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# WICK CALCULUS AND THE CAUTHY PROBLEM FOR SOME DISPERSIVE EQUATIONS

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## 1. Introduction

In this paper we first introduce the Wick calculus used by Lerner in [5] and investigate the algebra for the Wick calculus more precisely than there. Next, we consider the Cauchy problem for some dispersive equations as an application of the Wick calculus.

Let $g \in \mathcal{S}(\boldsymbol{R}^n)$ and set $g^{y,\eta}(x) = g^Y(x) = e^{ix\eta}g(x-y)$ where $Y = (y,\eta) \in \boldsymbol{R}^n_y \times \boldsymbol{R}^n_\eta$ and $i = \sqrt{-1}$. We define a windowed Fourier transform of $u \in L^2(\boldsymbol{R}^n)$ by

$$(Wu)(Y) = \int_{\boldsymbol{R}^n} \overline{g^Y(x)}u(x)dx.$$

By Plancherel's theorem we have

$$(Wu)(Y) = e^{-iy\eta}\mathcal{F}^{-1}\left[\overline{\widehat{g}(\cdot-\eta)}\widehat{u}(\cdot)\right](y),$$

where

$$\widehat{u}(\xi) = \int_{\boldsymbol{R}^n} e^{-ix\xi}u(x)dx, \quad (\mathcal{F}^{-1}u)(x) = (2\pi)^{-n}\int_{\boldsymbol{R}^n} e^{ix\xi}u(\xi)d\xi.$$

Using this formula and Plancherel's theorem, we get

$$\begin{aligned}(Wu, Wv)_{L^2(\boldsymbol{R}^{2n})} &= (2\pi)^{-n}\int_{\boldsymbol{R}^n} d\eta \int_{\boldsymbol{R}^n} |\widehat{g}(\xi-\eta)|^2\widehat{u}(\xi)\overline{\widehat{v}(\xi)}d\xi \\ &= (2\pi)^n\|g\|^2_{L^2(\boldsymbol{R}^n)}(u,v)_{L^2(\boldsymbol{R}^n)}\end{aligned}$$

for $u, v \in L^2(\boldsymbol{R}^n)$. If we take $\|g\|^2_{L^2(\boldsymbol{R}^n)} = (2\pi)^{-n}$, then $W$ is an isometric operator from $L^2(\boldsymbol{R}^n)$ into $L^2(\boldsymbol{R}^{2n})$ and we have $W^*W = I_d$ on $L^2(\boldsymbol{R}^n)$. Here $W^*$ is the adjoint operator of $W$, which is defined by

$$(W^*r)(x) = \int_{\boldsymbol{R}^{2n}} g^Y(x)r(Y)dY \quad \text{for} \quad r \in L^2(\boldsymbol{R}^{2n}).$$

Now, for $a(x,\xi) = a(X) \in L^\infty(\boldsymbol{R}^{2n})$, $X = (x,\xi) \in \boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi$, we define the bounded linear operator $a^{\mathrm{Wick}} = a^{\mathrm{Wick}}(x,D)$ on $L^2(\boldsymbol{R}^n)$ by

$$a^{\mathrm{Wick}}(x,D)u(x) = (W^* a^\mu W u)(x) \quad \text{for} \quad u \in L^2(\boldsymbol{R}^n), \tag{1.1}$$

where $a^\mu$ is the multiplication operator by $a(X)$. By the definition we see that:

**Proposition 1.1.** *Let $a(X) \in L^\infty(\boldsymbol{R}^{2n})$. Then we have*

$$\|a^{\mathrm{Wick}}\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))} \le \|a\|_{L^\infty(\boldsymbol{R}^{2n})}.$$

*Moreover, if $a(X)$ is a nonnegative function, then $a^{\mathrm{Wick}}$ is a nonnegative operator on $L^2(\boldsymbol{R}^n)$.*

Proof. For any $u, v \in L^2(\boldsymbol{R}^n)$ we have the following estimate

$$\left|(a^{\mathrm{Wick}}u, v)_{L^2(\boldsymbol{R}^n)}\right| = \left|(a^\mu Wu, Wv)_{L^2(\boldsymbol{R}^{2n})}\right| \le \|a\|_{L^\infty(\boldsymbol{R}^{2n})}\|Wu\|_{L^2(\boldsymbol{R}^{2n})}\|Wv\|_{L^2(\boldsymbol{R}^{2n})},$$

which gives the first part because $W$ is an isometric operator from $L^2(\boldsymbol{R}^n)$ into $L^2(\boldsymbol{R}^{2n})$. The second part follows from the equality $(a^{\mathrm{Wick}}u, u)_{L^2(\boldsymbol{R}^n)} = (a^\mu Wu, Wu)_{L^2(\boldsymbol{R}^{2n})}$ for any $u \in L^2(\boldsymbol{R}^n)$. □

In Section 2 we shall study the product formulas of Wick operators by taking $g(x)$ equal to the Gaussian function and introducing some symbol class of $a(X,\Lambda)$ with a parameter $\Lambda \ge 1$. In Section 3, the product formulas will be used to prove the well-posedness of the Cauchy problem for some dispersive equations, motivated by the similar problem for the Schrödinger type equation. The detail will be explained there.

## 2. Algebra for Wick calculus

We introduce a class of symbols with a large parameter.

DEFINITION 2.1. Let $\Lambda \ge 1$ be a large parameter and $m \in \boldsymbol{R}$, $\rho$, $\delta > 0$. Then we say that the function $a(x,\xi;\Lambda)$ on $\boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi$ with a large parameter $\Lambda$ belongs to the class $\mathcal{T}^m_{\rho,\delta}$ of symbols if $a(\cdot,\cdot;\Lambda)$ is in $C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ and satisfies

$$\gamma_k(a) := \sup_{\substack{x,\xi\in\boldsymbol{R}^n,\Lambda\ge1\\|\alpha+\beta|=k}} \left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi;\Lambda)\right| \Lambda^{-m+\rho|\alpha|+\delta|\beta|} < \infty \tag{2.1}$$

for all $k \in \boldsymbol{Z}_+ := \boldsymbol{N} \cup \{0\}$. We denote $\mathcal{T}^m_{1/2,1/2}$ simply by $\mathcal{T}^m$.

EXAMPLE 2.2. Let $m \in \boldsymbol{R}$, $\rho$, $\delta > 0$ and $a \in S^m_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$. We set

$$\widetilde{a}(x,\xi;\Lambda) = a(\Lambda^{-\delta}x, \Lambda^\delta\xi) \times \varphi(\Lambda^{-\rho}\xi),$$

where $\varphi$ is in $C^\infty(\boldsymbol{R}^n)$ with the support $\subset \{\xi \in \boldsymbol{R}^n; 1/2 \le |\xi| \le 2\}$. Then we have $\widetilde{a} \in \mathcal{T}^{(\rho+\delta)m}_{\rho,\delta}$.

From now on, we take $g(x) = (4\pi^3)^{-n/4}\exp(-|x|^2/2)$, which satisfies $\|g\|^2_{L^2(\boldsymbol{R}^n)} = (2\pi)^{-n}$.

**Theorem 2.3** (cf. [5; Proposition 2.1]). *Let $m \in \boldsymbol{R}$, $\rho$, $\delta > 0$ and $a \in \mathcal{T}^m_{\rho,\delta}$. Then*

$$a^{\mathrm{Wick}} = a^w + r^w \quad \textit{with } r \in \mathcal{T}^{m-2\sigma}_{\rho,\delta} \tag{2.2}$$

*where $\sigma = \min(\rho, \delta)$ and we denote the pseudo-differential operator of the Weyl symbol $b(x, \xi)$ by $b^w = b^w(x, D)$, that is*

$$b^w(x, D)u(x) = (2\pi)^{-n} \iint_{\boldsymbol{R}^{2n}} e^{i(x-y)\xi} b\left(\frac{x+y}{2}, \xi\right) u(y)dyd\xi. \quad (\textit{see } [3].)$$

*Furthermore, if $a$ is real valued then so is $r$.*

**Corollary 2.4.** *Let $\rho$, $\delta > 0$ and $a \in \mathcal{T}^m_{\rho,\delta}$. Set $\sigma = \min(\rho, \delta)$. For $l \ge 3$ we have*

$$a^w = \left(a + \sum_{k=1}^{[(l-1)/2]} \frac{1}{k!}\left(\frac{-\Delta_X}{4}\right)^k a\right)^{\mathrm{Wick}} + r^w \quad \textit{with } r \in \mathcal{T}^{m-\sigma l}_{\rho,\delta} \tag{2.3}$$

*where $\Delta_X = \sum_{j=1}^n \{(\partial/\partial x_j)^2 + (\partial/\partial \xi_j)^2\}$.*

Theorem 2.3 and Corollary 2.4 show that the Wick operator approximates the pseudodifferential operator of the Weyl symbol.

**Theorem 2.5.** *Let $N$ be a positive even integer and let $a \in \mathcal{T}^0$, $b \in \mathcal{T}^m$. If $m = N/2$ then we have the expansion formulas as follows*:

$$a^{\mathrm{Wick}} b^{\mathrm{Wick}} = \begin{cases} \left(ab - \frac{1}{2}a' \cdot b' + \frac{1}{2i}\{a, b\}\right)^{\mathrm{Wick}} + R_2 \quad \textit{if } N = 2, \\ \left(ab - \frac{1}{2}a' \cdot b' + \frac{1}{2i}\{a, b\}\right)^{\mathrm{Wick}} + \sum_{k=2}^{N/2} \frac{(-1)^k}{2^k k!} \\ \left(\left(\sum_{j=1}^{2n}\left(\partial_{X_j}\partial_{Z_j} + \frac{H_{X_j}}{i}\partial_{Z_j}\right)\right)^k a(X)b(Z)\Big|_{Z=X}\right)^{\mathrm{Wick}} \\ \qquad + R_N \quad \textit{if } N \ge 4, \end{cases} \tag{2.4}$$

$$
(2.5)\qquad b^{\mathrm{Wick}}a^{\mathrm{Wick}} = \begin{cases} \left(ba - \dfrac{1}{2}b'\cdot a' + \dfrac{1}{2i}\{b,a\}\right)^{\mathrm{Wick}} + \widetilde{R_2} \quad \text{if } N = 2, \\ \left(ba - \dfrac{1}{2}b'\cdot a' + \dfrac{1}{2i}\{b,a\}\right)^{\mathrm{Wick}} + \displaystyle\sum_{k=2}^{N/2}\frac{(-1)^k}{2^k k!} \\ \left(\left(\displaystyle\sum_{j=1}^{2n}\left(\partial_{X_j}\partial_{Z_j} - \frac{H_{X_j}}{i}\partial_{Z_j}\right)\right)^k a(X)b(Z)\Bigg|_{Z=X}\right)^{\mathrm{Wick}} \\ \qquad\qquad + \widetilde{R_N} \quad \text{if } N \geq 4, \end{cases}
$$

