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GROWTH PROPERTIES OF HYPERPLANE INTEGRALS OF SOBOLEV FUNCTIONS IN A HALF SPACE

Dedicated to Professor Masayuki Ito on the occasion of his sixtieth birthday

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1. Introduction

Let $\mathbf{D} \subset \mathbf{R}^n$ $(n \geq 2)$ denote the half space

$$\mathbf{D} = \{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}^1 : x_n > 0\}$$

and set

$$\mathbf{S} = \partial \mathbf{D};$$

we sometimes identify $x' \in \mathbf{R}^{n-1}$ with $(x', 0) \in \mathbf{S}$. We define the hyperplane integral $S_q(u)$ over $\mathbf{S}$ by

$$S_q(u) = \left(\int_{\mathbf{S}} |u(x')|^q \, dx' \right)^{1/q}$$

for a measurable function u on $\mathbf{S}$ and $q > 0$.

Set

$$U_r(x') = u(x', r) - \sum_{k=0}^{m-1} \frac{r^k}{k!} \left[\left(\frac{\partial}{\partial x_n} \right)^k u \right] (x', 0)$$

for quasicontinuous Sobolev functions u on $\mathbf{D}$, where the vertical limits

$$\left(\frac{\partial}{\partial x_n} \right)^k u(x', 0) = \lim_{x_n \to 0} \left(\frac{\partial}{\partial x_n} \right)^k u(x', x_n)$$

exist for almost every $x' = (x', 0) \in \partial \mathbf{D}$ and $0 \leq k \leq m - 1$ (see [8, Theorem 2.4, Chapter 8]).

Our main aim in this note is to study the existence of limits of $S_q(U_r)$ at $r = 0$. More precisely, we show (in Theorem 3.1 below) that

$$\lim_{r \to 0} r^{-\omega} S_q(U_r) = 0$$

for some $\omega > 0$.

Consider the Dirichlet problem for polyharmonic equation

$$\Delta^m u(x) = 0$$

with the boundary conditions

$$\left(\frac{\partial}{\partial x_n}\right)^k u(x', 0) = f_k(x') \quad (k = 0, 1, \ldots, m-1).$$

We show (in Corollary 3.1 below) that if $1 < p \leq q < \infty$, $n/p - (n-1)/q < 1$ and $u \in W^{m,p}(\mathbf{D})$ is a solution of the Dirichlet problem with $f_k(x') = (\partial/\partial x_n)^k u(x', 0)$ for $0 \leq k \leq m-1$, then

$$\lim_{r \to 0} r^{n/p-(n-1)/q-m} S_q(U_r) = 0,$$

where $U_r(x') = u(x', r) - \sum_{k=0}^{m-1} (r^k/k!) f_k(x')$.

To prove our results, we apply the integral representation in [6, 8]. For this purpose, we are concerned with K-potentials $U_K f$ defined by

$$U_K f(x) = \int K(x-y) f(y) dy$$

for functions f on $\mathbf{R}^n$ satisfying weighted L^p condition:

$$\int_{\mathbf{R}^n} |f(y)|^p |y_n|^\beta dy < \infty.$$

In connection with our integral representation, $K(x)$ is of the form $x^\lambda |x|^{-n}$ for a multi-index λ with length m. Our basic fact is stated as follows (see Theorem 2.1 below):

$$\lim_{r \to 0} r^{n/p-(n-1)/q-m} S_q(u_r) = 0,$$

where $u_r(x') = U_K f(x', r) - \sum_{k=0}^{m-1} (r^k/k!) [(\partial/\partial x_n)^k U_K f](x')$.

In the final section, we give growth estimates of higher differences of Sobolev functions.

For related results, see Gardiner [2], Stoll [14, 15, 16] and Mizuta [5, 6, 9]. We also refer the reader to Mizuta-Shimomura [10, 11] concerning monotone functions as a generalization of harmonic functions.

2. Hyperplane integrals of potentials

For a multi-index λ and $l > 0$, set

$$K(x) = \frac{x^\lambda}{|x|^l}.$$

We define the K-potential $U_K f$ by

$$U_K f(x) = \int_{\mathbf{R}^n} K(x-y) f(y) dy$$

for a measurable function f on $\mathbf{R}^n$ satisfying

$$\int_{\mathbf{R}^n} (1+|y|)^{|\lambda|-l} |f(y)| dy < \infty \tag{2.1}$$

and

$$\int_{\mathbf{R}^n} |f(y)|^p |y_n|^\beta dy < \infty, \quad y = (y_1, \ldots, y_n). \tag{2.2}$$

In particular, K is the Riesz α-kernel when $\lambda = 0$ and $l = n - \alpha$. In this case, $U_K f$ is written as $U_\alpha f$ with $\alpha = |\lambda| - l + n > 0$. Note here that (2.1) is equivalent to the condition that

$$U_\alpha |f| \not\equiv \infty. \tag{2.3}$$

Throughout this paper, let M denote various constants independent of the variables in question.

