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SYMMETRY OF THE TANGENTIAL CAUCHY-RIEMANN EQUATIONS AND SCALAR CR INVARIANTS

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Introduction

Let $\Omega = \{r > 0\}$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^{n+1}$ with smooth (C^∞) boundary and let K_Ω be the Bergman kernel defined on Ω. In [3], C. Fefferman proved

$$K_\Omega(Z, Z) = \frac{\phi_\Omega}{r^{n+2}} + \psi_\Omega \ln r,$$

where ϕ_Ω and ψ_Ω are functions that are C^∞ up to $\partial\Omega$.

$\partial\Omega$ inherits a geometric structure, called CR structure, from $\mathbb{C}^{n+1}$ which is relevant for the biholomorphic equivalence of Ω. Fefferman's program, initiated in [5], is to compute all the scalar CR invariants of $\partial\Omega$ and to express the asymptotic expansion of ϕ_Ω modulo $O(r^{n+2})$ and ψ_Ω modulo $O(r^\infty)$ in terms of scalar CR invariants of $\partial\Omega$.

Fefferman's invariant theory was developed further by T.N. Bailey, M.G. Eastwood, C.R. Graham, G. Komatsu and K. Hirachi, see [1], [7] and [8]. The main method is to obtain a defining function which is invariant under biholomorphic maps up to a power of determinants of biholomorphic maps and to construct a Kähler-Lorentz metric on a line bundle of Ω which is invariant under local biholomorphic maps and unique modulo $O(r^{n+1})$.

In present paper our approach is viewing the CR invariants of a real hypersurface M of $\mathbb{C}^{n+1}$ as a scalar function defined on the jet space of CR embedding $F : M \to \mathbb{C}^{n+1}$ which is invariant under deformation of embedding. We express necessary and sufficient condition for scalar CR invariants using symmetry of the tangential Cauchy-Riemann equations.

Let $M = \{r = 0\}$ be a C^∞ real hypersurface in $\mathbb{C}^{n+1}$ and let $\{L_j\}_{j=1,\dots,n}$ be a C^∞ basis of the CR structure bundle $H^{1,0}(M) = \mathbb{C}T(M) \cap T^{1,0}(\mathbb{C}^{n+1})$. A mapping $F = (f^1, \dots, f^{n+1}) : M \to \mathbb{C}^{n+1}$ is a CR embedding if

$$\overline{L}_j f^k = 0,\ j = 1, \dots, n,\ k = 1, \dots, n+1 \tag{0.1}$$

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and

$$df^1 \wedge \cdots \wedge df^{n+1} \neq 0.$$

(0.1) is called the tangential Cauchy-Riemann equations.

A symmetry of CR embedding equations transforms a CR embedding into another. Hence if a function is invariant under CR maps, then it is invariant under a symmetry of CR embeddings. We show that in the case of C^∞ real hypersurfaces in $\mathbb{C}^{n+1}$ of nondegenerate Levi form, all scalar CR invariants are invariant under the symmetry of CR embeddings up to a power of determinants of CR maps and vice versa.

The merit of using the symmetry of CR embedding equations is that one need not construct special defining functions such as Fefferman's defining functions to define scalar CR invariants with weights.

We organize this paper as follows.

In §1 we introduce the definition of scalar CR invariants. In §2 we review some basic notions of jet theory and symmetry of partial differential equations. In §3 we study the infinitesimal symmetry of CR embeddings. In §4 we state and prove our main result and in §5 we restate our result on a $\mathbb{C}^*$ bundle of M.

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1. Definition of the scalar CR invariants

Let $M \subset \mathbb{C}^{n+1}$ be a real analytic(C^ω) real hypersurface containing $0 \in M$ as a reference point. Let $Z = (z, z_{n+1}) = (z_1, \ldots, z_n, z_{n+1}) \in \mathbb{C}^{n+1}$ and $z_{n+1} = u + iv$. Define

$$\langle z, z\rangle = \sum_{i,j=1}^{n} g_{i\bar{j}} z_i \bar{z}_j,$$

where $(g_{i\bar{j}})_{i,j=1,\ldots,n}$ is an $n \times n$ hermitian matrix with $\det(g_{i\bar{j}}) \neq 0$.

DEFINITION 1.1. M is said to be in Moser's normal form if M is given by

$$v = \langle z, z\rangle - F(z, \bar{z}, u),$$

where

$$F(z, \bar{z}, u) = \sum_{\substack{|\alpha|,|\beta| \geq 2 \\ l \geq 0}} A^l_{\alpha\beta} z^\alpha \bar{z}^\beta u^l$$

with $A^l_{\alpha\beta} = \overline{A^l_{\beta\bar{\alpha}}}$ and

$$\operatorname{tr} A^l_{22} = \operatorname{tr}^2 A^l_{23} = \operatorname{tr}^3 A^l_{33} = 0 \text{ for all } l \geq 0,$$

where $A^l_{i\bar{j}} = (A^l_{\alpha\beta} : |\alpha| = i, |\beta| = j)$ and tr is the trace with respect to $(g_{i\bar{j}})_{i,j=1,\ldots,n}$.

We have

Theorem 1.2. ([2], [9]) *If M is a C^ω real hypersurface of nondegenerate Levi form, then there exists a local biholomorphic map Φ such that $\Phi(M)$ is in Moser's normal form.*

If M is in Moser's normal form, we write $M = N(A)$, where $A = (A^l_{\alpha\beta})$ is a collection of the coefficients of the defining function of M. In general, Moser's normal form of M is not unique. In fact, M has a unique Moser's normal form if and only if M is locally equivalent to a hyperquadric, a real hypersurface M_0 defined by $v = \langle z, z\rangle$.