*where*

$$
a'\cdot b' = \sum_{j=1}^{2n}\frac{\partial a}{\partial X_j}\frac{\partial b}{\partial X_j},
$$

$$
\{a,b\} = \sum_{j=1}^{n}\left(\frac{\partial a}{\partial \xi_j}\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial \xi_j}\right)
$$

*and $H_f$ denotes the Hamilton vector field of $f(X)$ (Note $\sum_{j=1}^{2n} H_{X_j}a\partial_{X_j}b = -\{a,b\}$). The remainder terms $R_N$, $\widetilde{R_N}$ are $L^2$ bounded operators uniformly with respect to a large parameter $\Lambda$ satisfying*

$$
(2.6)\quad \|R_N\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))}, \left\|\widetilde{R_N}\right\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))} \leq C_{(N,n)}\left(\gamma_0(a)\gamma_N(b) + \sum_{\substack{|\beta|<|\alpha|<N \\ |\alpha+\beta|=N}}\|a^{(\beta)}b^{(\alpha)}\|_{L^\infty}\right)
$$

*where $a^{(\alpha)} = \partial_X^\alpha a$ and $C_{(N,n)}$ is the constant depending only on $N$ and $n$.*

REMARK 2.6. In the case where $N = 2$ and symbols $a$ and $b$ are real valued, Lerner [5] proved that

$$
\mathrm{Re}\left(a^{\mathrm{Wick}}b^{\mathrm{Wick}}\right) = \left(ab - \frac{1}{2}a'\cdot b'\right)^{\mathrm{Wick}} + S
$$

where $S$ satisfies the estimate of the same type as (2.6) (see Proposition 2.3 of [5]). If $N = 2$ then the second term of the right hand side of (2.6) disappears.

**Theorem 2.7.** *Let $N$ be a positive even integer and let $a \in \mathcal{T}^0$, $b \in \mathcal{T}^m$. If $m = N/2$ then we have the expansion formulas as follows*:

$$
(2.7)\quad \left[a^{\mathrm{Wick}}, b^{\mathrm{Wick}}\right] = \begin{cases} \frac{1}{i}(\{a,b\})^{\mathrm{Wick}} + R_2' \quad \textit{if } N = 2, \\ \frac{1}{i}\left(\{a,b\} - \frac{1}{2}\sum_{j=1}^{2n}\left\{\frac{\partial a}{\partial X_j}, \frac{\partial b}{\partial X_j}\right\}\right)^{\mathrm{Wick}} + \sum_{k=3}^{N/2}\frac{(-1)^k}{2^{k-1}} \\ \frac{1}{i}\left(\sum_{l=0}^{[(k-1)/2]}\frac{(-1)^{l+1}}{2l+1}\sum_{\substack{|\alpha|=2l\\|\beta|=k-2l-1}}\frac{1}{\alpha!\beta!}\{H_X^\alpha \partial_X^\beta a, \partial_X^{\alpha+\beta} b\}\right)^{\mathrm{Wick}} \\ \qquad +R_N' \quad \textit{if } N \geq 4, \end{cases}
$$

*where $R_N'$ satisfies the estimate*

$$
(2.8)\quad \|R_N'\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \leq C'_{(N,n)}\left(\gamma_0(a)\gamma_N(b) + \sum_{\substack{|\beta|<|\alpha|<N-1\\|\alpha+\beta|=N-2}}\|\{a^{(\beta)}, b^{(\alpha)}\}\|_{L^\infty}\right),
$$

*where $C'_{(N,n)}$ is the constant depending only on $N$ and $n$.*

REMARK 2.8. If $N = 4$ then the Poisson bracket terms of the right hand side of (2.8) are equal to $\|\{a, b^{(\alpha)}\}\|_{L^\infty}$ with $|\alpha| = 2$. The expansion formulas of Theorems 2.5 and 2.7 hold for symbols $a, b \in \mathcal{T}^m_{\rho,\delta}$ and moreover for general symbols with a large parameter $\Lambda$. More general formulas will be given after the proof of the theorems.

For the proof of theorems, we define the operator $\Sigma_Y$ as

$$
(2.9)\quad (\Sigma_Y u)(x) = (Wu)(Y)g^Y(x) \quad \text{for } u \in L^2(\boldsymbol{R}^n).
$$

Then it follows from (1.1) that for $a \in L^\infty(\boldsymbol{R}^n)$

$$
(2.10)\quad a^{\mathrm{Wick}} = \int_{\boldsymbol{R}^{2n}} a(Y)\Sigma_Y dY.
$$

We prepare two lemmas.

**Lemma 2.9.** *Let $p_Y(X) = \pi^{-n}e^{-|X-Y|^2}$. Then we have*

$$
(2.11)\quad \Sigma_Y = p_Y^w(x, D).
$$

Proof. It follows from the definition that

$$
p_Y^w(x,D)u(x) = (2\pi^2)^{-n}\iint_{\boldsymbol{R}^n_z\times\boldsymbol{R}^n_\zeta} e^{i(x-z)\zeta}e^{-|(x+z)/2-y|^2-|\zeta-\xi|^2}u(z)dzd\zeta.
$$

Noting that

$$\int_{\boldsymbol{R}^n} e^{i(x-z)\zeta} e^{-|\zeta-\xi|^2} d\zeta = \pi^{n/2} e^{i(x-z)\xi} e^{-(1/4)|x-z|^2},$$

we get

$$\begin{aligned} p_Y^w(x, D)u(x) &= (4\pi^3)^{-n/2} \int_{\boldsymbol{R}^n} e^{i(x-z)\xi} e^{-|(x+z)/2-y|^2-(1/4)|x-z|^2} u(z)dz \\ &= (4\pi^3)^{-n/2} \int_{\boldsymbol{R}^n} e^{i(x-z)\xi} e^{-(1/2)|x-y|^2-(1/2)|z-y|^2} u(z)dz, \end{aligned}$$

which shows (2.11). □

**Lemma 2.10.** *Let* $\sigma(X, Y) = \sigma((x, \xi), (y, \eta)) = \xi y - x\eta$ *and set*

$$\begin{aligned} p_{Y,Z}(X) &= (2\pi^2)^{-n} e^{i\sigma(X-Z,X-Y)} e^{-(1/2)|X-Y|^2-(1/2)|X-Z|^2} \\ &= (2\pi^2)^{-n} e^{i\sigma(X-Z,X-Y)} e^{-(1/4)|Y-Z|^2-|X-(Y+Z)/2|^2}. \end{aligned}$$

*Then we have*

$$\Sigma_Y \Sigma_Z u(x) = (Wg^Z)(Y) g^Y(x)(Wu)(Z) = p^w_{Y,Z} u(x) \quad \text{for } u \in L^2(\boldsymbol{R}^n). \tag{2.12}$$

*Moreover, we get*

$$\Sigma_Y \Sigma_Y = (2\pi)^{-n} \Sigma_Y \quad \text{on} \quad L^2(\boldsymbol{R}^n), \tag{2.13}$$

$$\|\Sigma_Y \Sigma_Z\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))} \le (2\pi)^{-2n} e^{-(1/4)|Y-Z|^2}. \tag{2.14}$$

Proof. Put $p^w = \Sigma_Y \Sigma_Z = p^w_Y p^w_Z$. Then

$$\begin{aligned} p(X) &= \pi^{-2n} \iint p_Y(X + Y') p_Z(X + Z') e^{2i\sigma(Z',Y')} dY' dZ' \\ &= \pi^{-4n} \iint e^{-|X+Y'-Y|^2-|X+Z'-Z|^2} e^{2i\sigma(Z',Y')} dY' dZ' \\ &= \pi^{-4n} \iint e^{-|Y'|^2-|Z'|^2} e^{2i\sigma(Z'-X+Z,Y'-X+Y)} dY' dZ'. \end{aligned}$$

Here, in the last equality, we used the translation of variables $(Y', Z') \longmapsto (Y' - X + Y, Z' - X + Z)$. Since $\int e^{-|X|^2 \pm 2i\sigma(X,Y)} dX = \pi^n e^{-|Y|^2}$, it follows from Fubini's theorem that

$$\begin{aligned} p(X) &= \pi^{-4n} e^{2i\sigma(Z-X,Y-X)} \int e^{-|Y'|^2+2i\sigma(Z-X,Y')} dY' \int e^{-|Z'|^2+2i\sigma(Z',Y'-X+Y)} dZ' \\ &= \pi^{-4n} e^{2i\sigma(Z-X,Y-X)} \int e^{-|Y'|^2+2i\sigma(Z-X,Y')} dY' \cdot \pi^n e^{-|Y'-X+Y|^2} \end{aligned}$$

$$= \pi^{-3n} e^{2i\sigma(Z-X,Y-X)} \int e^{-2\left|Y'-(X-Y)/2\right|^2-(1/2)|X-Y|^2+2i\sigma(Z-X,Y')} dY'$$
$$= \pi^{-3n} e^{2i\sigma(Z-X,Y-X)-(1/2)|X-Y|^2} \int e^{-2|Y'|^2+2i\sigma(Z-X,Y'+(X-Y)/2)} dY'$$
$$= \pi^{-3n} e^{i\sigma(Z-X,Y-X)-(1/2)|X-Y|^2} \cdot \left(\frac{\pi}{2}\right)^n e^{-(1/2)|X-Z|^2}$$
$$= \pi^{-n}(2\pi)^{-n} e^{i\sigma(Z-X,Y-X)-(1/2)|X-Y|^2-(1/2)|X-Z|^2},$$

which shows (2.12). Here, in the fourth equality, we used the translation of variables $Y' \longmapsto Y' + (X - Y)/2$. (2.13) follows from (2.11) and (2.12). We shall show (2.14). Setting $Y = (y, \eta)$, $Z = (z, \zeta) \in \boldsymbol{R}^n \times \boldsymbol{R}^n$, we obtain

$$\begin{aligned}(Wg^Z)(Y) &= (4\pi^3)^{-n/2} \int_{\boldsymbol{R}^n} e^{ix(\zeta-\eta)} e^{-(1/2)|x-y|^2-(1/2)|x-z|^2} dx \\ &= (4\pi^3)^{-n/2} \int_{\boldsymbol{R}^n} e^{ix(\zeta-\eta)} e^{-(1/4)|y-z|^2-|x-(y+z)/2|^2} dx \\ &= (2\pi)^{-n} e^{i\{(y+z)/2\}(\zeta-\eta)} e^{-(1/4)|Y-Z|^2}.\end{aligned}$$

