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Osaka J. Math.
38 (2001), 827–837

ON THE CLASSIFICATION OF SIMPLE MODULES FOR CYCLOTOMIC HECKE ALGEBRAS OF TYPE $G(m,1,n)$ AND KLESHCHEV MULTIPARTITIONS

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(Received March 2, 2000)

1. Introduction

After Hecke algebras appeared, unexpectedly deep applications and results have been found in the representation theory of these algebras. Concerned with ordinary representations, Lusztig's cell theory is the main driving force. But we do not consider it here. The other interest is about the modular representation theory of these algebras. We are mainly working with Hecke algebras of type A and type B, and this research is driven by Dipper and James [5, 6]. Recently, a new type of Hecke algebras was introduced. We call them cyclotomic Hecke algebras of type $G(m,1,n)$ following [4]. Hecke algebras of type A and type B are special cases of these algebras. The author studied modular representations of the algebra in the case that parameters are roots of unity in the field of complex numbers [1]. In particular, it gives a classification of simple modules. Removal of the restriction on base fields was achieved in [3]. In the paper [3], we gave a classification of the simple modules of cyclotomic Hecke algebras in terms of the crystal graphs of integrable highest weight modules of certain quantum algebras. The result turns out to be useful for verifying a conjecture of Vigneras [30].

On the other hand, another approach was already proposed in [10, 7]. Main results in the theory are that we can define "Specht modules", and that each Specht module $S^{\underline{\lambda}}$ has a natural bilinear form, and each of $D^{\underline{\lambda}} := S^{\underline{\lambda}}/\operatorname{rad} S^{\underline{\lambda}}$ is an absolutely irreducible or zero module. Further, the theory claims that the set of non-zero $D^{\underline{\lambda}}$ is a complete set of simple modules.

But there is one drawback. The theory does not tell which $D^{\underline{\lambda}}$ are actually non-zero. We conjectured in [3] that the crystal graph description gave the criterion. Namely, we conjectured that $D^{\underline{\lambda}} \neq 0$ if and only if $\underline{\lambda}$ is a Kleshchev multipartition. The purpose of this paper is to prove the conjecture. It is achieved by interpreting the conjecture into a problem about canonical bases in Fock spaces. This part is based on [1] and [3]. Then the conjecture is easily verified by using a recent result of Uglov

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A.M.S. subject classification, 20C20, 20C33, 20G05
This work is a contribution to the JSPS-DFG Japanese-German Cooperative Science Promotion Program on "Representation Theory of Finite and Algebraic Groups"

[26, 27].

The author is grateful to A.Mathas for discussion he had at the early stage of the research. He also thanks B.Leclerc, Varagnolo and Vasserot.

2. Preliminaries

Let R be an integral domain, $u_1, \ldots, u_m$ be elements in R, and ζ be an invertible element. The Hecke algebra of type $G(m,1,n)$ is the R-algebra associated with these parameters defined by the following defining relations for generators a_i $(1 \leq i \leq n)$. We denote this algebra by $\mathcal{H}_n$.

$$
\begin{gathered}
(a_1 - u_1)\cdots(a_1 - u_m) = 0, \qquad (a_i - \zeta)(a_i + \zeta^{-1}) = 0 \quad (i \geq 2) \\
a_1a_2a_1a_2 = a_2a_1a_2a_1, \qquad a_ia_j = a_ja_i \quad (j \geq i+2) \\
a_ia_{i-1}a_i = a_{i-1}a_ia_{i-1} \; (3 \leq i \leq n)
\end{gathered}
$$

It is known that this algebra is R-free of rank $m^n n!$ as an R-module. This algebra is also known to be cellular in the sense of Graham and Lehrer [10], and thus has Specht modules. Following [7], we shall explain the theory. A *partiton* λ of size n is a sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ such that $\sum \lambda_i = n$. We write $|\lambda| = n$. A *multipartiton* of size n is a sequence of m partitions $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ such that $\sum_{k=1}^m |\lambda^{(k)}| = n$. If $n = 0$, we denote the multipartition by $\underline{\emptyset}$. The set of multipartitions of a given size has a poset structure. The partial order is the *dominance order*, which is defined as follows.