Let H be the isotropy group of the hyperquadric M_0 consisting of the holomorphic mappings leaving M_0 and the origin fixed and let $\mathcal{N}$ be the set of all Moser's normal form coefficients $A = (A^l_{\alpha\beta})$. Then there is a group action

$$\begin{aligned} H \times \mathcal{N} &\longrightarrow \mathcal{N} \\ (h, A) &\longrightarrow h \cdot A \end{aligned}$$

such that two hypersurfaces M and $\widetilde{M}$ are biholomorphically equivalent if and only if their Moser's normal form coefficients are in the same H-orbit.

DEFINITION 1.3. A polynomial $P(A)$ in $A \in \mathcal{N}$ is said to be a scalar CR invariant of weight w if

$$P(A) = |\det h'(0)|^{2w/(n+2)} P(h \cdot A)$$

for all $h \in H$.

Suppose P is a scalar CR invariant of weight w. Then for each C^ω hypersurface of nondegenerate Levi form M, we can define a real-valued C^ω function a_M as follows:

Let $p \in M$. Choose a local biholomorphic map Φ_p defined on a neighborhood of p such that $\Phi_p(p) = 0$ and $N(A) := \Phi_p(M)$ is in Moser's normal form. Define

$$a_M(p) := |\det {\Phi_p}'(p)|^{2w/(n+2)} P(A).$$

Then $a_M(p)$ is well-defined independently of the choice of Φ_p. Let

$$\Phi_p(Z) = \sum_{\widetilde{\alpha}} c_{\widetilde{\alpha}}(p)(Z - p)^{\widetilde{\alpha}}$$

be the power series expansion of Φ_p at p, where $\widetilde{\alpha} = (\alpha_1, \ldots, \alpha_{n+1})$ are $(n+1)$-tuples of nonnegative integers. Then for each $\widetilde{\alpha}$, $c_{\widetilde{\alpha}}(p)$ is determined by a finite jet of a defining function of M at p. In particular, if M is defined by

$$v - g(z, \bar{z}, u) = 0,$$

then $c_{\widetilde{\alpha}}(p)$ is a polynomial of the derivatives of g at p. Hence if M is C^ω, then $c_{\widetilde{\alpha}}(p)$ is also C^ω. Therefore a_M is C^ω. Furthermore, a_M satisfies a transformation law

$$a_M = |\det \Psi'|^{2w/(n+2)} a_{\Psi(M)} \circ \Psi$$

for any biholomorphic map Ψ.

Since $a_M(p)$ depends only on a finite jet of a defining function at p, we can define a smooth function a_M as above for any C^∞ hypersurfaces of nondegenerate Levi form.

2. Infinitesimal symmetry of differential equations

In this section we introduce basic notions of infinitesimal symmetry of differential equations. We refer [10] as a reference.

Let X be an open set of $\mathbb{R}^p$ and U be an open set of $\mathbb{R}^q$. Let $y : X \to U$ be a smooth map. By $(x, y^{(m)})$ we denote all the partial derivatives of $y = (y^1, \ldots, y^q)$ up to order m at x and by $j_x y$ we denote finite jet of y at x of unspecified order. The set $J^m(X, U) := \{(x, y^{(m)}) : (x, y) \in X \times U\}$, whose coordinates represent the independent variables, dependent variables and the derivatives of the dependent variables up to order m is called the m-th order jet space of the underlying space $X \times U$. A real valued smooth function $a(x, y^{(m)})$ defined on $J^m(X, U)$ is called a differential function of order m and denoted by $a[y]$.

Now consider a system of m-th order differential equations

$$\Delta^\nu(x, y^{(m)}) = 0\,, \quad \nu = 1, \ldots, l \tag{2.1}$$

for unknown functions $y = (y^1, \ldots, y^q)$ of p variables $x = (x_1, \ldots, x_p)$. For a p-tuple of integers $J = (j_1, \ldots, j_p)$, define $|J| = j_1 + \cdots + j_p$ and

$$D_J = \left(\frac{\partial}{\partial x_1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial x_p}\right)^{j_p}.$$

We consider an evolutionary vector field

$$V_Q = \sum_{a=1}^{q} Q_a[y] \frac{\partial}{\partial y^a},$$

where $Q = (Q_1[y], \ldots, Q_q[y])$ is a q-tuple of differential functions of unspecified order. The m-th prolongation of V_Q is an evolutionary vector field on $J^m(X, U)$ defined by

$$pr^{(m)}V_Q = V + \sum_a \sum_{1\le|J|\le m} \phi_J^a \frac{\partial}{\partial y_J^a},$$

where

$$\phi_J^a = D_J\phi^a$$

and

$$y_J^a = D_J y^a.$$

Let $\mathcal{I}$ be the set of all differential functions of the form

$$\sum_{|J|\ge 0} \sum_{\nu=1}^{l} H_\nu^J[y](D_J\Delta^\nu),$$

where $H_\nu^J[y]$ is a differential function of unspecified order. Then

DEFINITION 2.1. $V_Q = \sum_{a=1}^q Q_a[y](\partial/\partial y^a)$ is called a generalized infinitesimal symmetry of a system (2.1) if

$$(pr^{(m)}V_Q)\Delta^\nu = 0 \bmod \mathcal{I}$$

for all $\nu = 1, \dots, l$.

If $y = f(x)$ is a C^∞ solution of (2.1) and $V_Q = \sum_{a=1}^q Q_a[y](\partial/\partial u^a)$ is a generalized infinitesimal symmetry of (2.1), then V_Q evaluated on the jet of f

$$V_Q(j_x f) = \sum_{a=1}^q Q_a[j_x f]\frac{\partial}{\partial y^a}$$

is a C^∞ vector field on $f(X)$, which is an infinitesimal deformation of the solution f. We have

Theorem 2.2. *Suppose that $V_Q = \sum_{a=1}^q Q_a[y](\partial/\partial y^a)$ is a generalized infinitesimal symmetry of a system* (2.1) *and that $f(x)$ is a solution of* (2.1)*. Suppose a mapping*

$$y(\cdot, t) = (y^1, \dots, y^q) : X \times (-\epsilon, \epsilon) \to U$$

satisfies

$$\begin{cases} \dfrac{\partial y^a(x,t)}{\partial t} = Q_a[j_x y], \ a = 1, \ldots, q \\ \quad y(x,0) = f(x) \end{cases}$$

Then for each $t \in (-\epsilon, \epsilon)$, $y(\cdot, t)$ *is a solution of* (2.1).