Since $|(Wu)(Z)| \le \|g^Z\|_{L^2(\boldsymbol{R}^n)}\|u\|_{L^2(\boldsymbol{R}^n)}$ by Schwarz's inequality, we get from (2.12)

$$\begin{aligned}\|\Sigma_Y \Sigma_Z u\|_{L^2(\boldsymbol{R}^n)} &= |(Wg^Z)(Y)|\,|(Wu)(Z)|\,\|g^Y\|_{L^2(\boldsymbol{R}^n)} \\ &\le (2\pi)^{-n} e^{-(1/4)|Y-Z|^2} \|g\|^2_{L^2(\boldsymbol{R}^n)} \|u\|_{L^2(\boldsymbol{R}^n)},\end{aligned}$$

which shows (2.14) by $\|g\|^2_{L^2(\boldsymbol{R}^n)} = (2\pi)^{-n}$. □

Proof of Theorem 2.3 and Corollary 2.4. It follows from (2.10) and (2.11) that $a^{\mathrm{Wick}}(x, D) = b^w(x, D)$, with

$$b(X) = \pi^{-n} \int_{\boldsymbol{R}^{2n}} a(X + Y) e^{-|Y|^2} dY.$$

Using Taylor's formula

$$a(X + Y) = \sum_{|\alpha| \le N-1} a^{(\alpha)}(X) \frac{Y^\alpha}{\alpha!} + a_N(X, Y)$$

with

$$a_N(X, Y) = N \sum_{|\alpha|=N} \int_0^1 (1 - \theta)^{N-1} a^{(\alpha)}(X + \theta Y) d\theta \frac{Y^\alpha}{\alpha!},$$

we have

$$b(X) = \sum_{|\alpha| \le N-1} \frac{a^{(\alpha)}(X)}{\pi^n \alpha!} \int_{\boldsymbol{R}^{2n}} Y^\alpha e^{-|Y|^2} dY + r_N(X)$$

where $r_N(X) = \pi^{-n}\int_{\mathbf{R}^{2n}} a_N(X,Y)e^{-|Y|^2}dY \in \mathcal{T}^{m-\sigma N}_{\rho,\delta}$. Note that $\int_{\mathbf{R}^{2n}} Y^\alpha e^{-|Y|^2}dY = 0$ for $\alpha \notin (2\mathbf{Z}_+)^{2n}$. Hence we get (2.2) if we take $N = 2$. If $\alpha = (2\beta_1, \dots, 2\beta_{2n}) \in (2\mathbf{Z}_+)^{2n}$ then we have with $\beta = (\beta_1, \dots, \beta_{2n})$

$$\frac{1}{\alpha!}\int_{\mathbf{R}^{2n}} Y^\alpha e^{-|Y|^2}dY = \prod_{j=1}^{2n}\frac{1}{(2\beta_j)!}\int_{-\infty}^{\infty} Y_j^{2\beta_j}e^{-|Y_j|^2}dY_j = \frac{\pi^n}{4^{|\beta|}\beta!}.$$

Choosing $N = l$ we have

$$\begin{aligned} a^{\mathrm{Wick}} = b^w &= \sum_{|\beta|=0}^{[(l-1)/2]}\frac{1}{4^{|\beta|}\beta!}\left(\partial_{X_1}^{2\beta_1}\cdots\partial_{X_{2n}}^{2\beta_{2n}}a(X)\right)^w + r_N^w \\ &= \left(\sum_{k=0}^{[(l-1)/2]}\frac{1}{k!}\left(\frac{\Delta_X}{4}\right)^k a\right)^w + r_N^w \end{aligned}$$

with $r_N \in \mathcal{T}^{m-\sigma l}_{\rho,\delta}$. Replacing $a$ by $\sum_{k=0}^{[(l-1)/2]}(1/k!)(-\Delta_X/4)^k a$, we obtain (2.3) because

$$\left(\sum_{k=0}^{[(l-1)/2]}\frac{t^k}{k!}\right)\left(\sum_{k=0}^{[(l-1)/2]}\frac{(-t)^k}{k!}\right) = 1 + O(t^{[(l+1)/2]}). \qquad \square$$

Proof of Thoerems 2.5 and 2.7. By means of (2.10) we note that

$$a^{\mathrm{Wick}}b^{\mathrm{Wick}} = \iint_{\mathbf{R}_Y^{2n}\times\mathbf{R}_Z^{2n}} a(Y)b(Z)\Sigma_Y\Sigma_Z dYdZ.$$

We use Taylor's formula

$$b(Z) = \sum_{|\alpha|\le N-1} b^{(\alpha)}(Y)\frac{(Z-Y)^\alpha}{\alpha!} + b_N(Y,Z),$$

where

$$b_N(Y,Z) = N\sum_{|\alpha|=N}\int_0^1(1-\theta)^{N-1}b^{(\alpha)}(Y+\theta(Z-Y))d\theta\frac{(Z-Y)^\alpha}{\alpha!}.$$

Then we have

$$a^{\mathrm{Wick}}b^{\mathrm{Wick}} = \sum_{l=0}^{N-1}\sum_{|\alpha|=l}\Omega_\alpha + R_N^0,$$

where

$$\Omega_\alpha = \frac{1}{\alpha!}\iint_{\mathbf{R}^{2n}_Y\times\mathbf{R}^{2n}_Z} a(Y)b^{(\alpha)}(Y)(Z-Y)^\alpha \Sigma_Y\Sigma_Z dYdZ,$$
$$R^0_N = \iint_{\mathbf{R}^{2n}_Y\times\mathbf{R}^{2n}_Z} a(Y)b_N(Y,Z)\Sigma_Y\Sigma_Z dYdZ.$$

In the same way as in the proof of (2.19) of [5], it follows from (2.13), (2.14) and Cotlar's lemma that we have

$$\|R^0_N\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \le C_{(N,n)}\|a\|_{L^\infty}\sum_{|\alpha|=N}\|b^{(\alpha)}\|_{L^\infty} \le C'_{(N,n)}\gamma_0(a)\gamma_N(b).$$

The last inequality follows because $a\in\mathcal{T}^0_{1/2,1/2}$, $b\in\mathcal{T}^{N/2}_{1/2,1/2}$. By means of (2.12), together with the change of variables $(X-Y,X-Z)\longmapsto(-Y,-Z)$, we see that if $\sigma(\Omega_\alpha)$ denotes the Weyl symbol of $\Omega_\alpha$,

$$\sigma(\Omega_\alpha) = (2\pi^2)^{-n}\sum_{\alpha'+\alpha''=\alpha}\frac{1}{\alpha'!\alpha''!}\int_{\mathbf{R}^{2n}_Y} a(X+Y)b^{(\alpha)}(X+Y)(-Y)^{\alpha''}e^{-(1/2)|Y|^2} \left(\int_{\mathbf{R}^{2n}_Z} Z^{\alpha'}e^{i\sigma(Z,Y)}e^{-(1/2)|Z|^2}dZ\right)dY. \tag{2.15}$$

Note that $Z^{\alpha'}e^{i\sigma(Z,Y)} = (-i)^{|\alpha'|}H_Y^{\alpha'}e^{i\sigma(Z,Y)}$ and $\int_{\mathbf{R}^{2n}_Z} e^{i\sigma(Z,Y)}e^{-(1/2)|Z|^2}dZ = (2\pi)^n e^{-(1/2)|Y|^2}$. We have the formulas

$$\begin{cases} e^{-(1/2)|Y|^2}H_{Y_j}e^{-(1/2)|Y|^2} = \dfrac{1}{2}H_{Y_j}e^{-|Y|^2}, \\ e^{-(1/2)|Y|^2}H_{Y_j}H_{Y_k}e^{-(1/2)|Y|^2} = \left(\dfrac{1}{4}H_{Y_j}H_{Y_k} - \dfrac{\delta_{j,k}}{2}\right)e^{-|Y|^2}, \\ e^{-(1/2)|Y|^2}H_{Y_m}H_{Y_j}H_{Y_k}e^{-(1/2)|Y|^2} \\ \qquad = \left(\dfrac{1}{8}H_{Y_m}H_{Y_j}H_{Y_k} - \dfrac{1}{4}(\delta_{j,k}H_{Y_m}+\delta_{k,m}H_{Y_j}+\delta_{j,m}H_{Y_k})\right)e^{-|Y|^2}. \end{cases} \tag{2.16}$$

Furthermore, for any $\alpha$ we can show by induction on $|\alpha|$ that

$$e^{-(1/2)|Y|^2}\left(\frac{H_Y}{i}\right)^\alpha e^{-(1/2)|Y|^2} = 2^{-|\alpha|}\left(\sum_{q=0}^\infty \frac{\Delta_Z^q(Z^\alpha)}{q!}\bigg|_{Z=H_Y/i}\right)e^{-|Y|^2}. \tag{2.17}$$

In fact, if we write

$$e^{-(1/2)|Y|^2}\left(\frac{H_Y}{i}\right)^\alpha e^{-(1/2)|Y|^2}$$
$$= \left(e^{-(1/2)|Y|^2}\left(\frac{H_{Y_m}}{i}\right)e^{(1/2)|Y|^2}\right)\left(e^{-(1/2)|Y|^2}\left(\frac{H_Y}{i}\right)^{\alpha'}e^{-(1/2)|Y|^2}\right)$$

for some $m$ and $\alpha'$ with $|\alpha'| = |\alpha| - 1$, then it equals, by induction hypothesis,

$$2^{-|\alpha'|}\left(H_{Y_m}\left(\frac{|Y|^2}{2i}\right) + \left(\frac{H_{Y_m}}{i}\right)\right)\left(\sum_{q=0}^{\infty}\frac{\Delta_Z^q\left(Z^{\alpha'}\right)}{q!}\Bigg|_{Z=H_Y/i}\right)e^{-|Y|^2}$$
$$= 2^{-|\alpha|}\left\{\left(\sum_{q=0}^{\infty}\frac{\Delta_Z^q\left(Z^{\alpha'}\right)}{q!}\Bigg|_{Z=H_Y/i}\right)\left(H_{Y_m}\left(\frac{|Y|^2}{i}\right)\right)e^{-|Y|^2}\right.$$
$$\left. + \left(\sum_{q=0}^{\infty}\frac{2\partial_{Z_m}\Delta_Z^q\left(Z^{\alpha'}\right)}{q!}\Bigg|_{Z=H_Y/i}\right)e^{-|Y|^2} + 2\left(\sum_{q=0}^{\infty}\frac{Z_m\Delta_Z^q\left(Z^{\alpha'}\right)}{q!}\Bigg|_{Z=H_Y/i}\right)e^{-|Y|^2}\right\}$$

because $[H_{Y_m}(|Y|^2/2i), (H_Y/i)^\beta] = (\partial_{Z_m}Z^\beta)\big|_{Z=H_Y/i}$ for any $\beta$. The first term of the right hand side is cancelled by half of the third term because $(H_{Y_m}(|Y|^2))e^{-|Y|^2} = -H_{Y_m}e^{-|Y|^2}$. Finally we have

$$e^{-(1/2)|Y|^2}\left(\frac{H_Y}{i}\right)^{\alpha}e^{-(1/2)|Y|^2}$$
$$= 2^{-|\alpha|}\left\{\left(\sum_{q=0}^{\infty}\frac{2\partial_{Z_m}\Delta_Z^q\left(Z^{\alpha'}\right) + Z_m\Delta_Z^q\left(Z^{\alpha'}\right)}{q!}\Bigg|_{Z=H_Y/i}\right)\right\}e^{-|Y|^2},$$

which gives (2.17) by means of $\Delta_Z^q Z_m Z^{\alpha'} = 2q\partial_{Z_m}\Delta_Z^{q-1}Z^{\alpha'} + Z_m\Delta_Z^q Z^{\alpha'}$.