DEFINITION 2.1. Let $\underline{\lambda}$ and $\underline{\mu}$ be multipartitions of a same size. We say that $\underline{\lambda}$ dominates $\underline{\mu}$, and write $\underline{\lambda} \trianglerighteq \underline{\mu}$ if we have for all j, k that

$$
\sum_{l=1}^{k-1} |\lambda^{(l)}| + \sum_{i=1}^{j} \lambda_i^{(k)} \geq \sum_{l=1}^{k-1} |\mu^{(l)}| + \sum_{i=1}^{j} \mu_i^{(k)}.
$$

With each multipartition $\underline{\lambda}$ of size n, we can associate an $\mathcal{H}_n$-module $S^{\underline{\lambda}}$. Its concrete construction is explained in [7, (3.28)]. It is easy to see from the construction that it is free as an R-module. These modules are called *Specht modules*. Each Specht module is naturally equipped with a bilinear form. We set $D^{\underline{\lambda}} = S^{\underline{\lambda}} / \operatorname{rad} S^{\underline{\lambda}}$, where $\operatorname{rad} S^{\underline{\lambda}}$ is the radical of the bilinear form. It can be zero, but non-zero ones exhaust all simple $\mathcal{H}_n$-modules. We denote the projective cover of $D^{\underline{\lambda}}$ by $P^{\underline{\lambda}}$.

We remark that Graham and Lehrer have introduced the notion of cellular algebras and have developped general theory for classifying simple modules using "cell modules". In [10], the cellular bases for the cell modules are given by Kazhdan-Lusztig bases. Here, different cellular bases are used, but the strategy to classify simple modules is the same. Hence we call the following parametrization the Graham-Lehrer parametrization.

Theorem 2.2 ([7, Theorem 3.30]). *Suppose that R is a field. Then,*
(1) *Non-zero $D^{\underline{\lambda}}$ form a complete set of non-isomorphic simple $\mathcal{H}_n$-modules. Further, these modules are absolutely irreducible.*
(2) *Let $\underline{\lambda}$ and $\underline{\mu}$ be multipartitions of size n and suppose that $D^{\underline{\mu}} \neq 0$ and $[S^{\underline{\lambda}} : D^{\underline{\mu}}] \neq 0$. Then we have $\underline{\lambda} \trianglerighteq \underline{\mu}$.*
(3) $[S^{\underline{\lambda}} : D^{\underline{\lambda}}] = 1$.

Note that (2) is equivalent to the following (2').
(2') *Let $\underline{\lambda}$ and $\underline{\mu}$ be multipartitions of size n and suppose that $D^{\underline{\mu}} \neq 0$ and $[P^{\underline{\mu}} : S^{\underline{\lambda}}] \neq 0$. Then we have $\underline{\lambda} \trianglerighteq \underline{\mu}$.*

It is obvious since we have $[P^{\underline{\mu}} : S^{\underline{\lambda}}] = \dim \operatorname{Hom}_{\mathcal{H}_n}(P^{\underline{\mu}}, S^{\underline{\lambda}}) = [S^{\underline{\lambda}} : D^{\underline{\mu}}]$.

As is explained in [3, 1.2], the classification of simple $\mathcal{H}_n$-modules is reduced to the classification in the case that $u_1, \ldots, u_m$ are powers of ζ^2. This is a consequence of a result in [28, 2.13] (see also [9]). We can also assume that $\zeta^2 \neq 1$, since the case $\zeta^2 = 1$ is well understood. In the rest of the paper throughout, we assume that

$$u_i = \zeta^{2\gamma_i} \quad (i = 1, \ldots, m), \qquad \zeta^2 \neq 1.$$

If ζ^2 is a primitive r th root of unity for a natural number r, γ_i take values in $\mathbb{Z}/r\mathbb{Z}$. Otherwise, these take values in $\mathbb{Z}$.

Next we recall the notion of *Kleshchev multipartitions* associated with $(\gamma_1, \ldots, \gamma_m)$. To do this, we explain the notion of good nodes first.

We identify a multipartition $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ with the associated Young diagram, i.e. an m-tuple of the Young diagrams associated with $\lambda^{(1)}, \ldots, \lambda^{(m)}$. Let x be a node on the Young diagram which is located on the a th row and the b th column of $\lambda^{(c)}$. If $u_c\zeta^{2(b-a)} = \zeta^{2i}$, we say that the node x has *residue* i (with respect to $\gamma = (\gamma_1, \ldots, \gamma_m)$). We denote the residue by $r_\gamma(x)$. A node is called an *i-node* if its residue is i. Let $\underline{\lambda}$ and $\underline{\mu}$ be multipartitions. We first assume that $|\underline{\lambda}| + 1 = |\underline{\mu}|$, and the node $x := \underline{\mu}/\underline{\lambda}$ has $r_\gamma(x) \equiv i$. We then call x an *addable i-node* of $\underline{\lambda}$. If $|\underline{\lambda}| - 1 = |\underline{\mu}|$ and $x := \lambda/\underline{\mu}$ has $r_\gamma(x) \equiv i$, we call x a *removable i-node* of $\underline{\lambda}$.

