C is a Cantor set. We cut $\mathcal{L}'$ along $A_1 \cup A_2$ (Fig. 5.2 (c)) and

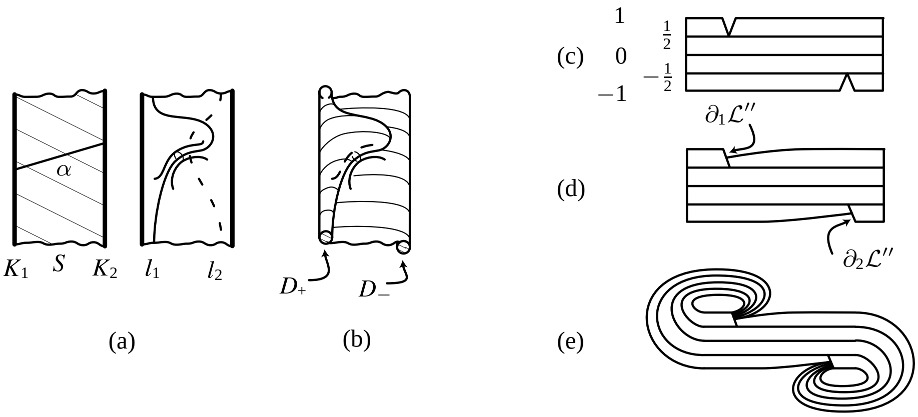


Fig. 5.2.

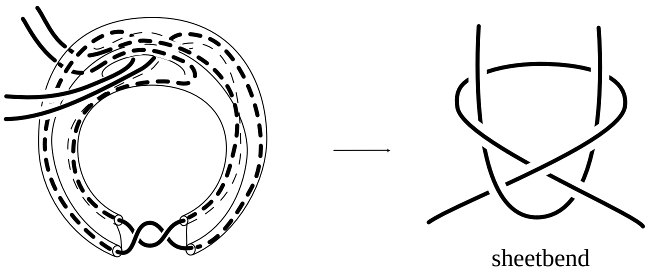


Fig. 5.3.

reglue it as in Fig. 5.2 (d) to obtain $\mathcal{L}''$, i.e., we identify the boundary components of $\mathcal{L}'$ contained in non-K_1-side of A_1 (resp. non-K_2-side of A_2) with the boundary components of $\mathcal{L}'$ contained in $l_1 \times [1/2, 3/4]$ (resp. $l_2 \times [-3/4, -1/2]$). Let $\partial_1\mathcal{L}''$ denote the union of boundary components of $\mathcal{L}''$ contained in $l_1 \times [3/4, 1]$ and $\partial_2\mathcal{L}''$ denote those contained in $l_2 \times [-1, -3/4]$ (Fig. 5.2 (d)). Curl $N(K_1 \times C, \mathcal{L}'')$ and identify $K_1 \times C$ with $\partial_1\mathcal{L}''$, and $K_2 \times C$ with $\partial_2\mathcal{L}''$ (Fig. 5.2 (e)). Then we obtain a closed lamination $\mathcal{L}$ which is fully carried by B_S. Therefore we see that B_S is essential with respect to K.

As a concrete example, we start with a Hopf band to obtain a branched surface in Fig. 5.3. By Corollary 2.5 we see that the sheet bend tangle is persistently laminar (see Fig. 5.3).

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