It follows from (2.17) that

$$\sigma(\Omega_\alpha) = \pi^{-n}\int_{\boldsymbol{R}_Y^{2n}} a(X+Y)b^{(\alpha)}(X+Y)$$
$$\sum_{q=0}^{\infty}\frac{\Delta_Z^q}{q!}\left(\sum_{\alpha'+\alpha''=\alpha}\frac{1}{\alpha'!\alpha''!}(-Y)^{\alpha''}\left(\frac{Z}{2}\right)^{\alpha'}\right)\Bigg|_{Z=H_Y/i} e^{-|Y|^2}dY$$
$$= \frac{\pi^{-n}}{\alpha!}\int_{\boldsymbol{R}_Y^{2n}} a(X+Y)b^{(\alpha)}(X+Y)\sum_{q=0}^{\infty}\frac{\Delta_Z^q}{q!}\left(-Y+\frac{Z}{2}\right)^{\alpha}\Bigg|_{Z=H_Y/i} e^{-|Y|^2}dY,$$

where we take a convention that all $Z$ in $(-Y+Z/2)^\beta$ for $\beta$ with $|\beta| = |\alpha| - 2q$ are ordered to the right hand side. Since $Y_jH_{Y_k} = \{Y_j, Y_k\} + H_{Y_k}Y_j$, we have

$$Y_jH_{Y_k} + Y_kH_{Y_j} = H_{Y_k}Y_j + H_{Y_j}Y_k.$$

The repeated use of this formula shows that if $W^\beta = W_{j_1}\cdots W_{j_{|\beta|}}$ and $S_{|\beta|}$ denotes the permutation group on $\{j_1, \ldots, j_{|\beta|}\}$ then for any $r$ $(0 < r < |\beta|)$ we have

$$\begin{aligned}\sum_{\substack{\beta'+\beta''=\beta\\|\beta'|=r,|\beta''|=|\beta|-r}} Y^{\beta'} H_Y^{\beta''} &= \frac{1}{r!(|\beta|-r)!}\sum_{\sigma\in S_{|\beta|}} Y_{\sigma(j_1)}\cdots Y_{\sigma(j_r)} H_{Y_{\sigma(j_{r+1})}}\cdots H_{Y_{\sigma(j_{|\beta|})}}\\ &= \frac{1}{r!(|\beta|-r)!}\sum_{\sigma\in S_{|\beta|}} Y_{\sigma(j_1)}\cdots Y_{\sigma(j_{r-1})} H_{Y_{\sigma(j_r)}} Y_{\sigma(j_{r+1})} H_{Y_{\sigma(j_{r+2})}}\cdots H_{Y_{\sigma(j_{|\beta|})}}\\ &= \cdots = \frac{1}{r!(|\beta|-r)!}\sum_{\sigma\in S_{|\beta|}} H_{Y_{\sigma(j_1)}}\cdots H_{Y_{\sigma(j_{|\beta|-r})}} Y_{\sigma(j_{|\beta|-r+1})}\cdots Y_{\sigma(j_{|\beta|})}\\ &= \sum_{\substack{\beta'+\beta''=\beta\\|\beta'|=r,|\beta''|=|\beta|-r}} H_Y^{\beta''} Y^{\beta'}.\end{aligned}$$

Hence we may regard that all $Z$ in $(-Y+Z/2)^\beta$ are ordered to the left hand side. Namely, we have

$$\begin{aligned}\sigma(\Omega_\alpha) &= \pi^{-n}\int_{\mathbf{R}_Y^{2n}} a(X+Y)b^{(\alpha)}(X+Y)\\ &\qquad \sum_{q=0}^{\infty}\frac{\Delta_Z^q}{q!}\left(\sum_{\alpha'+\alpha''=\alpha}\frac{1}{\alpha'!\alpha''!}\left(\frac{Z}{2}\right)^{\alpha'}\right)\Bigg|_{Z=H_Y/i}(-Y)^{\alpha''}e^{-|Y|^2}dY.\end{aligned}$$

Note that

$$(2.18)\quad \begin{cases} Y_j e^{-|Y|^2} = -\dfrac{1}{2}\partial_{Y_j} e^{-|Y|^2},\\ Y_j Y_k e^{-|Y|^2} = \left(\dfrac{1}{4}\partial_{Y_j}\partial_{Y_k} + \dfrac{\delta_{j,k}}{2}\right) e^{-|Y|^2}\\ Y_m Y_j Y_k e^{-|Y|^2} = -\left(\dfrac{1}{8}\partial_{Y_m}\partial_{Y_j}\partial_{Y_k} + \dfrac{\delta_{j,k}}{4}\partial_{Y_m} + \dfrac{\delta_{k,m}}{4}\partial_{Y_j} + \dfrac{\delta_{m,j}}{4}\partial_{Y_k}\right) e^{-|Y|^2}. \end{cases}$$

In the same way as in the proof of (2.17), we can show that for any $\alpha$

$$(2.19)\qquad Y^\alpha e^{-|Y|^2} = 2^{-|\alpha|}\left(\sum_{q=0}^{\infty}\frac{\Delta_W^q (W^\alpha)}{q!}\Bigg|_{W=-\partial_Y}\right) e^{-|Y|^2}$$

by induction on $|\alpha|$, if we note $[Y_m, (-\partial_Y)^\beta] = \left(\partial_{W_m} W^\beta\right)\big|_{W=-\partial_Y}$.

In view of

$$\left(\sum_{q=0}^{\infty}\frac{\Delta_Z^q}{q!}\right)\left(\sum_{q=0}^{\infty}\frac{\Delta_W^q}{q!}\right) = \sum_{q=0}^{\infty}\frac{(\Delta_Z+\Delta_W)^q}{q!},$$

it follows from (2.19) that

$$\sigma(\Omega_\alpha) = \frac{\pi^{-n}}{2^{|\alpha|}\alpha!}\int_{\mathbf{R}^{2n}_Y} a(X+Y)b^{(\alpha)}(X+Y) \sum_{q=0}^{\infty}\frac{(\Delta_Z+\Delta_W)^q}{q!}\,(W+Z)^\alpha\big|_{(W,Z)=(\partial_Y, H_Y/i)}\, e^{-|Y|^2}dY$$

$$= \frac{\pi^{-n}}{2^{|\alpha|}\alpha!}\int_{\mathbf{R}^{2n}_Y} a(X+Y)b^{(\alpha)}(X+Y)\left(\sum_{q=0}^{\infty}\frac{(2\Delta_Z)^q}{q!}Z^\alpha\Big|_{Z=\partial_Y+H_Y/i}\right)e^{-|Y|^2}dY.$$

By integration by parts we get

$$\sigma(\Omega_\alpha) = \frac{\pi^{-n}}{2^{|\alpha|}\alpha!} \times\int_{\mathbf{R}^{2n}_Y}\left\{\left(\sum_{q=0}^{\infty}(-1)^{|\alpha|-2q}\frac{(2\Delta_Z)^q}{q!}Z^\alpha\Big|_{Z=\partial_Y+H_Y/i}\right)a(Y)\partial_W^\alpha b(Y+W)\big|_{W=0}\right\}e^{-|X-Y|^2}dY.$$

Hence we obtain

$$\sum_{|\alpha|=l}\sigma(\Omega_\alpha) = \frac{(-1)^l\pi^{-n}}{2^l l!} \times\int_{\mathbf{R}^{2n}_Y}\left\{\sum_{q=0}^{\infty}\frac{(2\Delta_Z)^q}{q!}\left(\sum_{j=1}^{2n}Z_j\partial_{W_j}\right)^l\Bigg|_{Z=\partial_Y+H_Y/i} a(Y)b(Y+W)\right\}\Bigg|_{W=0} e^{-|X-Y|^2}dY$$

$$= \int_{\mathbf{R}^{2n}_Y}\left\{\sum_{q=0}^{[l/2]}\frac{(-2)^{q-l}}{q!(l-2q)!}\left(\sum_{j=1}^{2n}\left(\partial_{Y_j}+\frac{H_{Y_j}}{i}\right)\partial_{W_j}\right)^{l-2q}(-\Delta_W)^q a(Y)b(Y+W)\right\}\Bigg|_{W=0} e^{-|X-Y|^2}\frac{dY}{\pi^n},$$

because $\Delta_Z\big(\sum_{j=1}^{2n}Z_j\partial_{W_j}\big)^l = l(l-1)\big(\sum_{j=1}^{2n}Z_j\partial_{W_j}\big)^{l-2}\Delta_W$. By setting $l-q=k$, we have the following rearrangement

$$\sum_{l=0}^{N-1}\sum_{|\alpha|=l}\sigma(\Omega_\alpha)$$

$$= \sum_{k=0}^{N/2}\frac{(-1)^k}{2^k k!}\int_{\mathbf{R}^{2n}_Y}\left(-\Delta_W+\sum_{j=1}^{2n}\left(\partial_{Y_j}+\frac{H_{Y_j}}{i}\right)\partial_{W_j}\right)^k a(Y)b(Y+W)\Bigg|_{W=0} e^{-|X-Y|^2}\frac{dY}{\pi^n}$$

$$
\begin{aligned}
&-\frac{(-1)^{N/2}}{2^{N/2}(N/2)!}\int_{\boldsymbol{R}^{2n}_Y} a(Y)(-\Delta_W)^{N/2}b(Y+W)\Big|_{W=0} e^{-|X-Y|^2}\frac{dY}{\pi^n}\\
&+\sum_{k=N/2+1}^{N-1}\frac{(-1)^k}{2^k k!}\int_{\boldsymbol{R}^{2n}_Y}\left\{\sum_{q=0}^{N-k-1}\binom{k}{q}\left(\sum_{j=1}^{2n}\left(\partial_{Y_j}+\frac{H_{Y_j}}{i}\right)\partial_{W_j}\right)^{k-q}\right.\\
&\qquad\qquad\left.\times\, a(Y)(-\Delta_W)^q b(Y+W)\right\}\Bigg|_{W=0} e^{-|X-Y|^2}\frac{dY}{\pi^n}\\
&=\sigma(I_1)+\sigma(I_2)+\sigma(I_3).
\end{aligned}
$$

Since it follows that for any $\tilde{b}$

$$
\sum_{j=1}^{2n} H_{Y_j}\partial_{W_j}\tilde{b}(Y+W)=0 \text{ and } \left(-\Delta_W+\sum_{j=1}^{2n}\partial_{Y_j}\partial_{W_j}\right)\tilde{b}(Y+W)=0, \tag{2.20}
$$

we have

$$
\begin{aligned}
&\left\{\left(-\Delta_W+\sum_{j=1}^{2n}\left(\partial_{Y_j}+\frac{H_{Y_j}}{i}\right)\partial_{W_j}\right)^k a(Y)b(Y+W)\right\}\Bigg|_{W=0}\\
&\qquad=\left(\sum_{j=1}^{2n}\left(\partial_{Y_j}+\frac{H_{Y_j}}{i}\right)\partial_{W_j}\right)^k a(Y)b(W)\Bigg|_{W=Y}.
\end{aligned}\tag{2.21}
$$