For each residue i, we have the notion of normal i-nodes and good i-nodes. To define these, We read addable and removable i-nodes of λ in the following way. We start with the first row of $\lambda^{(1)}$, and we read rows in $\lambda^{(1)}$ downward. We then move to the first row of $\lambda^{(2)}$, and repeat the same procedure. We continue the procedure to $\lambda^{(3)}, \ldots, \lambda^{(m)}$. If we write A for an addable i-node, and similarly R for a removable i-node, we get a sequence of A and R. We then delete RA as many as possible. For example, if the sequence is $RRAAAARRRAARAR$, it ends up with $----AAR------R$. The remaining removable i-nodes in this sequence are called *normal i-nodes*. The node corresponding to the leftmost R is called the *good i-node*. If x is a good i-node for some i, we simply say that x is a *good node*. We can now define the set of Kleshchev multipartitions associated with $\gamma = (\gamma_1, \ldots, \gamma_m)$.

DEFINITION 2.3. We declare that $\underline{\emptyset}$ is Kleshchev. Assume that we have already defined the set of Kleshchev multipartitions of size n.

Let $\underline{\lambda}$ be a multipartition of size $n+1$. We say that $\underline{\lambda}$ is Kleshchev if and only if there is a good node x of $\underline{\lambda}$ such that $\underline{\mu} := \underline{\lambda} \setminus \{x\}$ is a Kleshchev multipartition.

We denote the set of Kleshchev multipartitions of size n by ${}^{\gamma}\mathcal{KP}_n$, and set ${}^{\gamma}\mathcal{KP} = \sqcup_{n\geq 0}{}^{\gamma}\mathcal{KP}_n$. The following theorem provides us another way to parametrize simple $\mathcal{H}_n$-modules.

Theorem 2.4 ([3, Theorem C]). *Suppose that ζ^2 and u_i satisfy the above condition $S^2 \neq 1$ and $u_i = S^{2\gamma_i}(1 \leq i \leq m)$. Then, the irreducible $\mathcal{H}_n$-modules are indexed by the set of Kleshchev multipartitions.*

Hence we have two parametrizations. One given in Theorem 2.2 and one given in Theorem 2.4. It is natural to ask, if these coincide. The main observation is the following conjecture, which will be proved in the last section. The conjecture was formulated by Mathas.

Conjecture ([3, 2.12]). These two parametrizations coincide. In particular, $D^{\underline{\lambda}} \neq 0$ if and only if $\underline{\lambda}$ is a Kleshchev multipartition.

To prove this, we use certain Fock spaces, which are modules of a quantum algebra*. In the next section, we recall necessary ingredients of these Fock spaces.

3. Fock spaces

Recall that the multiplicative order of ζ^2 is $r \geq 2$. We denote by U_v the quantum algebra of type $A^{(1)}_{r-1}$ if r is finite, and of type A_∞ if $r = \infty$. Let $\mathcal{F}_v^\gamma$ be the combinatorial Fock space: it is a U_v-module, whose basis elements are indexed by the set of all multipartitions. We identify the basis elements with the multipartitions. The size of multipartitions naturally makes it into a graded vector space.

We consider the U_v-submodule $\mathcal{M}_v^\gamma$ of $\mathcal{F}_v^\gamma$ generated by the empty multipartition $\underline{\emptyset}$. It is isomorphic to an irreducible highest weight module with highest weight $\Lambda = \Lambda_{\gamma_1} + \cdots + \Lambda_{\gamma_m}$, where Λ_i are fundamental weights. To describe its basis in a combinatorial way, we need the crystal graph theory of Kashiwara. In our particular setting, we can prove the following theorem using argument in [22]. The theorem explains the representation theoretic meaning of Kleshchev multipartitions.

*The idea to use such Fock spaces to study the modular representation theory of cyclotomic Hecke algebras first appeared in [1], generalizing and verifying a conjecture of Lascoux, Leclerc and Thibon [18].