For a nonnegative integer m, consider

$$K_m(x, y) = K(x-y) - \sum_{k=0}^{m} \frac{{x_n}^k}{k!} \left[\left(\frac{\partial}{\partial x_n} \right)^k K \right] (x' - y),$$

where $x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$; we sometimes identify x' with $(x', 0)$.

Lemma 2.1. *Let m be a nonnegative integer such that $|\lambda| - l < m + 1$.*

(1) *If $|x' - y| \geq x_n/2 > 0$ and $|x - y| \geq x_n/2 > 0$, then*

$$|K_m(x, y)| \leq M x_n^{m+1} |x' - y|^{|\lambda| - l - m - 1}.$$

(2) *If $|x - y| < x_n/2$, then $|K_m(x, y)| \leq M(x_n^{|\lambda|-l} + |x-y|^{|\lambda|-l})$.*

(3) *If $|x' - y| < x_n/2$, then $|K_m(x, y)| \leq M(x_n^{|\lambda|-l} + x_n^m |x' - y|^{|\lambda|-l-m})$.*

Proof. If $|x' - y| > 2x_n$, then by Taylor's theorem, we obtain

$$\begin{aligned} |K_m(x, y)| &\le M \frac{x_n^{m+1}}{(m+1)!} |(x', \theta x_n) - y|^{|\lambda|-l-m-1} \quad (0 < \theta < 1) \\ &\le M x_n^{m+1} |x' - y|^{|\lambda|-l-m-1}. \end{aligned}$$

If $x_n/2 < |x' - y| < 2x_n$ and $|x - y| \ge x_n/2 > 0$, then

$$\begin{aligned} |K_m(x, y)| &\le |K(x - y)| + \sum_{k=0}^{m} \left| \frac{{x_n}^k}{k!} \left[\left(\frac{\partial}{\partial x_n} \right)^k K \right] (x' - y) \right| \\ &\le M x_n^{|\lambda|-l} + M \sum_{k=0}^{m} \frac{{x_n}^k}{k!} |x' - y|^{|\lambda|-l-k} \\ &\le M x_n^{|\lambda|-l} \\ &\le M x_n^{m+1} |x' - y|^{|\lambda|-l-m-1}, \end{aligned}$$

so that (1) is proved.

If $|x' - y| < x_n/2$, then $x_n/2 < |x - y| < 3x_n/2$, so that

$$\begin{aligned} |K_m(x, y)| &\le |K(x - y)| + \sum_{k=0}^{m} \left| \frac{{x_n}^k}{k!} \left[\left(\frac{\partial}{\partial x_n} \right)^k K \right] (x' - y) \right| \\ &\le M x_n^{|\lambda|-l} + M \sum_{k=0}^{m} \frac{{x_n}^k}{k!} |x' - y|^{|\lambda|-l-k} \\ &\le M (x_n^{|\lambda|-l} + x_n^m |x' - y|^{|\lambda|-l-m}), \end{aligned}$$

which proves (3).

Finally, if $|x - y| < x_n/2$, then $x_n/2 < |x' - y| \le x_n + |x - y| < 3x_n/2$, so that

$$\begin{aligned} |K_m(x, y)| &\le |K(x - y)| + \sum_{k=0}^{m} \left| \frac{{x_n}^k}{k!} \left[\left(\frac{\partial}{\partial x_n} \right)^k K \right] (x' - y) \right| \\ &\le |x - y|^{|\lambda|-l} + M |x' - y|^{|\lambda|-l} \\ &\le M (x_n^{|\lambda|-l} + |x - y|^{|\lambda|-l}), \end{aligned}$$

which proves (2). Thus the present lemma is established. □

For a point $x \in \mathbf{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball with center at x and radius r.

Lemma 2.2 (cf. [9, Lemma 3.2]). *Let* $\beta > -1$, $q > 0$ *and* $|\lambda| - l + n/q > 0$. *Let* m *be a nonnegative integer such that*

$$m < |\lambda| - l + \frac{n+\beta}{q} < m+1.$$

Then

$$\left(\int |K_m(x,y)|^q |y_n|^\beta dy \right)^{1/q} \le M x_n^{|\lambda| - l + (n+\beta)/q}$$

for all $x = (x', x_n) \in \mathbf{D}$.