By means of (2.21) we get the desired expansion formula (2.4) if we show that $I_2$ and $I_3$ are $L^2$ bounded operators whose operator norms are bounded by the right hand side of (2.6). In fact, we have $\|I_2\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))}\le C\gamma_0(a)\gamma_N(b)$ because $I_2$ equals $(a\Delta^{N/2}b)^{\text{Wick}}$ with a constant factor. $I_3$ is the sum of the following terms

$$
\left(\int_{\boldsymbol{R}^{2n}_Y}\left(\partial_Y^{\beta'}H_Y^{\beta''}(a(Y)b^{(\alpha)}(Y))\right)e^{-|X-Y|^2}\frac{dY}{\pi^n}\right)^w := \Omega^{\alpha}_{\beta',\beta''}
$$

for $\alpha,\beta',\beta''$ with $N/2<|\alpha|<N$, $|\alpha|-|\beta'+\beta''|\in 2\boldsymbol{Z}_+$ and $\beta'+\beta'':=\beta\subset\alpha$. Here we denote $\beta\subset\alpha$ for $\beta=(\beta_1,\dots,\beta_{2n}),\alpha=(\alpha_1,\dots,\alpha_{2n})$ if $\beta_j\le\alpha_j$ for each $j=1,\dots,2n$. Since $a\in\mathcal{T}^0_{1/2,1/2}$, $b\in\mathcal{T}^{N/2}_{1/2,1/2}$, there exist $\tilde{\beta}'$, $\tilde{\beta}''$ satisfying $\tilde{\beta}'\subset\beta'$, $\tilde{\beta}''\subset\beta''$ and $|\tilde{\beta}'|+|\tilde{\beta}''|<N/2$ such that $B(Y)=\partial_Y^{\tilde{\beta}''}H_Y^{\tilde{\beta}'}(a(Y)b^{(\alpha)}(Y))$ belong to $L^\infty$ unformly with respect to $\Lambda$. By integration by parts again, we have with $\tilde{\beta}=\tilde{\beta}'+\tilde{\beta}''$

$$
\Omega^{\alpha}_{\beta',\beta''}=(-1)^{|\beta-\tilde{\beta}|}\left(\int_{\boldsymbol{R}^{2n}_Y}B(Y)\partial_Y^{\beta''-\tilde{\beta}''}H_Y^{\beta'-\tilde{\beta}'}\left(e^{-|X-Y|^2}\right)\frac{dY}{\pi^n}\right)^w.
$$

By this expression we can see

$$\|\Omega^{\alpha}_{\beta',\beta''}\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))} \leq C_{(N,n)}\|B\|_{L^\infty},$$

in the same way as in [5; (3.19)]. Now the proof of (2.4) is complete. □

It follows from (2.10) that

$$b^{\text{Wick}}a^{\text{Wick}} = \iint_{\boldsymbol{R}^{2n}_Y \times \boldsymbol{R}^{2n}_Z} a(Y)b(Z)\Sigma_Z\Sigma_Y dY dZ.$$

We use the same Taylor's formula of $b(Z)$ as above. If $\widetilde{\Omega}_\alpha$, $\widetilde{R}^0_N$ denote the corresponding terms to $\Omega_\alpha$, $R^0_N$ then they are defined only with $\Sigma_Y\Sigma_Z$ replaced by $\Sigma_Z\Sigma_Y$. Since the Weyl symbol of $\Sigma_Z\Sigma_Y$ equals $P_{Z,Y}(X) = \overline{P_{Y,Z}(X)}$ because of Lemma 2.10, $\widetilde{R}^0_N$ is estimated in the same way as for $R^0_N$. Using the change of variables $(X-Y, X-Z) \longmapsto (-Y, Z)$ we have instead of (2.15)

$$\begin{aligned}\sigma(\widetilde{\Omega}_\alpha) = (2\pi^2)^{-n} \sum_{\alpha'+\alpha''=\alpha} \frac{1}{\alpha'!\alpha''!} \int_{\boldsymbol{R}^{2n}_Y} a(X+Y)b^{(\alpha)}(X+Y)(-Y)^{\alpha''} e^{-(1/2)|Y|^2} \\ \times \left( \int_{\boldsymbol{R}^{2n}_Z} (-Z)^{\alpha'} e^{i\sigma(Z,Y)} e^{-(1/2)|Z|^2} dZ \right) dY.\end{aligned} \tag{2.22}$$

Since the difference between (2.15) and (2.22) is only the fact that $Z^{\alpha'}$ is replaced by $(-Z)^{\alpha'}$ we easily obtain (2.5) by replacing $H_Y/i$ by $-H_Y/i$. The formula (2.7) is obvious if we note

$$\left(A + \frac{B}{i}\right)^k - \left(A - \frac{B}{i}\right)^k = \frac{2}{i} \sum_{l=0}^{[(k-1)/2]} \binom{k}{2l+1} A^{k-2l-1}(-1)^l B^{2l+1}$$

and $\sum_{j=1}^{2n} H_{Y_j}\partial_{W_j} a(Y)b(W)|_{W=Y} = -\{a, b\}(Y)$. □

As stated in Remark 2.8 the expansion fomulas hold for more general symbols $a$, $b$ and any positive integer $N$.

**Theorem 2.11.** *Let $a \in L^\infty(\boldsymbol{R}^{2n})$ and $b \in \mathcal{T}^m_{\rho,\delta}$ with $\rho$, $\delta > 0$ and $m \in \boldsymbol{R}$. Set $\sigma = \min(\rho, \delta)$. Let $N$ be a positive integer with $N \geq m/\sigma$ and assume that*

$$\left( \sum_{j=1}^{2n} \left( \partial_{X_j}\partial_{Z_j} \pm \frac{H_{X_j}}{i}\partial_{Z_j} \right) \right)^k a(X)b(Z) \Bigg|_{Z=X} \in L^\infty \quad \text{for } 0 \leq k \leq k_0 \tag{2.23}$$

*for some nonnegative integer $k_0 \leq [N/2]$. Then we have the expansion formulas as* *follows*:

$$
(2.24)\quad a^{\mathrm{Wick}}b^{\mathrm{Wick}} = \begin{cases} (ab)^{\mathrm{Wick}} + R_1^0 & \text{if } N = 1,\\ \displaystyle\sum_{k=0}^{k_0} \frac{(-1)^k}{2^k k!}\left(\left(\sum_{j=1}^{2n}\left(\partial_{X_j}\partial_{Z_j} + \frac{H_{X_j}}{i}\partial_{Z_j}\right)\right)^k a(X)b(Z)\Big|_{Z=X}\right)^{\mathrm{Wick}} & \\ \displaystyle + \sum_{k=k_0+1}^{N-1}\sum_{|\alpha|=k}\sum_{\substack{\beta'+\beta''\subset\alpha\\ |\alpha|-|\beta'+\beta''|\in 2\mathbf{Z}_+}} C_{\alpha,\beta',\beta''}\Omega^{\alpha}_{\beta',\beta''} + R_N^0 & \text{if } N \geq 2,\end{cases}
$$

$$
(2.25)\quad b^{\mathrm{Wick}}a^{\mathrm{Wick}} = \begin{cases} (ab)^{\mathrm{Wick}} + \widetilde{R_1^0} & \text{if } N = 1,\\ \displaystyle\sum_{k=0}^{k_0} \frac{(-1)^k}{2^k k!}\left(\left(\sum_{j=1}^{2n}\left(\partial_{X_j}\partial_{Z_j} - \frac{H_{X_j}}{i}\partial_{Z_j}\right)\right)^k a(X)b(Z)\Big|_{Z=X}\right)^{\mathrm{Wick}} & \\ \displaystyle + \sum_{k=k_0+1}^{N-1}\sum_{|\alpha|=k}\sum_{\substack{\beta'+\beta''\subset\alpha\\ |\alpha|-|\beta'+\beta''|\in 2\mathbf{Z}_+}} \overline{C_{\alpha,\beta',\beta''}}\Omega^{\alpha}_{\beta',\beta''} + \widetilde{R_N^0} & \text{if } N \geq 2,\end{cases}
$$

*where* $C_{\alpha,\beta',\beta''} \in \mathbf{C}$ *are constants depending only on* $\alpha$, $\beta'$, $\beta''$ *and* $n$. *The remainder terms* $R_N^0$, $\widetilde{R_N^0}$ *are* $L^2$ *bounded operators uniformly with respect to a large parameter* $\Lambda$ *satisfying*

$$
(2.26)\qquad \|R_N^0\|_{\mathcal{L}(L^2(\mathbf{R}^n))}, \left\|\widetilde{R_N^0}\right\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \leq C_{(N,n)}\|a\|_{L^\infty(\mathbf{R}^{2n})}\gamma_N(b),
$$

*where* $C_{(N,n)}$ *is the constant depending only on* $N$ *and* $n$. *Here* $\Omega^{\alpha}_{\beta',\beta''}$ *is a pseudodifferential operator whose Weyl symbol is given by*

$$
(2.27)\qquad (-1)^{|\beta'+\beta''|}\int_{\mathbf{R}^{2n}_Y} a(Y)b^{(\alpha)}(Y)H_Y^{\beta'}\partial_Y^{\beta''}\left(e^{-|X-Y|^2}\right)\frac{dY}{\pi^n}.
$$

*Furthermore, if* $\partial_X^{\tilde\beta''}H_X^{\tilde\beta'}(a(X)b^{(\alpha)}(X)) \in L^\infty(\mathbf{R}^{2n})$ *for some* $\tilde\beta' \subset \beta'$, $\tilde\beta'' \subset \beta''$ *then there exists a constant* $C'_{(N,n)} > 0$ *depending only on* $N$ *and* $n$ *such that*

$$
(2.28)\qquad \|\Omega^{\alpha}_{\beta',\beta''}\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \leq C'_{(N,n)}\|\partial_X^{\tilde\beta''}H_X^{\tilde\beta'}(a(X)b^{(\alpha)}(X))\|_{L^\infty(\mathbf{R}^{2n})}.
$$