Theorem 3.1 ([3, Theorem 2.9, Corollary 2.11]). *Let R_v be the localized ring of $\mathbb{Q}[v]$ with respect to the prime ideal (v). We consider the R_v-lattice of $\mathcal{F}_v^\gamma$ generated by all multipartitions, and denote it by $\mathcal{L}_v^\gamma$. We set*

$$L(\Lambda) = \mathcal{L}_v^\gamma \cap \mathcal{M}_v^\gamma, \text{ and } B(\Lambda) = \{\underline{\lambda} \pmod{v\mathcal{L}_v^\gamma} \mid \underline{\lambda} \in {}^\gamma\mathcal{KP}\}.$$

Then, $(L(\Lambda), B(\Lambda))$ is a (lower) crystal base of $\mathcal{M}_v^\gamma$ in the sense of Kashiwara.

It is known that the canonical basis of U_v^- multiplied by the empty multipartition gives a crystal base of $\mathcal{M}_v^\gamma$ [11], which is unique up to a scalar multiple [13]. More precisely, the crystal lattice $L(\Lambda)$ is the R_v-lattice generated by these canonical basis elements of $\mathcal{M}_v^\gamma$, and $B(\Lambda)$ consists of the canonical basis elements modulo $vL(\Lambda)$. Hence, this theorem says that for each Kleshchev multipartition $\underline{\nu}$, there exists a unique canonical basis element $G(b)$ of $\mathcal{M}_v^\gamma$ such that

$$G(b) \pmod{vL(\Lambda)} = \underline{\nu} \pmod{v\mathcal{L}_v^\gamma}$$

and vice-versa.

To explain the U_v-module structure given to $\mathcal{F}_v^\gamma$, we first fix notations. Let $\underline{\lambda}$ be a multipartition and let x be a node on the associated Young diagram which is located on the a th row and the b th column of $\lambda^{(c)}$. Then we say that a node is *above* x if it is on $\lambda^{(k)}$ for some $k < c$, or if it is on $\lambda^{(c)}$ and its row number is strictly smaller than a. We denote the set of addable (resp. removable) i-nodes of $\underline{\lambda}$ which are above x by $A_i^a(x)$ (resp. $R_i^a(x)$). In a similar way, we say that a node is *below* x if it is on $\lambda^{(k)}$ for some $k > c$, or if it is on $\lambda^{(c)}$ and its row number is strictly greater than a. We denote the set of addable (resp. removable) i-nodes of $\underline{\lambda}$ which are below x by $A_i^b(x)$ (resp. $R_i^b(x)$). The set of all addable (resp. removable) i-nodes of $\underline{\lambda}$ is denoted by $A_i(\underline{\lambda})$ (resp. $R_i(\underline{\lambda})$).

In the similar way, we define the notion that a node is *left* to x (resp. *right* to x). We denote the set of addable i-nodes which are left to x (resp. right to x) by $A_i^l(x)$ (resp. $A_i^r(x)$). The set of removable i-nodes which are left to x (resp. right to x) is denoted by $R_i^l(x)$ (resp. $R_i^r(x)$). We then set

$$N_i^a(x) = |A_i^a(x)| - |R_i^a(x)|, \quad N_i^b(x) = |A_i^b(x)| - |R_i^b(x)|,$$
$$N_i(\underline{\lambda}) = |A_i(\underline{\lambda})| - |R_i(\underline{\lambda})|.$$

$N_i^l(x)$ and $N_i^r(x)$ are similarly defined. Finally, we denote the number of all 0-nodes in $\underline{\lambda}$ by $N_d(\underline{\lambda})$. Then the U_v-module structure of $\mathcal{F}_v^\gamma$ (called Hayashi action) is defined as follows.

$$e_i\underline{\lambda} = \sum_{r_\gamma(\underline{\lambda}/\underline{\mu})\equiv i} v^{-N_i^a(\lambda/\underline{\mu})}\underline{\mu}, \quad f_i\underline{\lambda} = \sum_{r_\gamma(\mu/\underline{\lambda})\equiv i} v^{N_i^b(\underline{\mu}/\underline{\lambda})}\underline{\mu},$$
$$v^{h_i}\underline{\lambda} = v^{N_i(\underline{\lambda})}\underline{\lambda}, \quad v^d\underline{\lambda} = v^{-N_d(\underline{\lambda})}\underline{\lambda}.$$