Proof. See [10]. □

3. Infinitesimal symmetry for CR embeddings

Let M be a C^∞ real hypersurface in $\mathbb{C}^{n+1}$ of nondegenerate Levi form and let $\{L_i\}_{i=1,\ldots,n}$ be a basis of $H^{1,0}(M)$. Let $F = (f^1, \ldots, f^{n+1}) : M \to \mathbb{C}^{n+1}$ be a C^∞ embedding into $\mathbb{C}^{n+1}$.

For each $i = 1, \ldots, n$ and $a = 1, \ldots, n+1$, let

$$\begin{cases} \Delta_i^{2a-1} = \operatorname{Re}(\overline{L}_i f^a) \\ \quad \Delta_i^{2a} = \operatorname{Im}(\overline{L}_i f^a) \end{cases}. \tag{3.1}$$

Denote

$$\frac{\partial}{\partial \zeta^a} = \frac{1}{2}\left(\frac{\partial}{\partial y^{2a-1}} - \sqrt{-1}\frac{\partial}{\partial y^{2a}}\right)$$
$$\frac{\partial}{\partial \overline{\zeta}^a} = \frac{1}{2}\left(\frac{\partial}{\partial y^{2a-1}} + \sqrt{-1}\frac{\partial}{\partial y^{2a}}\right)$$

for all $a = 1, \ldots, n+1$. Then an evolutionary vector field

$$V_Q = \sum_{a=1}^{n+1} Q_a[y]\frac{\partial}{\partial \zeta^a} + \overline{Q}_a[y]\frac{\partial}{\partial \overline{\zeta}^a},$$

where $Q_a[y] = q_a^1[y] + \sqrt{-1}q_a^2[y]$, $\overline{Q}_a[y] = q_a^1[y] - \sqrt{-1}q_a^2[y]$ for some differential functions $q_a^1[y]$ and $q_a^2[y]$, $a = 1, \ldots, n+1$, is a generalized infinitesimal symmetry of (3.1) if and only if

$$\overline{L}_i Q_a[y] = 0 \bmod \mathcal{I}$$

for all $i = 1, \ldots, n$ and $a = 1, \ldots, n+1$, where $\mathcal{I}$ is the ideal generated by

$$\sum_{|J|\geq 0} \sum_{\nu=1}^{2n+2} H_\nu^J[y](D_J \Delta^\nu).$$

If $F : M \to \mathbb{C}^{n+1}$ is a C^∞ CR embedding, then V_Q evaluated on the jet of F is a C^∞ vector field $\sum_{a=1}^{n+1} \phi^a(\partial/\partial\zeta^a) + \sum_{a=1}^{n+1} \overline{\phi^a}(\partial/\partial\overline{\zeta}^a)$ on $F(M)$ such that ϕ^a is a C^∞ CR function defined on $F(M)$ for all $a = 1, \ldots, n+1$. On the other hand, if $V = \sum_{a=1}^{n+1} \phi^a(\partial/\partial\zeta^a)$ is a holomorphic vector field on $\mathbb{C}^{n+1}$, where ϕ^a, $a = 1, \ldots, n+1$, are holomorphic functions, then $V + \overline{V}$ is an infinitesimal symmetry of (3.1).

Now let T be a C^∞ real vector field on $M = \{r = 0\}$ such that $\partial r(\mathrm{T}) \neq 0$. Then $\{L_j\}_{j=1,\ldots,n}$ together with $L_{n+1} := \mathrm{J}(\mathrm{T}) + \sqrt{-1}\mathrm{T}$ span $T^{1,0}(\mathbb{C}^{n+1})$ along M, where J is the complex structure on $\mathbb{C}^{n+1}$. Hence there exists $A(x) = \left(A_a^b(x)\right)_{a,b=1,\ldots,n+1}$ such that each $A_a^b(x)$ is C^∞ on M and

$$\begin{pmatrix} & & \\ & A_a^b & \\ & & \end{pmatrix} \cdot \begin{pmatrix} L_1 \\ \vdots \\ L_{n+1} \end{pmatrix} = \begin{pmatrix} \partial/\partial z_1 \\ \vdots \\ \partial/\partial z_{n+1} \end{pmatrix}$$

along M.

For $(n+1)$-tuple of holomorphic functions $\phi = (\phi^1, \ldots, \phi^{n+1})$ on a neighborhood of M, define

$$V_\phi = \sum_a Q_a \frac{\partial}{\partial\zeta^a} + \overline{Q}_a \frac{\partial}{\partial\overline{\zeta}^a},$$

where

$$Q_a = \sum_{b=1}^{n+1} \phi^b \left(\sum_{j=1}^{n} A_b^j L_j \zeta^a + \sqrt{-1} A_b^{n+1} \mathrm{T}\zeta^a \right).$$

Then we have

Proposition 3.1. *V_ϕ is an infinitesimal symmetry of* (3.1). *Moreover, if a differential function $a[y] = a[y^{(m)}]$ which is holomorphic in its arguments satisfies*

$$(pr^{(m)} V_\phi) a[y] = 0 \bmod \mathcal{I} \tag{3.2}$$

for all $(n+1)$-tuples of holomorphic functions ϕ, then $a[y] = b(x)$ modulo $\mathcal{I}$, where $b(x)$ is a C^∞ function of x variables only.

Proof. Since ϕ^a, $a = 1, \ldots, n+1$, are holomorphic on a neighborhood of M, we can easily show that V_ϕ is an infinitesimal symmetry of (3.1).