Proof. Even if $N$ is odd, we have the same formula as one just before (2.20) with $N/2$ repalced by $[N/2]$ and $\sigma(I_2) = 0$. If $a$ is not smooth enough, the integrand of $\sigma(I_1)$ and $\sigma(I_3)$ must be replaced, in view of the integration by parts, by

$$a(Y)b(Y+W)\left(\left(-\Delta_W-\sum_{j=1}^{2n}\left(\partial_{Y_j}+\frac{H_{Y_j}}{i}\right)\partial_{W_j}\right)^k e^{-|X-Y|^2}\right)\Bigg|_{W=0}$$

and

$$\sum_{q=0}^{N-k-1}\binom{k}{q}a(Y)(-\Delta_W)^q b(Y+W)\left(\left(-\sum_{j=1}^{2n}\left(\partial_{Y_j}+\frac{H_{Y_j}}{i}\right)\partial_{W_j}\right)^{k-q} e^{-|X-Y|^2}\right)\Bigg|_{W=0},$$

respectively, where we regard that $\partial_W$ and $\Delta_W$ operate $b(Y+W)$ even if they are put on the right hand side. Hence $\Omega^{\alpha}_{\beta',\beta''}$ in the proof of Theorem 2.7 must be written as in the form (2.27). If (2.23) holds then we can use the formula (2.21) and obtain (2.24) in the same way as in the proof of Theorem 2.7. □

**Theorem 2.12.** *Let $a\in L^\infty(\boldsymbol{R}^{2n})$ and $b\in\mathcal{T}^m_{\rho,\delta}$ with $\rho$, $\delta>0$ and $m\in\boldsymbol{R}$. Assume that $\{a,b\}\in L^\infty(\boldsymbol{R}^{2n})$. Set $\sigma=\min(\rho,\delta)$. Then, for a positive integer $N\geq m/\sigma$ we have*

$$[a^{\mathrm{Wick}},b^{\mathrm{Wick}}]=\begin{cases}\frac{1}{i}(\{a,b\})^{\mathrm{Wick}}+R^0_2 & \text{if } N=2,\\ \frac{1}{i}(\{a,b\})^{\mathrm{Wick}}+\frac{1}{i}\sum_{k=2}^{N-1}\sum_{|\alpha|=k}\sum_{\substack{\beta'+\beta''\subset\alpha,0\neq|\beta'|odd\\ |\alpha|-|\beta'+\beta''|\in 2\boldsymbol{Z}_+}}C'_{\alpha,\beta',\beta''}\Omega^{\alpha}_{\beta',\beta''} & \\ \qquad+R^0_N \quad \text{if } N\geq 3,\end{cases} \tag{2.29}$$

*where $C'_{\alpha,\beta',\beta''}\in\boldsymbol{R}$ are constants depending only on $\alpha$, $\beta'$, $\beta''$ and $n$. Here $R^0_N$ and $\Omega^{\alpha}_{\beta',\beta''}$ satisfy the same property as* Theorem 2.11. *In particular, when $N=3$ we have*

$$[a^{\mathrm{Wick}},b^{\mathrm{Wick}}]=\left(\frac{1}{i}\{a,b\}\right)^{\mathrm{Wick}}+R_3, \tag{2.30}$$

*where $R_3$ is an $L^2$ bounded operator satisfying*

$$\|R_3\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))}\leq C_{(N,n)}\left(\|a\|_{L^\infty(\boldsymbol{R}^{2n})}\gamma_3(b)+\sum_{j=1}^{2n}\|\{a,\partial_{X_j}b\}\|_{L^\infty(\boldsymbol{R}^{2n})}\right), \tag{2.31}$$

*provided that $\{a,\partial_{X_j}b\}\in L^\infty(\boldsymbol{R}^{2n})$ for $j=1,\dots,2n$.*

## 3. Application

In this last section we shall apply the Wick calculus to the Cauchy problem for some dispersive equations. Let us consider the following Cauchy problem

$$(\text{C.P.})_\kappa \qquad \begin{cases} (\partial_t + i a_2^w(D_x) + a_1^w(x, D_x) + a_0^w(x, D_x))u = f \quad \text{in} \quad \mathcal{D}'((0, T) \times \boldsymbol{R}_x^n) \\ u(0, x) = u_0(x) \end{cases}$$

where $a_j \in S_{1,0}^{2\kappa-2+j}$ $(j = 1, 2)$, $a_0 \in S_{1,0}^0$, $1/2 < \kappa \leq 1$ and $T > 0$. We assume that

$$a_2(\xi) = \frac{1}{2\kappa}|\xi|^{2\kappa}, \quad a_1(x, \xi) = \sum_{j=1}^n b_j(x)\xi_j|\xi|^{2\kappa-2}$$

for $|\xi| \geq 1/4$ and $b_j(x) \in \mathcal{B}^\infty(\boldsymbol{R}^n)$ for $j = 1, \dots, n$.

For the $L^2$ well-posedness of the Cauchy problem $(\text{C.P.})_\kappa$, in the case of $\kappa = 1$, the following necessary condition

$$\sup_{\substack{(x,\omega)\in\boldsymbol{R}^n\times S^{n-1} \\ T>0}} \left| \int_0^T \sum_{j=1}^n \operatorname{Re} b_j(x - \omega\theta)\omega_j d\theta \right| < +\infty \tag{3.1}$$

is shown by Mizohata (see [6]). As to the sufficiency of the $L^2$ well-posedness of the Cauchy problem $(\text{C.P.})_\kappa$, in the case of $\kappa = 1$, there are works such as [1], [2], [4], [7], [8], etc. Here we quote only the sufficient conditions of the Cauchy problem $(\text{C.P.})_\kappa$ in the case of $\kappa = 1$ from [2] and [7] for the comparison with our results which will be mentioned later.

**Theorem** (*Mizohata*, see [7]). *Let* $\kappa = 1$. *Suppose* (3.1) *and the condition* (3.2);

$$\sup_{\substack{(x,\omega)\in\boldsymbol{R}^n\times S^{n-1} \\ T>0}} \left| \int_0^T \sum_{j=1}^n \partial_x^\alpha b_j(x - \omega\theta)\omega_j d\theta \right| < \infty \quad \textit{for all } \alpha \in \boldsymbol{Z}_+^n \setminus \{0\}. \tag{3.2}$$

*Then the Cauchy problem* $(\text{C.P.})_\kappa$ *is* $L^2$ *well-posed.*

We remark that, in [7], the following condition

$$\sup_{\substack{(x,\omega)\in\boldsymbol{R}^n\times S^{n-1} \\ T>0}} \int_0^T |\partial_x^\alpha b_j(x - \omega\theta)| \, d\theta < \infty \quad \text{for } j = 1, \dots, n, \tag{3.3}$$

which is more restrictive than (3.2), is assumed. But it is not difficult to see that we can replace (3.3) with (3.2) in the proof of the theorem for the sufficiency in [7].

**Theorem** (*Doi*, see [2]). *Let* $\kappa = 1$. *If there exists a positive non-increasing function* $\lambda(t)$ *in* $C([0,\infty)) \cap L^1((0,\infty))$ *satisfying*

$$|\operatorname{Re} b_j(x)| \le \lambda(|x|) \quad \textit{for all } x \in \boldsymbol{R}^n,\ j = 1, \ldots, n, \tag{3.4}$$

*then the Cauchy problem* $(\text{C.P.})_\kappa$ *is* $L^2$ *well-posed.*

About the sufficiency of the Cauchy problem $(\text{C.P.})_\kappa$ in the case of $\kappa = 1$, the above Theorem (Doi) is the simplest result and powerful. But it is not able to handle the delicate case treated in [7]. We have an example which satisfies neither (3.2) nor (3.4) but (3.1).

EXAMPLE 3.1.

$$\operatorname{Re} b_j(x) = c_j \frac{\sin x_j}{\langle x \rangle}, \quad \operatorname{Im} b_j(x) = d_j \frac{x_j}{\langle x \rangle} \tag{3.5}$$

for $x \in \boldsymbol{R}^n$, $j = 1, \ldots, n$, where $c = (c_1, \ldots, c_n)$, $d = (d_1, \ldots, d_n) \in \boldsymbol{R}^n \setminus \{0\}$ with $d_j \neq d_k$ $(j \neq k)$ are the constant vectors and $\langle x \rangle = (1 + |x|^2)^{1/2}$.

When $\kappa = 1$, we don't yet know the answer for the question that in the case where $b_j(x)$ $(j = 1, \ldots, n)$ are given by (3.5) the Cauchy problem $(\text{C.P.})_\kappa$ is $L^2$ well-posed or not. However, in the case of $1/2 < \kappa \le 5/6$ we can get some results by using the Wick calculus. Our main theorem in this paper is as follows.

**Theorem 3.2.** (i) *Suppose* $1/2 < \kappa \le 2/3$. *Then the Cauchy problem* $(\text{C.P.})_\kappa$ *is* $L^2$ *well-posed if* (3.1) *is fulfilled.*
(ii) *Suppose* $2/3 < \kappa \le 5/6$. *In addtion to* (3.1), *assume that the following conditions are satisfied*:

$$\sup_{\substack{(x,\omega)\in \boldsymbol{R}^n \times S^{n-1} \\ T>0}} \left| \int_0^T \sum_{j=1}^n \partial_x^\alpha \operatorname{Re} b_j(x - \omega\theta)\omega_j d\theta \right| < \infty \quad \textit{for } |\alpha| = 1, \tag{3.6}$$

$$\sup_{x \in \boldsymbol{R}^n} \langle x \rangle\, |\partial_x^\alpha b_j(x)| < \infty \quad \textit{for } |\alpha| = 1,\ \ j = 1, \ldots, n. \tag{3.7}$$

*Then the Cauchy problem* $(\text{C.P.})_\kappa$ *is* $L^2$ *well-posed.*

REMARK 3.3. $b_j(x)$ $(j = 1, \ldots, n)$ given by (3.5) satisfy (3.1), (3.6) and (3.7). So Theorem 3.2 (ii) is applicable. In fact, taking the change of variable; $s = \theta - x \cdot \omega$,

$$\int_0^T \frac{\sin(x_j - \omega_j\theta)}{\langle x - \omega\theta \rangle} \omega_j d\theta = \int_{-x\cdot\omega}^{T - x\cdot\omega} \frac{\sin(x_j - \omega_j(s + x\cdot\omega))}{(1 + \rho + s^2)^{1/2}} \omega_j ds = I_1 - I_2,$$

where $\rho = |x|^2 - (x \cdot \omega)^2 \geq 0$,

$$I_1 = \int_0^{T-x\cdot\omega} \frac{\sin(x_j - \omega_j(s + x\cdot\omega))}{(1+\rho+s^2)^{1/2}}\omega_j ds, \quad I_2 = \int_0^{-x\cdot\omega} \frac{\sin(x_j - \omega_j(s + x\cdot\omega))}{(1+\rho+s^2)^{1/2}}\omega_j ds.$$

By the second mean value theorem of integration, there exists $s_1 \in \boldsymbol{R}$ such that

$$\begin{aligned} I_1 &= \frac{1}{(1+\rho)^{1/2}} \int_0^{s_1} \sin(x_j - \omega_j(s + x\cdot\omega))\omega_j ds \\ &\quad + \frac{1}{(1+\rho+(T - x\cdot\omega)^2)^{1/2}} \int_{s_1}^{T-x\cdot\omega} \sin(x_j - \omega_j(s + x\cdot\omega))\omega_j ds \\ &= \frac{1}{(1+\rho)^{1/2}} \int_{x_j-\omega_j(x\cdot\omega)}^{x_j-\omega_j(s_1+x\cdot\omega)} \sin t dt \\ &\quad + \frac{1}{(1+\rho+(T - x\cdot\omega)^2)^{1/2}} \int_{x_j-\omega_j(s_1+x\cdot\omega)}^{x_j-\omega_j T} \sin t dt, \end{aligned}$$

where in the last equality we changed the variable of integration by $t = x_j - \omega_j(s + x\cdot\omega)$. Since

$$\left| \int_a^b \sin t dt \right| \leq 2 \quad \text{for } a, b \in \boldsymbol{R},$$

we have $|I_1| \leq 4$. Similary, we can get $|I_2| \leq 4$. Thus (3.1) are satisfied. In the same way, it is easy to see that (3.5) satisfies (3.6).