To compare it with other Fock spaces, we introduce another U_v-module $\mathcal{F}_{v^{-1}}^{-\gamma}$. It is also the space with basis indexed by multipartitions, but the action is given by the following.

$$e_i\underline{\lambda} = \sum_{r_{-\gamma}(\underline{\lambda}/\underline{\mu})\equiv i} v^{N_i^l(\underline{\lambda}/\underline{\mu})}\underline{\mu}, \quad f_i\underline{\lambda} = \sum_{r_{-\gamma}(\underline{\mu}/\underline{\lambda})\equiv i} v^{-N_i^r(\underline{\mu}/\underline{\lambda})}\underline{\mu}$$

$$v^{h_i}\underline{\lambda} = v^{N_i(\underline{\lambda})}\underline{\lambda}, \quad v^d\underline{\lambda} = v^{-N_d(\underline{\lambda})}\underline{\lambda}$$

Multipartitions constitute "basis at $v = \infty$" in the sense of Lusztig. We denote by $\mathcal{M}_{v^{-1}}^{-\gamma}$ its U_v-submodule generated by $\underline{\emptyset}$.

For each partition λ we denote its transpose by λ'. For a multipartition $\underline{\lambda}$, we denote $(\lambda^{(1)\prime}, \dots, \lambda^{(m)\prime})$ by $\underline{\lambda}^{\mathrm{T}}$ and call it the *transpose* of $\underline{\lambda}$.

Let $\xi : \mathcal{F}_v^{\gamma} \to \mathcal{F}_{v^{-1}}^{-\gamma}$ be a semilinear map which sends $\underline{\lambda}$ to $\underline{\lambda}^{\mathrm{T}}$. Then an addable (resp. removable) i-node of $\underline{\lambda}$ corresponds to an addable (resp. removable) $-i$-node of $\underline{\lambda}^{\mathrm{T}}$. Hence, the action of f_i on $\mathcal{F}_v^{\gamma}$ corresponds to the action of f_{-i} on $\mathcal{F}_{v^{-1}}^{-\gamma}$. Since the involution $f_i \mapsto f_{-i}$ of U_v^- permutes the canonical basis elements, we have that if $G(b)$ is a canonical basis element of $\mathcal{F}_v^{\gamma}$, then $\xi(G(b))$ is a canonical basis element of $\mathcal{F}_{v^{-1}}^{-\gamma}$.

We now recall Takemura-Uglov Fock spaces. In [25], Takemura and Uglov have constructed higher level Fock spaces generalizing [14, Proposition 1.4]. Let $\{u_k\}_{i\in\mathbb{Z}}$ be the basis vectors of an infinite dimensional space. More precisely, the space is originally $\mathbb{Q}(v)^r \otimes \mathbb{Q}(v)^m[z, z^{-1}]$, and if we denote the basis elements by $e_a \otimes e_b z^N$, we identify u_k with $e_a \otimes e_b z^N$ through $k = a + r(b - 1 - mN)$ as in [27]. We warn that there are differences between definitions in [25], [26] and [27]. We follow [27] here. Since $\mathbb{Q}(v)^r[z, z^{-1}]$ is naturally a U_v'-module, this space is also a U_v'-module. We now consider semi-infinite wedges of the form $u_I = u_{i_1} \wedge u_{i_2} \wedge \cdots$ with $i_k = c - k + 1$ for all $k \gg 0$. These are called semi-infinite wedges of charge c. The space spanned by semi-infinite wedges of charge c is denoted by $\mathcal{F}_c$. To make $\mathcal{F}_c$ into a U_v-module, we use the following coproduct. (Compare it with [27, 3.5])

$$\Delta^{(l)}(f_i) = f_i \otimes 1 + v^{-h_i} \otimes f_i$$

A wedge u_I is called *normally ordered* if the indices i_k are in descending order. Straightening laws are given in [27, Proposition 3.16], and the normally ordered semi-infinite wedges of charge c form a basis of $\mathcal{F}_c$ [27, Proposition 4.1].