Proof. For fixed $x \in \mathbf{D}$, consider the sets

$$E_1 = B\left(x, \frac{x_n}{2}\right), \quad E_2 = B\left(x', \frac{x_n}{2}\right), \quad E_3 = \mathbf{R}^n - (E_1 \cup E_2).$$

Since $|\lambda| - l + (n+\beta)/q - m - 1 < 0$, applying the polar coordinates about x', we have by Lemma 2.1(1)

$$\begin{aligned} \left(\int_{E_3} |K_m(x,y)|^q |y_n|^\beta dy \right)^{1/q} &\le M x_n^{m+1} \left(\int_{E_3} |x' - y|^{(|\lambda| - l - m - 1)q} |y_n|^\beta dy \right)^{1/q} \\ &\le M x_n^{m+1} \left(\int_{x_n/2}^{\infty} r^{(|\lambda| - l - m - 1)q + \beta} r^{n-1} dr \right)^{1/q} \\ &= M x_n^{|\lambda| - l + (n+\beta)/q}. \end{aligned}$$

Similarly, since $|\lambda| - l + n/q > 0$, we have by Lemma 2.1(2)

$$\begin{aligned} \left(\int_{E_1} |K_m(x,y)|^q |y_n|^\beta dy \right)^{1/q} &\le M x_n^{\beta/q} \left(\int_{E_1} (x_n^{|\lambda| - l} + |x - y|^{|\lambda| - l})^q dy \right)^{1/q} \\ &= M x_n^{|\lambda| - l + (n+\beta)/q}. \end{aligned}$$

Finally, since $|\lambda| - l + (n+\beta)/q - m > 0$, we obtain by Lemma 2.1(3)

$$\begin{aligned} \left(\int_{E_2} |K_m(x,y)|^q |y_n|^\beta dy \right)^{1/q} &\le M \left(\int_{E_2} (x_n^{|\lambda| - l} + x_n^m |x' - y|^{|\lambda| - l - m})^q |y_n|^\beta dy \right)^{1/q} \\ &\le M x_n^{|\lambda| - l + (n+\beta)/q} + M x_n^m \left(\int_0^{x_n/2} r^{(|\lambda| - l - m)q + \beta} r^{n-1} dr \right)^{1/q} \\ &= M x_n^{|\lambda| - l + (n+\beta)/q}. \end{aligned}$$

The required inequality now follows. □

Lemma 2.3 (cf. [9, Lemma 3.4]). *Let $q > 0$ and m be a nonnegative integer such that*

$$m < |\lambda| - l + \frac{n-1}{q} < m + 1.$$

If $x = (x', x_n) \in \mathbf{D}$ and $y = (y', y_n) \in \mathbf{R}^n$, then

$$\left(\int_{\mathbf{R}^{n-1}} |K_m(x, y)|^q dx' \right)^{1/q} \leq M x_n^{m+1} (x_n + |y_n|)^{|\lambda| - l - m - 1 + (n-1)/q}.$$

Proof. Let $x = (x', x_n) \in \mathbf{D}$ and $y = (y', y_n) \in \mathbf{R}^n$. If $|y_n| \geq 2x_n$, then, since $|\lambda| - l - m - 1 + (n-1)/q < 0$, we have by Lemma 2.1(1)

$$\begin{aligned} \left(\int_{\mathbf{R}^{n-1}} |K_m(x, y)|^q dx' \right)^{1/q} &\leq M x_n^{m+1} \left(\int_{\mathbf{R}^{n-1}} |x' - y|^{(|\lambda| - l - m - 1)q} dx' \right)^{1/q} \\ &= M x_n^{m+1} \left(\int_0^{\infty} (r^2 + {y_n}^2)^{(|\lambda| - l - m - 1)q/2} r^{n-2} dr \right)^{1/q} \\ &= M x_n^{m+1} |y_n|^{|\lambda| - l - m - 1 + (n-1)/q}. \end{aligned}$$