Proof of Theorem 3.2. First we shall prove (ii) of the theorem. For the proof it suffices to show the following *a priori* estimate

$$\|u(t)\| \leq C\left(\|u(0)\| + \int_0^t \|f(s)\| ds\right) \tag{3.8}$$

for $t \in [0, T]$, $u \in C([0, T]; H^2(\boldsymbol{R}^n)) \cap C^1([0, T]; L^2(\boldsymbol{R}^n))$ with the constant $C = C(T) > 0$ where $f = (\partial_t + i a_2^w(D_x) + a_1^w(x, D_x) + a_0^w(x, D_x))u$ and $\|\cdot\| = \|\cdot\|_{L^2(\boldsymbol{R}^n)}$. Take $\varphi_0$, $\varphi$, $\psi_1$, $\psi_2 \in C_0^\infty(\boldsymbol{R}^n)$ of real value such that

$$\begin{cases} 0 \leq \varphi \leq 1, \quad \operatorname{supp}\varphi \subset \left\{\frac{1}{2} \leq |\xi| \leq 2\right\}, \\ \varphi_0(\xi)^2 + \sum_{\nu=1}^{\infty} \varphi(2^{-\nu}\xi)^2 = 1 \quad \text{for } \xi \in \boldsymbol{R}^n, \end{cases} \tag{3.9}$$

$$\begin{cases} \psi_1(\xi) = 1 \quad \text{on } \left\{\frac{1}{2} \leq |\xi| \leq 2\right\}, \quad \operatorname{supp}\psi_1 \subset \left\{\frac{1}{4} \leq |\xi| \leq \frac{9}{4}\right\}, \\ \psi_2(\xi) = 1 \quad \text{on } \left\{\frac{1}{4} \leq |\xi| \leq \frac{9}{4}\right\}, \quad \operatorname{supp}\psi_2 \subset \left\{\frac{1}{8} \leq |\xi| \leq \frac{9}{4} + \frac{1}{8}\right\}. \end{cases} \tag{3.10}$$

Put $\varphi_0^w = \varphi_0^w(D_x)$, $\varphi_\nu^w = \varphi_\nu^w(D_x) = \varphi^w(2^{-\nu}D_x)$ for $\nu \geq 1$. In view of a Littlewood-Paley decomposition $\sum_{\nu=0}^\infty \varphi_\nu^w(D_x)^2 = I_d$ we consider

$$(\partial_t + ia_{2,\nu}^w + a_{1,\nu}^w)\varphi_\nu^w u + r_\nu^w u = \varphi_\nu^w f \quad \text{for } \nu = 0, 1, 2, \dots, \tag{3.11}$$

where

$$\begin{aligned} &a_{2,\nu}^w = a_{2,\nu}^w(D_x), \quad a_{2,\nu}(\xi) = a_2(\xi)\psi_2(2^{-\nu}\xi),\\ &a_{1,\nu}^w = a_{1,\nu}^w(x, D_x), \quad a_{1,\nu}(x,\xi) = a_1(x,\xi)\psi_1(2^{-\nu}\xi),\\ &r_\nu^w = \left[\varphi_\nu^w, a_{1,\nu}^w\right] + \varphi_\nu^w a_0^w + \sum_{j=1}^2 i^{j-1}\varphi_\nu^w((1-\psi_j(2^{-\nu}\cdot))a_j)^w \quad \text{for } \nu \geq 0. \end{aligned}$$

We shall show that

$$\sum_{\nu=0}^\infty \|r_\nu^w u(t)\|^2 \leq C_0\|u(t)\|^2 \tag{3.12}$$

with some constant $C_0 > 0$. By $\sum_{\nu=0}^\infty(\varphi_\nu^w)^2 = I_d$, we have

$$\sum_{\nu=0}^\infty \|\varphi_\nu^w a_0^w u\|^2 = \|a_0^w u\|^2 \leq C_{0,0}\|u\|^2 \tag{3.13}$$

with some constant $C_{0,0} > 0$. Since $\operatorname{supp}\varphi_\nu \cap \operatorname{supp}((1-\psi_j(2^{-\nu}\cdot))a_j) = \emptyset$ for $j = 1, 2$, we get

$$\left\|\sum_{j=1}^2 \varphi_\nu^w((1-\psi_j(2^{-\nu}\cdot))a_j)^w\right\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))} \leq C_N 2^{-N\nu} \quad \text{for } N \in \boldsymbol{N}$$

with some constant $C_N > 0$ independent of $\nu$, which implies

$$\sum_{\nu=0}^\infty \left\|\sum_{j=1}^2 \varphi_\nu^w((1-\psi_j(2^{-\nu}\cdot))a_j)^w u\right\|^2 \leq C_{0,1}\|u\|^2 \tag{3.14}$$

with some constant $C_{0,1} > 0$. Noting that

$$[\varphi_\nu^w, a_{1,\nu}^w] = \left(\frac{1}{i}\{\varphi_\nu, a_{1,\nu}\}\right)^w + r_\nu' = \frac{1}{i}\sum_{k=1}^n \left(\frac{\partial a_{1,\nu}}{\partial x_k}\right)^w \left(\frac{\partial \varphi_\nu}{\partial \xi_k}\right)^w + r_\nu'',$$

where $\|r_\nu'\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))}$, $\|r_\nu''\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))} \leq C_{0,2}2^{-\nu}$ with some constant $C_{0,2} > 0$ independent of $\nu$ and

$$\sum_{\nu=0}^\infty \left\|\left(\frac{\partial a_{1,\nu}}{\partial x_k}\right)^w \left(\frac{\partial \varphi_\nu}{\partial \xi_k}\right)^w u\right\|^2 \leq C_{0,3}\sum_{\nu=0}^\infty \left(2^\nu \left\|\left(\frac{\partial \varphi_\nu}{\partial \xi_k}\right)^w u\right\|\right)^2 \leq C_{0,4}\|u\|^2$$

for $k = 1, \dots n$ with some constants $C_{0,3}$, $C_{0,4} > 0$, we have

$$\sum_{\nu=0}^{\infty} \left\| \left[ \varphi_\nu^w, a_{1,\nu}^w \right] \right\|^2 \leq C_{0,5} \|u\|^2 \tag{3.15}$$

with some constant $C_{0,5} > 0$. Thus from (3.13), (3.14) and (3.15) we obtain (3.12).

Set $\delta = 1/3$ and $\rho = 1 - \delta$. Changing $(x, \xi) \longmapsto (\Lambda^{-\delta} x, \Lambda^{\delta} \xi)$ with $\Lambda = \Lambda(\nu) = 2^\nu$, we have

$$M_{\Lambda^\delta} \left( \partial_t + i a_{2,\Lambda}^w + a_{1,\Lambda}^w \right) M_{\Lambda^{-\delta}} \varphi_\nu^w u + r_\nu^w u = \varphi_\nu^w f,$$

where $M_\lambda$ is the scaling operator defined by $(M_\lambda u)(x) = u(\lambda x)$ and

$$a_{2,\Lambda}(\xi) = a_{2,\nu}(\Lambda^\delta \xi), \quad a_{1,\Lambda}(x, \xi) = a_{1,\nu}(\Lambda^{-\delta} x, \Lambda^\delta \xi).$$

Since $\kappa \leq 5/6$, we see $a_{2,\Lambda} \in \mathcal{T}_{\rho,\delta}^{1+\rho}$ and $a_{1,\Lambda} \in \mathcal{T}_{\rho,\delta}^{\rho}$. Set

$$u_\Lambda = \Lambda^{-\delta n/2} M_{\Lambda^{-\delta}} \varphi_\nu^w u, \quad f_\Lambda = \Lambda^{-\delta n/2} M_{\Lambda^{-\delta}} \varphi_\nu^w f.$$

Since $\|u_\Lambda\|^2 = \|\varphi_\nu^w u\|^2$, $\|f_\Lambda\|^2 = \|\varphi_\nu^w f\|^2$ and $\sum_{\nu=0}^{\infty} (\varphi_\nu^w)^2 = I_d$, we note that

$$\sum_{\nu=0}^{\infty} \|u_{\Lambda(\nu)}(t)\|^2 = \|u(t)\|^2, \quad \sum_{\nu=0}^{\infty} \|f_{\Lambda(\nu)}(t)\|^2 = \|f(t)\|^2, \tag{3.16}$$

$$\left( \partial_t + i a_{2,\Lambda}^w + a_{1,\Lambda}^w \right) u_\Lambda + \Lambda^{-\delta n/2} M_{\Lambda^{-\delta}} r_\nu^w u = f_\Lambda. \tag{3.17}$$

Apply Corollary 2.4 with $l = 4$ to $a_{2,\Lambda}^w$, noting that its symbol depends only on $\xi$. Furthermore, using Theorem 2.3 for $a_{1,\Lambda}^w$, we can see that (3.17) are reduced to the equations

$$\left( \partial_t + i a_{2,\Lambda}^{\mathrm{Wick}} + \widetilde{a_{1,\Lambda}}^{\mathrm{Wick}} \right) u_\Lambda + \Lambda^{-\delta n/2} M_{\Lambda^{-\delta}} r_\nu^w u + \widetilde{a_{0,\Lambda}}^{w} u_\Lambda = f_\Lambda, \tag{3.18}$$

where $a_{2,\Lambda}^{\mathrm{Wick}} = a_{2,\Lambda}^{\mathrm{Wick}}(D_x)$ and $\widetilde{a_{1,\Lambda}}^{\mathrm{Wick}} = \widetilde{a_{1,\Lambda}}^{\mathrm{Wick}}(x, D_x)$ with

$$\widetilde{a_{1,\Lambda}}(x, \xi) = a_{1,\Lambda}(x, \xi) - \frac{i}{4} \left( \Delta_\xi a_{2,\Lambda} \right)(\xi) \quad \in \mathcal{T}_{\rho,\delta}^{\rho} \quad , \tag{3.19}$$

and $\widetilde{a_{0,\Lambda}} \in \mathcal{T}_{\rho,\delta}^{0}$. Set

$$r_\Lambda u = \Lambda^{-\delta n/2} M_{\Lambda^{-\delta}} r_\nu^w u + \widetilde{a_{0,\Lambda}}^{w} u_\Lambda.$$

From (3.18) we have

$$(\partial_t + i a_{2,\Lambda}^{\mathrm{Wick}} + \widetilde{a_{1,\Lambda}}^{\mathrm{Wick}}) u_\Lambda + r_\Lambda u = f_\Lambda. \tag{3.20}$$

Since $\|\Lambda^{-\delta n/2} M_{\Lambda^{-\delta}} r_\nu^w u\|^2 = \|r_\nu^w u\|^2$ and $\widetilde{a_{0,\Lambda}} \in \mathcal{T}^0_{\rho,\delta}$, it follows from (3.12) and (3.16) that

$$\sum_{\nu=0}^{\infty} \|r_{\Lambda(\nu)} u(t)\|^2 \le C_1 \|u(t)\|^2 \tag{3.21}$$

with some constant $C_1 > 0$.