Now suppose there is $(x_0, y^{(m)})$ such that $a[x_0, y^{(m)}] \neq a[x_0, id^{(m)}]$ modulo $\mathcal{I}$. We may assume that $y = F$ is a C^∞ embedding. Furthermore we may assume that there exist C^∞ embeddings $F_t = F + tH$, $0 \leq t \leq 1$, such that

$$\left.\frac{d}{dt}\right|_{t=0} a[x_0, F_t^{(m)}] \neq 0.$$

Choose a holomorphic mapping $\widetilde{F}$ and $\widetilde{H}$ such that

$$F = \widetilde{F} + O\big(|x - x_0|^{m+1}\big)$$

and

$$H = \widetilde{H} + O\big(|x - x_0|^{m+1}\big).$$

Then

$$\frac{d}{dt}\Big|_{t=0} a[x_0, F_t^{(m)}] = \sum_{b,J} \widetilde{H}_J^b(x_0) \frac{\partial a}{\partial \zeta_J^b}[x_0, \widetilde{F}^{(m)}],$$

where

$$\widetilde{H}_J^b(x) = \left(\frac{\partial}{\partial Z}\right)^J \widetilde{H}^b(x) := \left(\frac{\partial}{\partial z_1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial z_{n+1}}\right)^{j_{n+1}} \widetilde{H}^b(x).$$

Define

$$\phi^a = \sum_b \widetilde{H}^b \widetilde{G}_b^a,$$

where $\widetilde{G} = (\widetilde{G}_b^a)$ is the inverse matrix of $\widetilde{F} = (\widetilde{F}_b^a) = (\partial \widetilde{F}^a / \partial z^b)_{a,b=1,\dots,n+1}$ and define $V_\phi = \sum_a Q_a(\partial/\partial\zeta^a) + \overline{Q}_a(\partial/\partial\overline{\zeta}^a)$ as above.

Then

$$\widetilde{H}_J^a(x) = \left(\frac{\partial}{\partial Z}\right)^J Q_a[x, F^{(m)}].$$

Hence we have

$$(pr^{(m)} V_\phi) a[x_0, F^{(m)}] \neq 0,$$

which contradicts the assumption (3.2). □

4. Scalar invariants for CR embeddings and scalar CR invariants

Let M be an $(2n+1)$-dimensional C^∞ CR manifold and let $x = (x_1, \dots, x_{2n+1})$ be a coordinate system on M. Let $y : M \to \mathbb{C}^{n+1}$ be a C^∞ map such that $dy^1 \wedge \cdots \wedge dy^{2n+1} \neq 0$ on M. Then the image $y(M)$ is a graph $y^{2n+2} = g(y^1, \dots, y^{2n+1})$ of some C^∞ function g. For each positive integer m, we define a map π from an open subset of $\Omega^m \subset J^m(M, \mathbb{R}^{2n+2})$ to the m-th jet space $J^m(\mathbb{R}^{2n+1}, \mathbb{R})$ as follows:

For $m = 1$, consider the chain rule

$$\frac{\partial y^{2n+2}}{\partial x_i} = \sum_{k=1}^{2n+1} \frac{\partial g}{\partial y^k} \frac{\partial y^k}{\partial x_i} \quad i = 1, \dots, 2n+1. \tag{4.1}$$

Let Ω^1 be the subset of $J^m(M, \mathbb{R}^{2n+2})$ on which $[(\partial y^k/\partial x_i)]_{i,k=1,\ldots,2n+1}$ is non-singular. Then on Ω^1, we can solve (4.1) for $(\partial g/\partial y^k)$, $k = 1, \ldots, 2n+1$, in terms of $\partial y^a/\partial x_i$, $i = 1, \ldots, 2n+1$ and $a = 1, \ldots, 2n+2$. So define $\pi : \Omega^1 \to J^1(\mathbb{R}^{2n+1}, \mathbb{R})$ by $\pi(x, y^{(1)}) = (y', g^{(1)})$, where $y' = (y^1, \ldots, y^{2n+1})$. We define $\pi : \Omega^m \to J^m(\mathbb{R}^{2n+1}, \mathbb{R})$ inductively for each positive integer m.

From now on we only consider the case that M is a C^∞ real hypersurface in $\mathbb{C}^{n+1}$ of nondegenerate Levi form. Let θ be a non-vanishing real-valued 1-form of M which annihilates $H^{1,0}(M)+H^{0,1}(M)$. Since M is of nondegenerate Levi form, we can choose a unique θ up to sign such that $\theta \wedge (d\theta)^n = dV_M$, where dV_M is a volume form of M defined by $dV_M = n \rfloor dV$, where n is a unit normal vector field on M and dV is a volume form of $\mathbb{C}^{n+1}$.

Now consider a differential function P of m-th jet space $J^m(\mathbb{R}^{2n+1}, \mathbb{R})$ of g which is analytic in its arguments on a neighborhood of the m-th jet of $g = \langle z, z \rangle$ at 0 in $J^m(\mathbb{R}^{2n+1}, \mathbb{R})$. Let $\{L_1, \ldots, L_n\}$ be a basis of $H^{1,0}(M)$ and T be a real vector field of M such that $\theta(\mathrm{T}) = 1$. Assume that

$$d\theta = \sqrt{-1} \sum_{j,k=1}^{n} g_{j\bar{k}} \theta^j \wedge \overline{\theta}^k \bmod \theta,$$

where $\{\theta, \theta^j, \overline{\theta}^j\}_{j=1,\ldots,n}$ is the dual basis of $\{\mathrm{T}, L_j, \overline{L}_j\}$. Let $\mathcal{I}$ be the ideal as in Section 3. Then

DEFINITION 4.1. Let $F = (f^1, \ldots, f^{n+1}) : M \to \mathbb{C}^{n+1}$ be a C^∞ embedding and let P be a holomorphic function on $J^m(\mathbb{R}^{2n+1}, \mathbb{R})$. P is a scalar invariant of CR embedding if for all C^∞ embedding F,

$$P \circ \pi(x, F^{(m)}) = \left\{ c_{n+1}^2 \|\theta\| \cdot \left| \det(g_{j\bar{k}}) \right|^{-1} \cdot \left| \det(X_a f^b) \right|^2 \right\}^{w/(n+2)} a(x), \bmod \mathcal{I}$$

for some function $a(x)$ of only x variables, where $\|\theta\|$ is the Euclidean norm of θ, $c_{n+1} dz_1 \wedge d\bar{z}_1 \cdots dz_{n+1} \wedge \overline{d} z_{n+1} = dV_{2n+2}$ and $X_i = L_i$, $i = 1, \ldots, n$, $X_{n+1} = \mathrm{T}$.