For a normally ordered wedge, we locate its indices on an abacus with rm runners. On each runner, larger numbers appear in upper location, and the row containing 1 is read $1, \dots, rm$ from left to right. We divide the set of these runners into m blocks. Then we have m abacuses each of which has r runners. By reading i_k's in each block, we have m semi-infinite wedges. We now assume that these are of the form $u_I^{(k)} := u_{j_1^{(k)}} \wedge u_{j_2^{(k)}} \wedge \cdots$ such that $j_i^{(k)} = -\tilde{\gamma}_k - i + 1$ for all k and $i \gg 0$. We then identify $u_I^{(k)}$ with a multipartition $\lambda^{(k)}$ by $j_i^{(k)} = -\tilde{\gamma}_k + \lambda^{(k)}{}_i - i + 1$. We consider the subspace of $\mathcal{F}_c$ ($c = -\sum \tilde{\gamma}_k$) spanned by the wedges u_I whose $u_I^{(k)}$ have this form, and denote it by $\mathcal{F}_{v^{-1}}^{-\tilde{\gamma}}$. This is a U_v-submodule of $\mathcal{F}_c$. We call it Takemura-Uglov Fock space. This Fock space is not isomorphic to $\mathcal{F}_{v^{-1}}^{-\gamma}$, but we again have that multipartitions constitute "basis at $v = \infty$". We denote by $\mathcal{M}_{v^{-1}}^{-\tilde{\gamma}}$ the U_v- submodule generated by $\underline{\emptyset}$. To clarify the relation between $\mathcal{F}_{v^{-1}}^{-\tilde{\gamma}}$ and $\mathcal{F}_{v^{-1}}^{-\gamma}$, we introduce the following notion.

DEFINITION 3.2. We say that $\tilde{\gamma} = (\tilde{\gamma}_1, \dots, \tilde{\gamma}_m)$ is a *lift* of $\gamma = (\gamma_1, \dots, \gamma_m)$ if $\tilde{\gamma}_i \pmod r = \gamma_i$ for all i. If $r = \infty$, we set $\tilde{\gamma} = \gamma$.

We then have the following lemma. It follows from the definition of the coproduct $\Delta^{(l)}$. The size of a canonical basis element $b \in U_v^-$ is by definition the height of its root.

Lemma 3.3. *For $n \in \mathbb{N}$, we take a lift $\tilde{\gamma}$ of γ such that $-\tilde{\gamma}_k \ll -\tilde{\gamma}_{k+1}$ for all k. Then for any canonical basis element $b \in U_v^-$ of size less than n, two canonical basis elements $b\underline{\emptyset} \in \mathcal{F}_{v^{-1}}^{-\tilde{\gamma}}$ and $b\underline{\emptyset} \in \mathcal{F}_{v^{-1}}^{-\gamma}$ are identical as linear combination of multipartitions.*

Proof. Let x, y be two nodes such that x is located on the (a, b)-th entry of $\lambda^{(c)}$ and y is located on the (a', b')-th entry of $\lambda^{(c')}$. Then we write $x < y$ if one of the following holds.

$$-\tilde{\gamma}_c - a + b < -\tilde{\gamma}_{c'} - a' + b', \quad -\tilde{\gamma}_c - a + b = -\tilde{\gamma}_{c'} - a' + b' \text{ and } c < c'$$

Assume that $\underline{\mu}$ appears in $f_i\underline{\lambda}$ in $\mathcal{F}_{v^{-1}}^{-\tilde{\gamma}}$. Then its coefficient has the form $v^{-N_i^{>}(x)}$ where $x = \mu/\lambda$ and $N_i^{>}(x)$ is the number of addable i-nodes $y > x$ minus the number of removable i-nodes $y > x$. By comparing it with the action of f_i on $\mathcal{F}_{v^{-1}}^{-\gamma}$, we have the result. $\square$

The use of the notation $\mathcal{F}_{v^{-1}}^{-\tilde{\gamma}}$ is not misleading, since these are in fact modules of the quantum algebra of type A_∞ as is previously defined. There is a standard way to make them into U_v^--modules [31].

The advantage to use Takemura-Uglov Fock spaces is that we have bar operations on these Fock spaces. This is a generalization of the bar operation on level one modules introduced in [16, Proposition 3.1], [17, 5.1–5.9].

We state the properties of the bar operation due to Uglov. For level one modules, these are stated in [16, Theorem 3.2, Theorem 3.3]. (The proof is given in [17, 7.1–7.4].)

Note that if we transfer the dominance order on $\mathcal{F}_v^{\gamma}$ to $\mathcal{F}_{v^{-1}}^{-\gamma}$, it reads columns of multipartitions from left to right. If we read the columns from right to left, we have the reversed dominance order. We denote it by $\underline{\lambda} \geq \underline{\mu}$.