**Lemma 3.4.** *Assume* (3.1) *and* (3.6). *Then for any* $M > 0$ *there exists* $p_\Lambda(x,\xi) \in C^\infty(\boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi)$ *of real value with a large parameter* $\Lambda$ *such that we have*

$$\{a_{2,\Lambda}, p_\Lambda\}(x,\xi) - \operatorname{Re} a_{1,\Lambda}(x,\xi) = 0 \tag{3.22}$$

*and moreover*

$$|\partial_x^\alpha p_\Lambda(x,\xi)| \le C\Lambda^{-\delta|\alpha|} \quad \text{if } |\alpha| \le 1, \tag{3.23}$$

$$|\partial_\xi^\alpha p_\Lambda(x,\xi)| \le C' \langle \Lambda^{-\delta} x\rangle \Lambda^{-\rho} \quad \text{if } |\alpha| = 1 \tag{3.24}$$

*for* $x,\ \xi \in \boldsymbol{R}^n$ *with some constants* $C,\ C' > 0$.

Proof. Set

$$p(x,\xi) = \sum_{j=1}^{n} \omega_j \int_0^{x\cdot\omega} \operatorname{Re} b_j(x-\omega\theta)d\theta, \quad \omega = \frac{\xi}{|\xi|}. \tag{3.25}$$

Then we get

$$\{a_2, p\}(x,\xi) = \operatorname{Re} a_1(x,\xi).$$

Setting $p_\Lambda(x,\xi) = p(\Lambda^{-\delta}x, \Lambda^\delta \xi)\psi_1(\Lambda^{-\rho}\xi)$ we obtain (3.22) in view of (3.10). The estimates (3.23) and (3.24) easily follow from (3.1) and (3.6). □

Let's return to the proof of Theorem 3.2. Set $K_\Lambda(x,\xi) = e^{p_\Lambda(x,\xi)}$ with $p_\Lambda(x,\xi)$ of Lemma 3.4. Then we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|K_\Lambda^{\mathrm{Wick}} u_\Lambda(t)\|^2 \\ &= \operatorname{Re}\left(K_\Lambda^{\mathrm{Wick}} \frac{\partial u_\Lambda}{\partial t}, K_\Lambda^{\mathrm{Wick}} u_\Lambda\right) \\ &= \operatorname{Re}\left(K_\Lambda^{\mathrm{Wick}} f_\Lambda, K_\Lambda^{\mathrm{Wick}} u_\Lambda\right) \\ &\quad - \operatorname{Re}\left(\left(i\left[K_\Lambda^{\mathrm{Wick}}, a_{2,\Lambda}^{\mathrm{Wick}}\right] + K_\Lambda^{\mathrm{Wick}} \tilde{a}_{1,\Lambda}^{\mathrm{Wick}}\right) u_\Lambda, K_\Lambda^{\mathrm{Wick}} u_\Lambda\right) \\ &\quad - \operatorname{Re}\left(K_\Lambda^{\mathrm{Wick}} r_\Lambda u, K_\Lambda^{\mathrm{Wick}} u_\Lambda\right)\end{aligned} \tag{3.26}$$

because $(a_{2,\Lambda}^{\rm Wick})^* = a_{2,\Lambda}^{\rm Wick}$, which follows from the fact that $a_{2,\Lambda}(\xi)$ is a real valued function. Here $(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\boldsymbol{R}^n)}$. By means of (3.23) with $\alpha = 0$ and Proposition 1.1 we have the estimate

$$C_2 \le \|K_\Lambda^{\rm Wick}\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))} \le C_3 \quad \text{for } \Lambda \ge 1 \tag{3.27}$$

with some constants $C_2,\ C_3 > 0$. Note that

$$K_\Lambda^{\rm Wick}\tilde{a}_{1,\Lambda}^{\rm Wick} = K_\Lambda^{\rm Wick}({\rm Re}\,a_{1,\Lambda})^{\rm Wick} + i({\rm Im}\,\tilde{a}_{1,\Lambda})^{\rm Wick}K_\Lambda^{\rm Wick} + i[K_\Lambda^{\rm Wick}, ({\rm Im}\,\tilde{a}_{1,\Lambda})^{\rm Wick}].$$

Using the expansion formula (2.24) with $a = K_\Lambda$, $b = {\rm Re}\,a_{1,\Lambda}$, $N = 2$ and $k_0 = 0$, we obtain

$$K_\Lambda^{\rm Wick}({\rm Re}\,a_{1,\Lambda})^{\rm Wick} = \big(K_\Lambda({\rm Re}\,a_{1,\Lambda})\big)^{\rm Wick} + R_\Lambda. \tag{3.28}$$

where $\|R_\Lambda\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))} \le C_4$ for $\Lambda \ge 1$ with some constant $C_4 > 0$. In fact, by means of (2.28) we have for $\alpha = (\alpha_x, \alpha_\xi)$ with $\alpha_\xi \neq 0$

$$\|\Omega^\alpha_{\beta',\beta''}\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))} \le C\|K_\Lambda({\rm Re}\,a_{1,\Lambda})^{(\alpha)}\|_{L^\infty} \le C'$$

because ${\rm Re}\,a_{1,\Lambda} \in \mathcal{T}^\rho_{\rho,\delta}$, and it follows from (3.23) and (3.7) that for $\alpha = (\alpha_x, 0)$ and $\beta' \neq 0$

$$\|\Omega^\alpha_{\beta',\beta''}\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))} \le C(\|(\partial_\xi K_\Lambda)\langle\Lambda^{-\delta}x\rangle^{-1}\Lambda^{\rho-\delta}\|_{L^\infty} + \|K_\Lambda\Lambda^{\rho-1}\|_{L^\infty} \le C'.$$

In view of (2.28) we see that for $\alpha = (\alpha_x, 0)$ and $\beta' = 0$ $(\beta'' \neq 0)$

$$\|\Omega^\alpha_{\beta',\beta''}\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))} = \|\partial_x^{\beta''}(K_\Lambda({\rm Re}\,a_{1,\Lambda})^{(\alpha)}))\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))} \le C',$$

because $\partial_x^{\beta''}(K_\Lambda({\rm Re}\,a_{1,\Lambda})^{(\alpha)})$ also belongs to $L^\infty$. Using the expansion formula (2.29) with $N = 2$ we get similarly

$$\|[K_\Lambda^{\rm Wick}, ({\rm Im}\,\tilde{a}_{1,\Lambda})^{\rm Wick}]\|_{\mathcal{L}(L^2(\boldsymbol{R}^n))} \le C'. \tag{3.29}$$

If we use (2.30) of Theorem 2.12 with $a = K_\Lambda$, $b = a_{2,\Lambda}$ and $N = 3$ then the remainder term $R_3$ is $L^2$ bounded uniformly with respect to $\Lambda$ on account of (3.23) with $|\alpha| = 1$. In view of (3.22), it follows from (3.28) and (3.29) that

$$-\,{\rm Re}\left(\left(i\left[K_\Lambda^{\rm Wick}, a_{2,\Lambda}^{\rm Wick}\right] + K_\Lambda^{\rm Wick}\tilde{a}_{1,\Lambda}^{\rm Wick}\right)u_\Lambda, K_\Lambda^{\rm Wick}u_\Lambda\right) \le C_5\|u_\Lambda\|^2 \tag{3.30}$$

where $C_5 > 0$ is a constant independent of $\Lambda \ge 1$. In view of (3.27) and (3.30) it follows from (3.26) that

$$\frac{1}{2}\frac{d}{dt}\|K_\Lambda^{\rm Wick}u_\Lambda(t)\|^2 \tag{3.31}$$

$$\leq \|K_\Lambda^{\text{Wick}} u_\Lambda(t)\| \left( \|K_\Lambda^{\text{Wick}} f_\Lambda(t)\| + \frac{C_5}{C_2} \|u_\Lambda(t)\| + C_6 \|r_\Lambda u(t)\| \right)$$

with some constant $C_6 > 0$. Set

$$U(t) = \left( \sum_{\nu=0}^{\infty} \|K_{\Lambda(\nu)}^{\text{Wick}} u_{\Lambda(\nu)}(t)\|^2 \right)^{1/2}, \quad F(t) = \left( \sum_{\nu=0}^{\infty} \|K_{\Lambda(\nu)}^{\text{Wick}} f_{\Lambda(\nu)}(t)\|^2 \right)^{1/2}$$

for $t \in [0, T]$. By virtue of (3.16) and (3.27) we get

$$C_7\|u(t)\| \leq U(t) \leq C_8\|u(t)\|, \quad C_7\|f(t)\| \leq F(t) \leq C_8\|f(t)\| \tag{3.32}$$

for $t \in [0, T]$ with some constants $C_7$, $C_8 > 0$. Sum up (3.31) with respect to $\nu$ and make use of Schwarz's inequality. In view of (3.15), (3.16), (3.21) and (3.32) we have the estimate

$$\frac{d}{dt} U(t) \leq C_9 U(t) + F(t) \quad \text{for } t \in [0, T] \tag{3.33}$$

with some constant $C_9 > 0$. Consequently we can obtain the desired *a priori* estimate (3.8) from (3.33) by using the Gronwall's inequality and noting (3.32).

We shall prove the assersion (i) of the theorem. Set $\delta = 1/3$ and $\rho = 1 - \delta$. Since $\kappa \leq 2/3$, we see $a_{2,\Lambda} \in \mathcal{T}_{\rho,\delta}^{2\rho}$ and $\tilde{a}_{1,\Lambda} \in \mathcal{T}_{\rho,\delta}^{\delta}$. We use (2.23) and (2.24) with $N = 1$ in estimating the product of $K_\Lambda^{\text{Wick}}$ and $a_{1,\Lambda}^{\text{Wick}}$. Furtheremore, we use them to get (3.29). About the commutator of $K_\Lambda^{\text{Wick}}$ and $a_{2,\Lambda}^{\text{Wick}}$ we may only use (2.29) with $N = 2$. Hence we get the conclusion without (3.6) and (3.7). □

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