If $|y_n| < 2x_n$, then we have as in the proof of Lemma 2.2

$$\begin{aligned} \left(\int_{\mathbf{R}^{n-1}} |K_m(x, y)|^q dx' \right)^{1/q} &\leq M \left(\int_{\{x' : y \in E_1\}} (x_n^{|\lambda| - l} + |x - y|^{|\lambda| - l})^q dx' \right)^{1/q} \\ &\quad + M \left(\int_{\{x' : y \in E_2\}} (x_n^{|\lambda| - l} + x_n^m |x' - y|^{|\lambda| - l - m})^q dx' \right)^{1/q} \\ &\quad + M x_n^{m+1} \left(\int_{\{x' : y \in E_3\}} |x' - y|^{(|\lambda| - l - m - 1)q} dx' \right)^{1/q} \\ &\leq M x_n^{|\lambda| - l + (n-1)/q} + M \left(\int_{B(y', x_n/2)} |x' - y'|^{(|\lambda| - l)q} dx' \right)^{1/q} \\ &\quad + M x_n^m \left(\int_{B(y', x_n/2)} |x' - y'|^{(|\lambda| - l - m)q} dx' \right)^{1/q} \\ &\quad + M x_n^{m+1} \left(\int_{\mathbf{R}^{n-1}} (x_n + |x' - y'|)^{(|\lambda| - l - m - 1)q} dx' \right)^{1/q} \\ &= M x_n^{|\lambda| - l + (n-1)/q}. \end{aligned}$$

Therefore the required inequality now follows. □

Lemma 2.4 (cf. [1, Theorem 13.5], [8, Sections 6.5 and 8.2]). *Let* $\alpha = |\lambda| - l + n$, $p > 1$, $\alpha p > 1$, $\alpha p > 1 + \beta$ *and* $-1 < \beta < p - 1$. *If* f *is a measurable function on* $\mathbf{R}^n$ *satisfying* (2.2) *and* (2.3), *then* $U_K f$ *has the* (ACL) *property*; *in particular,* $U_K f(x', x_n)$ *is absolutely continuous on* $\mathbf{R}$ *for almost every* $x' \in \mathbf{R}^{n-1}$. *Moreover, in case* m *is a positive integer such that* $(\alpha - m)p > 1$ *and* $(\alpha - m)p > 1 + \beta$,

$$\left(\frac{\partial}{\partial x_n}\right)^m U_K f(x', x_n) = \int \left(\frac{\partial}{\partial x_n}\right)^m K(x - y) f(y) dy$$

is absolutely continuous on $\mathbf{R}$ *for almost every* $x' \in \mathbf{R}^{n-1}$.

Theorem 2.1 (cf. [5, Theorem 2.1] and [9, Theorem 2.1]). *Let* $\alpha = |\lambda| - l + n$ *satisfy* $m + 1/p < \alpha < m + n$. *Let* $1 < p \le q < \infty$, $-1 < \beta < p - 1$ *and*

$$\frac{n - \alpha p}{p(n - \alpha)} < \frac{n - 1}{q(n - \alpha + m)} \quad \text{when } n - \alpha > 0.$$

Further suppose $m < \omega < m + 1$, *where* $\omega = (n - 1)/q - (n - \alpha p + \beta)/p$. *If* f *is a nonnegative measurable function on* $\mathbf{R}^n$ *satisfying* (2.2) *and* (2.3), *then*

$$\lim_{r \to 0} r^{-\omega} S_q(u_r) = 0,$$

where $u_r(x') = U_K f(x', r) - \sum_{k=0}^{m} (r^k/k!)[(\partial/\partial x_n)^k U_K f](x', 0)$.

Proof. Under the assumptions on p, α, β, q and m in Theorem 2.1, we can take (δ, γ) such that

$$\beta < \gamma < p(n - \alpha + m + 1)\delta + \beta - \frac{p(n-1)}{q}, \tag{2.4}$$

$$p(n - \alpha + m + 1)\delta + (\alpha - m - 1)p - n < \gamma < p(n - \alpha + m)\delta + (\alpha - m)p - n, \tag{2.5}$$

$$\beta < \gamma < p - 1, \quad 0 < \delta < 1, \tag{2.6}$$

$$\delta p(n - \alpha) > n - \alpha p \tag{2.7}$$

and

$$\frac{n - 1}{q(n - \alpha + m + 1)} < \delta < \frac{n - 1}{q(n - \alpha + m)} \tag{2.8}$$