Theorem 3.4. *There exists a semilinear endomorphism of $\mathcal{F}_{v^{-1}}^{-\tilde{\gamma}}$, called the bar operation on $\mathcal{F}_{v^{-1}}^{-\tilde{\gamma}}$, such that it has the following properties.*
(1) $\overline{f_i \underline{\lambda}} = f_i \overline{\underline{\lambda}}$, *and* $\overline{\underline{\emptyset}} = \underline{\emptyset}$.
(2) *For $n \in \mathbb{N}$, we take a lift $\tilde{\gamma}$ of γ such that $-\tilde{\gamma}_k \ll -\tilde{\gamma}_{k+1}$ for all k. Then for any multipartition $\underline{\lambda}$ of size less than n, $\overline{\underline{\lambda}}$ has the form $\underline{\lambda} + \sum_{\underline{\mu}<\underline{\lambda}} c_{\lambda,\underline{\mu}}(v)\underline{\mu}$.*

Proof. (1) See [27, Lemma 4.10] for the welldefinedness of the bar operation. In the same page, $\overline{\underline{\emptyset}} = \underline{\emptyset}$ is also proved. See [27, Proposition 4.12] for $\overline{f_i \underline{\lambda}} = f_i \overline{\underline{\lambda}}$.
(2) Let u_I be a normally ordered wedge of charge c corresponding to $\underline{\lambda}$. We associate a partition λ by setting $i_k = c + \lambda_k - k + 1$. The definition of the bar operation and the straightening laws imply that $\overline{\underline{\lambda}}$ has terms $\underline{\mu}$ corresponding to $\mu \trianglelefteq \lambda$. By our assumption, it implies that $\underline{\mu} \leq \underline{\lambda}$. □

4. The proof of the conjecture

We first interprete the conjecture into a problem about canonical bases on Fock spaces. To do this, we use the direct sum of the Grothendieck groups of projective $\mathcal{H}_n$-modules ($n = 0, 1, \ldots$). We always assume that the coefficients are extended to the field of rational numbers. If $\mathcal{H}_n$ is semisimple, all $S^{\underline{\lambda}}$ are irreducible, and we identify the direct sum with $\mathcal{F}_{v=1}^{\gamma}$, which is by definition a based $\mathbb{Q}$-vector space whose basis elements are indexed by multipartitions, and nodes of multipartitions are given residues. If $\mathcal{H}_n$ is not semisimple, we have a proper subspace of $\mathcal{F}_{v=1}^{\gamma}$ by lifting idempotents argument. It is proved in [1] that it coincides with $\mathcal{M}_{v=1}^{\gamma}$.

Recall that simple modules are obtained as factor modules of Specht modules. To distinguish between simple modules over different base rings, we write $D_R^{\underline{\lambda}}$ when the base ring is R. Let (K, R, F) be a modular system. We assume that there is an invertible element $\zeta \in R$ such that its multiplicative order in K and F is the same. Then $D_K^{\underline{\lambda}}$ is obtained from $D_R^{\underline{\lambda}}$ by extension of coefficients, and $D_F^{\underline{\lambda}}$ is obtained from $D_R^{\underline{\lambda}}$ by taking the unique simple factor module of $D_R^{\underline{\lambda}} \otimes F$. The proof of Theorem 2.4 implies that these give the correspondence between simple modules over fields of positive characteristics and fields of characteristic 0, and $D_F^{\underline{\lambda}} \neq 0$ if and only if $D_K^{\underline{\lambda}} \neq 0$. Further, still assuming that the multiplicative order is the same, the proof given in [3] also shows that $D_K^{\underline{\lambda}} \neq 0$ if and only if $D_{\mathbb{C}}^{\underline{\lambda}} \neq 0$. In particular, to know which $D^{\underline{\lambda}}$ are non-zero, it is enough to consider the case that the base field is $\mathbb{C}$.

Now assume that we are in the case that the base field is $\mathbb{C}$. We identify the direct sum of the Grothendieck groups of projective $\mathcal{H}_n$-modules with $\mathcal{M}_{v=1}^{\gamma}$ as before. The main theorem in [1] asserts that the canonical basis evaluated at $v = 1$ consists of indecomposable projective $\mathcal{H}_n$-modules ($n = 0, 1 \ldots$). Hence we have a bijection between canonical basis elements of $\mathcal{M}_v^{\gamma}$ and indecomposable projective $\mathcal{H}_n$-modules $P^{\underline{\lambda}}$ for various n, and thus a bijection between canonical basis elements of $\mathcal{M}_v^{\gamma}$ and simple $\mathcal{H}_n$-modules $D^{\underline{\lambda}}$ for various n.