Note that if $F : U \to V$ is a local biholomorphic map on a neighborhood of M, then $c_{n+1}^2 \|\theta\| \cdot |\det(g_{j\bar{k}})|^{-1} \cdot |\det(X_a f^b)|^2 = |\det F'|^2$. Hence we have

Theorem 4.2. *P is a scalar invariant of CR embeddings of weight w if and only if P is a scalar CR invariant of weight w.*

Suppose M is defined by $y^{2n+2} - g(y^1, \ldots, y^{2n+1}) = 0$ and P is a scalar invariant of CR embeddings of weight w. Since $P \circ \pi(x, id^{(m)}) = a(x)$, $a(x)$ is a scalar CR invariant of weight w by Theorem 4.2.

Proof of Theorem 4.2. Suppose $0 \in M$ is a C^∞ real hypersurface in $\mathbb{C}^{n+1}$ of nondegenerate Levi form. Let P be a scalar invariant of CR embeddings of weight w. If $F : U \to V$ is a (formal) local biholomorphic map on a neighborhood of M such that $F(M)$ is in Moser's normal form, then $P \circ \pi(x, F^{(m)})|_{x=0} = |\det F'(0)|^{2w/(n+2)} P'(A)$, where P' is a holomorphic function defined on $J^m(\mathbb{R}^{2n+1}, \mathbb{R})$ such that $P'(h \cdot A) = |\det h|^{-2w/(n+2)} P'(A)$.

Now let

$$h_\lambda(z, z_{n+1}) := (\lambda z_1, \ldots, \lambda z_n, \lambda^2 z_{n+1})$$

for some $\lambda > 0$. Then

$$P'(h_\lambda \cdot A) = \lambda^{-2w} P'(A).$$

Hence P' is a weighted homogeneous polynomial in $(A^l_{\alpha\beta})$ with weight

$$wt(A^l_{\alpha\beta}) := |\alpha| + |\beta| + 2l - 2.$$

Conversely, if P' is a polynomial in the coefficients of Moser's normal form $(A^l_{\alpha\beta})$ such that $P'(h \cdot A) = |\det h|^{-2w/(n+2)} P'(A)$, then define $P(y', g^{(m)})$ as follows:

Consider a C^∞ real hypersurface $\widetilde{M} = \{y^{2n+2} - g(y^1, \ldots, y^{2n+1}) = 0\}$ in $\mathbb{R}^{2n+2}$ with nondegenerate Levi form. Let $y_0 \in M$. After a holomorphic change of coordinates given by a quadratic map, we may assume that $y_0 = 0$ and

$$g(y^1, \ldots, y^{2n+1}) = \sum_{j,k=1}^{n} g_{j\bar{k}} \zeta^j \overline{\zeta}^k + o(|y^{2n+1}| + |\zeta|^2),$$

where $\zeta^j = y^{2j-1} + \sqrt{-1}\zeta^{2j}$ and $\zeta = (\zeta^1, \ldots, \zeta^n)$. Choose $\Phi : \widetilde{U} \to \widetilde{V}$, a formal series of local biholomorphic map on a neighborhood of $\widetilde{M}$ to a neighborhood $\widetilde{V}$ of $\Phi(\widetilde{M}) = \widetilde{N}$ which is in Moser's normal form, with the properties

$$d\Phi(0) = Id, \quad \frac{\partial^2 \Phi^{n+1}}{\partial \zeta^j \partial \zeta^k}(0) = 0, \quad j, k = 1, \ldots, n$$

and

$$\mathrm{Im}\left(\frac{\partial^2 \Phi^{n+1}}{\partial^2 \zeta^{n+1}}(0)\right) = 0,$$

where $\zeta^{n+1} = y^{2n+1} + \sqrt{-1}y^{2n+2}$.

Let $\widetilde{N} = \widetilde{N}(A)$. Then $A^l_{\alpha\beta}$, $l + |\alpha| + |\beta| \leq m$ are holomorphic functions in $g^{(m)}$ at $(y_0^1, \ldots, y_0^{2n+1})$. Define $P(y_0', g^{(m)}) = P'(A)$, where $y_0' = (y_0^1, \ldots, y_0^{2n+1})$. Then

$P(y_0', g^{(m)})$ is a holomorphic function in its arguments on a neighborhood of the m-th jet of $\langle z, z\rangle$ at 0 in $J^m(\mathbb{R}^{2n+1}, \mathbb{R})$ such that

$$P \circ \pi(x, F^{(m)}) = |\det F'|^{2w/(n+2)} a(x) \bmod \mathcal{I}$$

for all (local) biholomorphic map F. Since $P \circ \pi$ depends only on a finite jet of F at a reference point, this implies that P is a scalar invariant of CR embeddings of weight w. □

Now we will give an equivalent condition of Definition 4.1 by using infinitesimal symmetries of tangential Cauchy-Riemann equations.