(if $\alpha \ge n$, then (2.7) clearly holds). Set $a = (1 - \delta)p'$ and $b = -\gamma p'/p$, where $p' = p/(p - 1)$. Then, by (2.6), we have

$$b > -1. \tag{2.9}$$

In case $\alpha \ge n$, we clearly find

$$\alpha - n + \frac{n}{a} > 0, \tag{2.10}$$

and in case $\alpha < n$, (2.10) also holds by (2.7). Further, (2.5) implies

$$m < \alpha - n + \frac{n+b}{a} < m+1. \tag{2.11}$$

By the fact that $m + 1/p < \alpha$, we have

$$\alpha p > 1. \tag{2.12}$$

Since $\omega > m$, we have

$$(\alpha - m)p > 1 + \beta. \tag{2.13}$$

By (2.12), (2.13) and Lemma 2.4, we first note that

$$\begin{aligned} u_{x_n}(x') &= U_K f(x) - \sum_{k=0}^{m} \frac{x_n^k}{k!} \left[\left(\frac{\partial}{\partial x_n} \right)^k U_K f \right] (x', 0) \\ &= \int K_m(x, y) f(y) dy. \end{aligned}$$

Using Hölder's inequality, we have

$$|u_{x_n}(x')| \le \left(\int |K_m(x, y)|^a |y_n|^b dy \right)^{(1-\delta)/a} \left(\int |K_m(x, y)|^{\delta p} f(y)^p |y_n|^\gamma dy \right)^{1/p}.$$

By (2.9)–(2.11) and Lemma 2.2, we have

$$|u_{x_n}(x')| \le M x_n^{(\alpha-n)(1-\delta)+n/p'-\gamma/p} \left(\int |K_m(x, y)|^{\delta p} f(y)^p |y_n|^\gamma dy \right)^{1/p}.$$

In view of Minkowski's inequality for integral we have

$$\begin{aligned} S_q(u_{x_n}) \le\ & M x_n^{(\alpha-n)(1-\delta)+n/p'-\gamma/p} \\ & \times \left\{ \int \left(\int_{R^{n-1}} |K_m(x, y)|^{\delta q} dx' \right)^{p/q} f(y)^p |y_n|^\gamma dy \right\}^{1/p}. \end{aligned}$$

Here, noting (2.8), we have by Lemma 2.3

$$\left(\int_{\mathbf{R}^{n-1}} |K_m(x, y)|^{\delta q} dx' \right)^{p/q} \le M [x_n^{m+1} (x_n + |y_n|)^{\alpha-n-m-1+(n-1)/\delta q}]^{\delta p}.$$

Consequently

$$S_q(u_r) \le Mr^{(\alpha-n)(1-\delta)+n/p'-\gamma/p+(m+1)\delta} \times \left\{ \int [(r+|y_n|)^{\alpha-n-m-1+(n-1)/\delta q}]^{\delta p} |y_n|^{\gamma-\beta} f(y)^p |y_n|^\beta dy \right\}^{1/p}.$$

Consider the function

$$k(r, y_n) = r^{p[(n-\alpha p+\beta)/p-(n-1)/q]} r^{p[(\alpha-n)(1-\delta)+n/p'-\gamma/p+(m+1)\delta]} \times [(r+|y_n|)^{\alpha-n-m-1+(n-1)/\delta q}]^{\delta p} |y_n|^{\gamma-\beta}.$$

Then

$$r^{-\omega} S_q(u_r) \le M \left\{ \int k(r, y_n) f(y)^p |y_n|^\beta dy \right\}^{1/p},$$

where $\omega = (n-1)/q - (n-\alpha p+\beta)/p$. It follows from (2.4) that

$$r^{-\omega} r^{(\alpha-n)(1-\delta)+n/p'-\gamma/p+(m+1)\delta} = r^{(n-\alpha+m+1)\delta+(\beta-\gamma)/p-(n-1)/q} \longrightarrow 0$$

as $r \to 0$. If $r < |y_n|$, then

$$k(r, y_n) \le M \left(\frac{r}{|y_n|} \right)^{(n-\alpha+m+1)\delta p+(\beta-\gamma)-p(n-1)/q} \le M;$$

if $|y_n| \le r$, then

$$k(r, y_n) \le M \left(\frac{|y_n|}{r} \right)^{\gamma-\beta} \le M.$$

Hence Lebesgue's dominated convergence theorem implies that

$$\lim_{r\to 0} r^{-\omega} S_q(u_r) = 0.$$

Now the proof of Theorem 2.1 is completed. □

3. Sobolev functions

For an open set $G \subset \mathbf{R}^n$, we denote by $BL_m(L^p_{\mathrm{loc}}(G))$ the Beppo Levi space

$$BL_m(L^p_{\mathrm{loc}}(G)) = \{u \in L^p_{\mathrm{loc}}(G) : D^\lambda u \in L^p_{\mathrm{loc}}(G) \quad (|\lambda| = m)\}$$