Then Theorem 3.1 asserts that with each canonical basis element $G(b)$, we can uniquely associate a multipartition $\underline{\nu} \in {}^{\gamma}\mathcal{K}P$. To summarize, we have the following.

For each non-zero $D^{\underline{\lambda}}$, there exists a unique canonical basis element $G(b) \in \mathcal{F}_v^{\gamma}$ such that we have $G(b)_{v=1} = P^{\underline{\lambda}}$ and $G(b) \equiv \underline{\nu}$.

This is the way to compare two parametrizations. Hence our aim to show that $\underline{\nu} = \underline{\lambda}$ holds in general.

Lemma 4.1. *Assume that for every canonical basis element $G(b) \in \mathcal{F}_v^{\gamma}$, there exists a unique maximal element among the multipartitions appearing in $\xi(G(b))$ with respect to the reversed dominance order, and assume that it has coefficient* 1. *Then we have that two parametrizations coincide and $D^{\underline{\lambda}} \neq 0$ if and only if $\underline{\lambda}$ is a Kleshchev multipartition.*

Proof. Recall that $\mathcal{M}_{v=1}^{\gamma}$, the sum of Grothendieck groups of projective $\mathcal{H}_n$-modules, is embedded into $\mathcal{F}_{v=1}^{\gamma}$ by sending $S^{\underline{\lambda}}$ to $\underline{\lambda}$. Hence, Theorem 2.2 implies that $\xi(P^{\underline{\lambda}})$ has the form $\xi(P^{\underline{\lambda}}) = \underline{\lambda}^{\mathrm{T}} + \sum_{\underline{\mu}<\underline{\lambda}^{\mathrm{T}}} c_{\underline{\mu}}\,\underline{\mu}$. In particular, among multipartitons appearing in $\xi(P^{\underline{\lambda}})$, $\underline{\lambda}^{\mathrm{T}}$ is the maximal element with respect to the reversed dominance order.

We take the canonical basis element $G(b)$ satisfying $P^{\underline{\lambda}} = G(b)_{v=1}$. By applying the assumption to $\xi(G(b))$, we know that multipartitions appearing in $\xi(G(b))$ has a unique maximal element with coefficient 1. Since $\xi(G(b))$ is a canonical basis element and the coefficient of the maximal element is 1, this maximal element must be the transpose of a Kleshchev multipartiton $\underline{\nu}$ with $G(b) \equiv \nu$. We specialize $\xi(G(b))$ to $v = 1$. Note that ν^{T} does not vanish. Since both $\underline{\lambda}^{\mathrm{T}}$ and $\underline{\nu}^{\mathrm{T}}$ are maximal elements, we have $\nu = \underline{\lambda}$. Hence the two parametrizations given in Theorem 2.2 and Theorem 2.4 coincide. $\square$

Hence it is enough to know that the assumption of the lemma holds. Recall that we are given a bar operation which has the properties stated in theorem 3.4(1). Its consequence is that each canonical basis element of $\mathcal{M}_{v^{-1}}^{-\tilde{\gamma}}$ is fixed by the bar operation. We also have that it coincides with the transpose of a Kleshchev multipartition modulo v^{-1}. It is well-known and easy to see that these two properties uniquely determine the canonical basis element. Combined with theorem 3.4(2), it implies that Gaussian elimination algorithm computes the canonical basis element, and it has the form

$$\underline{\lambda} + \sum_{\underline{\mu}<\underline{\lambda}} c_{\lambda,\underline{\mu}}(v)\underline{\mu}$$

if we take a lift $\tilde{\gamma}$ of γ such that $-\tilde{\gamma}_k \ll -\tilde{\gamma}_{k+1}$ for all k. By Lemma 3.3, it gives the required property of the canonical basis elements $\xi(G(b))$. Therefore, we have reached the following theorem, which verifies the conjecture.

Theorem 4.2. *$D^{\underline{\lambda}} \neq 0$ if and only if $\underline{\lambda}$ is a Kleshchev multipartition.*

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