Lemma 4.3. *Let* $y^{(1)} \in \Omega^1$. *If* $V_Q = \sum_{a=1}^{n+1} Q_a[y](\partial/\partial\zeta^a) + \overline{Q}_a[y](\partial/\partial\overline{\zeta}^a)$ *is an infinitesimal symmetry of* (3.1), *then*

$$\begin{aligned} &pr^{(1)} V_Q \left(\|\theta\| \cdot \left| \det(g_{j\bar{k}}) \right|^{-1} \cdot \left| \det(X_a f^b) \right|^2 \right) \\ &= \left(\operatorname{tr}(Q'[y]) + \overline{\operatorname{tr}(Q'[y])} \right) \left(\|\theta\| \cdot \left| \det(g_{j\bar{k}}) \right|^{-1} \cdot \left| \det(X_a f^b) \right|^2 \right) \bmod \mathcal{I}, \end{aligned}$$

where $(Q'[y]) = (X_a Q_b[y])_{a,b=1,\ldots,n+1}$ *for* $X_i = L_i$, $i = 1, \ldots, n$, $X_{n+1} = \mathrm{T}$ *and* tr *is the trace with respect to* $\left(X_a f^b\right)^{-1}_{a,b=1,\ldots,n+1}$.

Proof. Suppose there exists a one-parameter family F_t of local biholomorphic maps such that

$$\begin{cases} \left.\dfrac{\partial F_t^a(x)}{\partial t}\right|_{t=0} = Q_a(x, F^{(m)}), a = 1, \ldots, n+1 \\ F_0(x) = F(x) \end{cases}$$

Then

$$pr^{(1)} V_Q(\det F') = \left.\frac{\partial}{\partial t}\right|_{t=0} (\det F_t{}').$$

Let $h_t := F^{-1} \circ F_t$. Then $h_t : M \to \mathbb{C}^{n+1}$ is a one-parameter family of local biholomorphic maps such that $h_0 = id$. Hence

$$\begin{aligned} \left.\frac{\partial}{\partial t}\right|_{t=0} (\det F_t{}') &= (\det F') \left.\frac{\partial}{\partial t}\right|_{t=0} (\det h_t{}') \\ &= (\det F') \left(\sum_{a=1}^{n+1} \left.\frac{\partial}{\partial t}\right|_{t=0} \frac{\partial h_t^a}{\partial z^a} \right). \end{aligned}$$

Since

$$\frac{\partial}{\partial t}\Big|_{t=0}\frac{\partial h_t^a}{\partial z_a}=\frac{\partial}{\partial t}\Big|_{t=0}\left(\sum_{b=1}^{n+1}\frac{\partial(F^{-1})^a}{\partial\zeta^b}\frac{\partial F_t^b}{\partial z^a}\right)$$
$$=\sum_{b=1}^{n+1}\frac{\partial(F^{-1})^a}{\partial\zeta^b}\frac{\partial}{\partial z^a}\frac{\partial}{\partial t}\Big|_{t=0}F_t^b$$
$$=\sum_{b=1}^{n+1}\frac{\partial(F^{-1})^a}{\partial\zeta^b}\frac{\partial}{\partial z^a}(Q_b[F^{(m)}]),$$

we have

$$\frac{\partial}{\partial t}\Big|_{t=0}(\det F_t{}') = (\det f')\,\mathrm{tr}\left(\frac{\partial Q_b[F^{(m)}]}{\partial z_a}\right)_{a,b=1,\ldots,n+1},$$

where tr is the trace with respect to $(\partial f^b/\partial z_a)^{-1}_{a,b=1,\ldots,n+1}$. This implies that

$$pr^{(1)}V_Q(|\det F'|^2) = |\det F'|^2\left(\mathrm{tr}\left(\frac{\partial Q_b[F^{(m)}]}{\partial z_a}\right)+\overline{\mathrm{tr}\left(\frac{\partial Q_b[F^{(m)}]}{\partial z_a}\right)}\right),$$

which completes the proof. $\square$

Theorem 4.4. *P is a scalar CR invariant of weight w if and only if P satisfies*

$$pr^{(m)}V_Q(P\circ\pi(x,y^{(m)}))$$
$$=\frac{w}{n+2}\left(\mathrm{tr}(Q'[y])+\overline{\mathrm{tr}(Q'[y])}\right)P\circ\pi(x,y^{(m)}) \bmod \mathcal{I}$$

for any infinitesimal symmetry $V_Q=\sum_{a=1}^{n+1}Q_a[y](\partial/\partial\zeta^a)+\overline{Q}_a[y](\partial/\partial\overline{\zeta}^a)$ *of* (3.1).

Proof. Suppose P is a scalar CR invariant of weight w. Then

$$P\circ\pi(x,y^{(m)}) = \left\{c_{n+1}^2\|\theta\|\cdot\left|\det(g_{j\bar{k}})\right|^{-1}\cdot\left|\det(X_a f^b)\right|^2\right\}^{w/(n+2)}a(x) \bmod \mathcal{I}.$$

Hence by Lemma 4.3,

$$pr^{(m)}V_Q(P\circ\pi(x,y^{(m)}))$$
$$=\frac{w}{n+2}\left(\mathrm{tr}(Q'[y])+\overline{\mathrm{tr}(Q'[y])}\right)P\circ\pi(x,y^{(m)}) \bmod \mathcal{I}.$$

Now suppose P satisfies

$$
\begin{aligned}
& pr^{(m)} V_Q(P \circ \pi(x, y^{(m)})) \\
&= \frac{w}{n+2} \left(\mathrm{tr}(Q'[y]) + \overline{\mathrm{tr}(Q'[y])} \right) P \circ \pi(x, y^{(m)}) \bmod \mathcal{I}.
\end{aligned}
$$