(see [8, Chapter 6]). Set $K_\lambda(x) = x^\lambda |x|^{-n}$ and

$$\tilde{K}_{\lambda,m}(x, y) = \begin{cases} K_\lambda(x-y), & y \in B(0,1), \\ K_\lambda(x-y) - \displaystyle\sum_{|\mu|\le m-1} \frac{x^\mu}{\mu!} \left[\left(\frac{\partial}{\partial x} \right)^\mu K_\lambda \right](-y), & y \in \mathbf{R}^n - B(0,1). \end{cases}$$

In view of [8, Theorem 7.2, Chapter 6], each $u \in BL_m(L^p_{\text{loc}}(\mathbf{D}))$ satisfying

$$\int_{\mathbf{D}} |\nabla_m u(x)|^p x_n^\beta dx < \infty \tag{3.1}$$

has an (m, p)-quasicontinuous representative $\tilde{u}$, where $|\nabla_m u(x)| = (\sum_{|\mu|=m} |D^\mu u(x)|^2)^{1/2}$, $1 < p < \infty$ and $-1 < \beta < p - 1$. Moreover, $\tilde{u}$ is given by

$$\tilde{u}(x) = \sum_{|\lambda|=m} a_\lambda \int \tilde{K}_{\lambda,m}(x, y) D^\lambda \overline{u}(y) dy + P(x),$$

where $\overline{u}$ is an extension of u to $\mathbf{R}^n$, $P(x)$ is a polynomial of degree at most $m - 1$. Note further from Lemma 2.4 that for each k with $0 \le k \le m - 1$ and for almost every $x' \in \mathbf{R}^{n-1}$,

$$\left(\frac{\partial}{\partial x_n}\right)^k \int \tilde{K}_{\lambda,m}(x, y) D^\lambda \overline{u}(y) dy = \int \left(\frac{\partial}{\partial x_n}\right)^k \tilde{K}_{\lambda,m}(x, y) D^\lambda \overline{u}(y) dy$$

holds for $x_n \in \mathbf{R}$, where $x = (x', x_n)$.

Since $Q(x) - \sum_{k=0}^{m-1} ({x_n}^k/k!)[(\partial/\partial x_n)^k Q](x') = 0$ for any polynomial Q of degree at most $m - 1$, we have

$$\begin{aligned} U(x) &\equiv \tilde{u}(x) - \sum_{k=0}^{m-1} \frac{{x_n}^k}{k!} \left(\frac{\partial}{\partial x_n}\right)^k \tilde{u}(x') \\ &= \sum_{|\lambda|=m} a_\lambda \int K_{\lambda,m}(x, y) D^\lambda \overline{u}(y) dy = \tilde{u}(x) - P(x) \end{aligned}$$

for $x \in \mathbf{D}$, where $K_{\lambda,m}(x, y) = K_\lambda(x - y) - \sum_{k=0}^{m-1} ({x_n}^k/k!)[(\partial/\partial x_n)^k K_\lambda](x' - y)$.

Theorem 2.1 now gives the following result.

Theorem 3.1. *Let* $1 < p \le q < \infty$,

$$\frac{n - mp}{p(n - m)} < \frac{1}{q} \quad \textit{when } n - m > 0$$

and

$$\frac{n - p + \beta}{p(n - 1)} < \frac{1}{q} < \frac{n + \beta}{p(n - 1)}.$$

If $u \in BL_m(L^p_{\text{loc}}(\mathbf{D}))$ *satisfying* (3.1) *for* $-1 < \beta < p - 1$ *is* (m, p)*-quasicontinuous on* $\mathbf{D}$, *then*

$$\lim_{r \to 0} r^{(n-mp+\beta)/p-(n-1)/q} S_q(U_r) = 0,$$

where $U_r(x') = u(x', r) - \sum_{k=0}^{m-1}(r^k/k!)[(\partial/\partial x_n)^k u](x', 0)$.

Consider the Dirichlet problem for polyharmonic equation:

$$\Delta^m u(x) = 0$$

with the boundary conditions

$$\left(\frac{\partial}{\partial x_n}\right)^k u(x', 0) = f_k(x') \quad (k = 0, 1, \ldots, m-1).$$

We denote by $W^{m,p}(G)$ the Sobolev space

$$W^{m,p}(G) = \{u \in L^p(G) : D^\lambda u \in L^p(G) \quad (|\lambda| \le m)\}$$

(see Stein [13, Chapter 6]). If $u \in W^{m,p}(\mathbf{D})$ is a solution of the Dirichlet problem, then the vertical limit $(\partial/\partial x_n)^k u(x', 0)$ exists for almost every $x' = (x', 0) \in \partial\mathbf{D}$ and $0 \le k \le m-1$ (see [6], [7]).