Since

$$
\begin{aligned}
& pr^{(m)} V_Q \left(\|\theta\| \cdot \left| \det(g_{j\bar{k}}) \right|^{-1} \cdot \left| \det(X_a f^b) \right|^2 \right)^{w/(n+2)} \\
&= \frac{w}{n+2} \left(\mathrm{tr}(Q'[y]) + \overline{\mathrm{tr}(Q'[y])} \right) \left(\|\theta\| \cdot \left| \det(g_{j\bar{k}}) \right|^{-1} \cdot \left| \det(X_a f^b) \right|^2 \right)^{w/(n+2)} \bmod \mathcal{I},
\end{aligned}
$$

by multiplying $(\|\theta\| \cdot |\det(g_{j\bar{k}})|^{-1} \cdot \det(X_a f^b)|^2)^{-w/(n+2)}$, we have

$$
\begin{aligned}
& \left(\|\theta\| \cdot \left| \det(g_{j\bar{k}}) \right|^{-1} \cdot \left| \det(X_a f^b) \right|^2 \right)^{-w/(n+2)} pr^{(m)} V_Q \left(P \circ \pi(x, y^{(m)}) \right) \\
& - pr^{(m)} V_Q \left(\|\theta\| \cdot \left| \det(g_{j\bar{k}}) \right|^{-1} \cdot \left| \det(X_a f^b) \right|^2 \right)^{-w/(n+2)} P \circ \pi(x, y^{(m)}) \\
& = 0 \bmod \mathcal{I}
\end{aligned}
$$

for any infinitesimal symmetry V_Q of (3.1). This implies that

$$
\begin{aligned}
& pr^{(m)} V_Q \left(\left(\|\theta\| \cdot \left| \det(g_{j\bar{k}}) \right|^{-1} \cdot \left| \det(X_a f^b) \right|^2 \right)^{-w/(n+2)} P \circ \pi(x, y^{(m)}) \right) \\
& = 0 \bmod \mathcal{I}
\end{aligned}
$$

for any infinitesimal symmetry V_Q of (3.1). Hence by Proposition 3.1 we have

$$
\left(\|\theta\| \cdot \left| \det(g_{j\bar{k}}) \right|^{-1} \cdot \left| \det(X_a f^b) \right|^2 \right)^{-w/(n+2)} P \circ \pi(x, y^{(m)}) = a(x) \bmod \mathcal{I}
$$

for some function $a(x)$ of only x variables. □

5. Scalar invariants of CR embeddings on a $\mathbb{C}^*$ bundle

In Section 4 we have to choose θ such that $\theta \wedge (d\theta)^n = dV_M$. In this section we restate our main theorem without choosing θ by considering a $\mathbb{C}^*$ bundle $\mathbb{C}^* \times M$ of M as in [4] and [5].

Suppose $\widetilde{M} \subset \mathbb{C}^{n+1}$ is defined by

$$
\rho = y^{2n+2} - g(y^1, \ldots, y^{2n+1}) = 0. \tag{5.1}
$$

Now consider $\mathbb{C}^* \times \widetilde{M}$. We regard this as a real hypersurface in $\mathbb{C}^* \times \mathbb{C}^{n+1}$ defined by (5.1).

Let $(\zeta^0, \zeta) = (\zeta^0, \zeta^1, \ldots, \zeta^{n+1}) \in \mathbb{C}^* \times \mathbb{C}^{n+1}$. Let $J^m(\mathbb{C}^* \times \mathbb{R}^{2n+1}, \mathbb{R})$ be the m-th jet space of C^∞ functions $g_\sharp : \mathbb{C}^* \times \mathbb{R}^{2n+1} \to \mathbb{R}$. If $g : \mathbb{R}^{2n+1} \to \mathbb{R}$, then define

$$g_\sharp = |\zeta^0|^{-2/(n+2)} g(y^1, \ldots, y^{2n+1}).$$

Let U be a neighborhood of M. Consider the m-th jet space $\mathcal{S}_\sharp^m \subseteqq J^m(\mathbb{C}^* \times M, \mathbb{C}^* \times \mathbb{C}^{n+1})$ consisting of restrictions of local biholomorphic maps

$$\phi_\sharp : \mathbb{C}^* \times U \to \mathbb{C}^* \times \mathbb{C}^{n+1}$$

such that $\det \phi_\sharp \equiv 1$.

If $F : U \to \mathbb{C}^{n+1}$ is a local biholomorphic map, then define

$$F_\sharp((z_0, Z)) = (z_0(\det F'(Z))^{-1}, F(Z)), \tag{5.2}$$

where $(z_0, Z) \in \mathbb{C}^* \times U$. Then $F_\sharp^{(m)} \in \mathcal{S}_\sharp^m$.

For $\phi_\sharp : \mathbb{C}^* \times M \to \mathbb{C}^* \times \mathbb{C}^{n+1}$, let $y'_\sharp = (\phi_\sharp^0, \ldots, \phi_\sharp^n, \mathrm{Re}(\phi_\sharp^{n+1}))$. Now consider $\pi_\sharp : J^m(\mathbb{C}^* \times M, \mathbb{C}^* \times \mathbb{C}^{n+1}) \to J^m(\mathbb{C}^* \times \mathbb{R}^{2n+1}, \mathbb{R})$ defined by $\pi_\sharp((z_0, Z), \phi_\sharp^{(m)}) = (y'_\sharp, g_\sharp^{(m)})$. Let $P_\sharp$ be a differential function which is holomorphic on a neighborhood of the m-th jet of $g_\sharp = |\zeta^0|^{-2/(n+1)} \sum_{j,k=1}^n g_{j\bar{k}} \zeta^j \overline{\zeta}^k$ at $(\zeta^0, \zeta) = (1, 0)$. Then

Theorem 5.1. *If $P_\sharp \circ \pi_\sharp = a(z_0, Z, \bar{z}_0, \overline{Z})$ on $\mathcal{S}_\sharp^m$ for some function a then*

$$P_\sharp = \sum_w |\zeta^0|^{-2w/(n+2)} P_w,$$

where P_w is a scalar CR invariant of weight w.