We also see that every function in $W^{m,p}(\mathbf{D})$ can be extended to a function in $W^{m,p}(\mathbf{R}^n)$ (see Stein [13, Theorem 5, Chapter 6]). Hence Theorem 3.1 gives the following result.

Corollary 3.1. *Let* $1 < p \le q < \infty$ *and*

$$(\,0 <\,)\, \frac{n}{p} - \frac{n-1}{q} < 1.$$

If $u \in W^{m,p}(\mathbf{D})$ *is a solution of the Dirichlet problem with* $f_k(x') = (\partial/\partial x_n)^k u(x', 0)$ *for* $0 \le k \le m-1$, *then*

$$\lim_{r \to 0} r^{n/p-(n-1)/q-m} S_q(U_r) = 0,$$

where $U_r(x') = u(x', r) - \sum_{k=0}^{m-1}(r^k/k!) f_k(x')$.

4. Higher differences

For $r > 0$ and a function u, we define the first difference

$$\Delta_r u(t) = \Delta_r^1 u(t) = u(t + r) - u(t)$$

and the m-th difference

$$\Delta_r^m u(t) = \Delta_r^{m-1}\left(\Delta_r u(\cdot)\right)(t).$$

It is easy to see that

$$\Delta_r^m u(t) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} u(t+kr).$$

As in Section 2, we consider

$$K(x) = \frac{x^\lambda}{|x|^l}$$

and define

$$u_r(x') = \Delta_r^m U_K f(x', \cdot)(0) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} U_K f(x', kr).$$

Theorem 4.1. *Let* $\alpha = |\lambda| - l + n$, $1 < p \le q < \infty$, $\beta < p - 1$ *and*

$$\frac{n - \alpha p}{p(n-1)} < \frac{1}{q} \quad (\textit{when } n - \alpha > 0).$$

Further suppose $0 < \omega < m$, *where* $\omega = (n-1)/q - (n - \alpha p + \beta)/p$. *If* f *is a nonnegative measurable function on* $\mathbf{R}^n$ *satisfying* (2.2) *and* (2.3), *then*

$$\lim_{r \to 0} r^{-\omega} S_q(u_r) = 0,$$

where $u_r(x') = \Delta_r^m U_K f(x', \cdot)(0)$.

To prove this, we have only to prepare the following two lemmas instead of Lemmas 2.2 and 2.3.

Lemma 4.1. *Let* $\beta > -1$, $q > 0$ *and* $|\lambda| - l + n/q > 0$. *Let* m *be a positive integer such that*

$$0 < |\lambda| - l + \frac{n+\beta}{q} < m.$$

Then

$$\left(\int |K_m^*(x,y)|^q |y_n|^\beta dy \right)^{1/q} \le M x_n^{|\lambda| - l + (n+\beta)/q}$$

for all $x = (x', x_n) \in \mathbf{D}$, *where* $K_m^*(x,y) = \Delta_{x_n}^m K(x' - y', \cdot - y_n)(0)$ *for* $x = (x', x_n) \in \mathbf{D}$ *and* $y = (y', y_n) \in \mathbf{R}^n$.

Proof. For $x = (x', x_n) \in \mathbf{D}$, write

$$\left(\int |K_m^*(x, y)|^q |y_n|^\beta dy \right)^{1/q} = U'(x_n) + U''(x_n),$$

where

$$U'(x_n) = \left(\int_{\{y=(y',y_n):|x'-y|\geq(m+2)x_n\}} |K_m^*(x, y)|^q |y_n|^\beta dy \right)^{1/q},$$
$$U''(x_n) = \left(\int_{\{y=(y',y_n):|x'-y|\leq(m+2)x_n\}} |K_m^*(x, y)|^q |y_n|^\beta dy \right)^{1/q}.$$

If $|x' - y| \geq (m+2)x_n$, then we obtain by Taylor's theorem,

$$|K_m^*(x, y)| \leq M x_n^m |x' - y|^{|\lambda|-l-m}. \tag{4.1}$$

Since $|\lambda| - l - m + (n+\beta)/q < 0$, applying the polar coordinates about x', we have

$$\begin{aligned} |U'(x_n)| &\leq M x_n^m \left(\int_{\{y=(y',y_n):|x'-y|\geq(m+2)x_n\}} |x' - y|^{(|\lambda|-l-m)q} |y_n|^\beta dy \right)^{1/q} \\ &= M x_n^m \left(\int_{(m+2)x_n}^\infty r^{(|\lambda|-l-m)q+\beta} r^{n-1} dr \right)^{1/q} \\ &= M x_n^{|\lambda|-l+(n+\beta)/q}. \end{aligned}$$

On the other hand, since $|\lambda| - l + n/q > 0$ and $|\lambda| - l + (n+\beta)/q > 0$, we have by Lemma 2.2

$$\begin{aligned} |U''(x_n)| &\leq M \sum_{k=0}^m \left(\int_{\{y=(y',y_n):|x'-y|\leq(m+2)x_n\}} |x' - y + k x_n e|^{(|\lambda|-l)q} |y_n|^\beta dy \right)^{1/q} \\ &\leq M x_n^{|\lambda|-l+(n+\beta)/q}, \end{aligned}$$

where $e = (0, \ldots, 0, 1)$. □

Lemma 4.2. *Let $q > 0$ and m be a positive integer such that*

$$0 < |\lambda| - l + \frac{n-1}{q} < m.$$

If $x = (x', x_n) \in \mathbf{D}$ and $y = (y', y_n) \in \mathbf{R}^n$, then

$$\left(\int_{\mathbf{R}^{n-1}} |K_m^*(x, y)|^q dx' \right)^{1/q} \leq M x_n^m (x_n + |y_n|)^{|\lambda|-l-m+(n-1)/q}.$$

Proof. Let $x = (x', x_n) \in \mathbf{D}$ and $y = (y', y_n) \in \mathbf{R}^n$. If $|y_n| \geq (m+2)x_n$, then, since $|\lambda| - l - m + (n-1)/q < 0$, we have by (4.1)

$$\begin{aligned}\left(\int_{\mathbf{R}^{n-1}} |K_m^*(x,y)|^q dx'\right)^{1/q} &\leq M x_n^m \left(\int_{\mathbf{R}^{n-1}} |x' - y|^{(|\lambda|-l-m)q} dx'\right)^{1/q} \\ &= M x_n^m |y_n|^{|\lambda|-l-m+(n-1)/q}.\end{aligned}$$

If $|y_n| < (m+2)x_n$, then we have by (4.1) and Lemma 2.3

$$\begin{aligned}&\left(\int_{\mathbf{R}^{n-1}} |K_m^*(x,y)|^q dx'\right)^{1/q} \\ &\leq M x_n^m \left(\int_{\{x' : |x'-y| \geq 2(m+2)x_n\}} |x' - y|^{(|\lambda|-l-m)q} dx'\right)^{1/q} \\ &\quad + M \sum_{k=0}^{m} \left(\int_{\{x' : |x'-y| \leq 2(m+2)x_n\}} |x' - y + kx_n e|^{(|\lambda|-l)q} dx'\right)^{1/q} \\ &\leq M x_n^{|\lambda|-l+(n-1)/q}.\end{aligned}$$

Therefore the required inequality now follows. □

Theorem 4.2. *Let* $1 < p \leq q < \infty$,

$$\frac{n-mp}{p(n-1)} < \frac{1}{q} \quad \textit{when } n - m > 0$$

and

$$\frac{n-mp+\beta}{p(n-1)} < \frac{1}{q} < \frac{n+\beta}{p(n-1)}.$$

If $u \in BL_m(L^p_{\mathrm{loc}}(\mathbf{D}))$ *satisfying* (3.1) *for* $-1 < \beta < p-1$ *is* (m,p)*-quasicontinuous on* $\mathbf{D}$, *then*

$$\lim_{r \to 0} r^{(n-mp+\beta)/p-(n-1)/q} S_q(U_r) = 0,$$

where $U_r(x') = \Delta_r^m u(x', \cdot)(0)$ *for* $r > 0$.

In fact, since $\Delta_r^m Q = 0$ for any polynomial Q of degree at most $m-1$, we have

$$U(x) \equiv \Delta_{x_n}^m u(x', \cdot)(0) = \sum_{|\lambda|=m} a_\lambda \int K^*_{\lambda,m}(x,y) D^\lambda \overline{u}(y) dy,$$

where $K^*_{\lambda,m}(x,y) = \Delta_{x_n}^m K_\lambda(x'-y', \cdot - y_n)(0)$ with $K_\lambda(x) = x^\lambda |x|^{-n}$. Now we can apply Theorem 4.1 to obtain the present result.

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