Proof. Let $J_\sharp^m \subset J^m(\mathbb{C}^* \times \mathbb{R}^{2n+1}, \mathbb{R})$ be the m-th jet space of functions of the form

$$g_\sharp = |\zeta^0|^{-2/(n+1)} g(y^1, \ldots, y^{2n+1}).$$

Then $P_\sharp|_{J_\sharp^m}$ is a holomorphic function in

$$(\zeta^0)^{-w_1/(n+2)} (\overline{\zeta^0})^{-w_2/(n+2)} D_J g(y^1, \ldots, y^{2n+1}), \quad w_1 + w_2 + |J| \le m.$$

Thus

$$P_\sharp = \sum_{w_1, w_2} (\zeta^0)^{-w_1/(n+2)} (\overline{\zeta^0})^{-w_2/(n+2)} P_{w_1 w_2},$$

where $P_{w_1 w_2}$ is holomorphic in $g^{(m)}$.

Since $P_\sharp$ is real, $w_1 = w_2$. Hence

$$P_\sharp = \sum_w |\zeta^0|^{-2w/(n+2)} P_w,$$

where P_w is holomorphic in $g^{(m)}$ and

$$P_\sharp \circ \pi_\sharp((z_0, Z), F_\sharp^{(m)}) = \sum_w \left(|z_0|^{-1} |\det F'|\right)^{2w/(n+2)} P_w \circ \pi(Z, F^{(m)})$$

for all $F_\sharp$ defined by (5.2).

On the other hand, if $P_\sharp \circ \pi_\sharp = a(z_0, Z, \bar{z}_0, \overline{Z})$ on $\mathcal{S}_\sharp^m$, then

$$a(z_0, Z, \bar{z}_0, \overline{Z}) = \sum_w |z_0|^{-2w/(n+2)} a_w(x)$$

on $\mathcal{S}_\sharp^m$, where a_w is a C^∞ function defined on M. Hence

$$P_w \circ \pi(x, F^{(m)}) = |\det F'|^{2w/(n+2)} a_w(x)$$

for all local biholomorphic map F, which implies that P_w is a scalar CR invariant of weight w.

□

Suppose $V_\sharp = \sum_{\tilde{a}=0}^{n+1} \phi_\sharp^{\tilde{a}} (\partial/\partial\zeta^{\tilde{a}})$ is a holomorphic vector field on $\mathbb{C}^* \times \mathbb{C}^{n+1}$ such that $\phi_\sharp^{\tilde{a}}$, $\tilde{a} = 0, \ldots, n+1$, are holomorphic functions on $\mathbb{C}^* \times \mathbb{C}^{n+1}$. Then $V_\sharp$ is an infinitesimal symmetry of $\mathcal{S}_\sharp^m$ if and only if $\operatorname{tr}\phi'_\sharp = 0$, where $\operatorname{tr}\phi'_\sharp$ is the trace of $(\partial(\phi_\sharp^0, \ldots, \phi_\sharp^{n+1})/\partial(z_0, \ldots, z_{n+1}))$.

If $V = \sum_{a=1}^{n+1} \phi^a (\partial/\partial\zeta^a)$ is a holomorphic vector field on $\mathbb{C}^{n+1}$, then define

$$V_\sharp = \sum_{\tilde{a}=0}^{n+1} \phi_\sharp^{\tilde{a}} \frac{\partial}{\partial\zeta^{\tilde{a}}}$$

with

$$\phi_\sharp^{\tilde{a}} = \begin{cases} -\zeta^0 \operatorname{tr}\phi'(\zeta) & \tilde{a} = 0 \\ \phi^{\tilde{a}}(\zeta) & \tilde{a} = 1, \ldots, n+1 \end{cases},$$

where $\operatorname{tr}\phi'$ is the trace of $\left(\partial(\phi^1, \ldots, \phi^{n+1})/\partial(z_1, \ldots, z_{n+1})\right)$. Then $V_\sharp$ is a holomorphic vector field on $\mathbb{C}^* \times \mathbb{C}^{n+1}$ such that $\operatorname{tr}\phi'_\sharp = 0$.

Consider $\mathcal{S}_\sharp^m$ as a subbundle of $J^m(\mathbb{C}^* \times M, \mathbb{C}^* \times \mathbb{C}^{n+1})$ over $\mathbb{C}^* \times M$. Then the set of vectors

(5.3)
$$\left\{ pr^{(m)}(V_\sharp + \overline{V}_\sharp) : V_\sharp = \sum_{\tilde{a}=0}^{n+1} \phi_\sharp^{\tilde{a}} \frac{\partial}{\partial\zeta^{\tilde{a}}}, \quad \operatorname{tr}\phi'_\sharp = 0, \quad \phi_\sharp^{\tilde{a}} : \text{ holomorphic functions } \right\}$$

spans the vertical tangent spaces of $\mathcal{S}_\sharp^m$.

Theorem 5.2. *If* $pr^{(m)}(V_\sharp + \overline{V}_\sharp)P_\sharp \circ \pi_\sharp = 0$ *on* $\mathcal{S}_\sharp^m$ *for all infinitesimal symmetry of* $\mathcal{S}_\sharp^m$ *of the form* (5.3), *then*

$$P_\sharp = \sum_w |\zeta^0|^{-2w/(n+2)} P_w,$$

where P_w *is a scalar CR invariant of weight* w.

Proof. Suppose $pr^{(m)}(V_\sharp + \overline{V}_\sharp)P_\sharp \circ \pi_\sharp = 0$ on $\mathcal{S}_\sharp^m$ for all infinitesimal symmetry of $\mathcal{S}_\sharp^m$ of the form (5.3). Since $pr^{(m)}(V_\sharp + \overline{V}_\sharp)$ of the form (5.3) span the vertical vector spaces of $\mathcal{S}_\sharp^m$ over $\mathbb{C}^* \times M$,

$$P_\sharp \circ \pi_\sharp = a(z_0, Z, \overline{z}_0, \overline{Z})$$

on $\mathcal{S}_\sharp^m$. Thus by Theorem 5.1,

$$P_\sharp = \sum_w |\zeta^0|^{-2w/(n+2)} P_w \qquad \square$$

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