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ON MARKOV CHAINS INDUCED FROM STOCK PROCESSES HAVING BARRIERS IN FINANCE MARKET

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1. Introduction

In Taiwan's market, there are lower and upper bounds on every day's stock price. The lower bound of today's stock price is defined by 93 % of the final price of yesterday's stock. And the upper bound of today's stock price is defined by 107 % of the final price of yesterday's stock. Under this background, we are interested in the effect of the lower bound and upper bound that cause every day's stock price in a long term. In words, what is the influence of the bounds on every day's stock price?

On the other hand, by the empirical studies [1] [2] [6], if the distribution of financial time series such as stocks returns are compared with the normal distribution, then fatter tails are observed. Besides, the standardized fourth moment for a normal distribution is 3 whereas for many financial time series a value well above 3 is observed by Mandelbrot [6]. Many other researchers [3] [4] [5] [7] [8] also report this feature and adopt the model with fat tail property to research financial problems. However, those researches above did not give the definite reasons of leading fat tails. Thus verifying the reason theoretically is a difficult problem but is a crucial research.

In order to research those problems in Taiwan's market, we use some kinds of difussion processes $\{S_t\}_{t\geq 0}$ to drive the price of the stock. Furthermore we suppose that the stock price must be stopped at the bounds until the end of that day when the process hits the bounds. From this restriction to diffusions, we get a discrete Markov chain $\{X_n\}_{n\geq 0}$ in $(0,\infty)$. The rigorous definition of the Markov chain is given in the following section.

Due to the motivation above, we attempt to probe the relationship between the bounds (lower and upper) and the asymptotic behavior of $\{X_n\}_{n\geq 0}$. And if the invariant probability measure $\mu(\cdot)$ of $\{X_n\}_{n\geq 0}$ exists, we are interested in the tail of $\mu(\cdot)$.

Therefore, the purpose of this paper is to research the (positive) recurrence and transience of $\{X_n\}_{n\geq 0}$. Also we compare the tail of the invariant probability measure of $\{X_n\}_{n\geq 0}$ with $\{S_t\}_{t\geq 0}$. Our results imply that if $\{S_t\}_{t\geq 0}$ is recurrent and the bounds satisfy some conditions, then the effect of lower and upper bounds gives a phenomenon of fat tails. Indeed, in other countries, the governments also give a restriction on stock processes when stock market falls down. But the restriction is not so clear as Taiwan's market. Also the restriction is sometimes ambiguous and is difficult to describe faithfully. Hence financial researchers often ignored the effect of the restriction in their visionary mathematical model. Is this the one of the reasons of leading the fat tails in empirical studies? In our framework of modelling Taiwan's market, we verify theoretically that the effect of the lower and upper bounds is the one of the reasons to give the fat tail property. The details of our results for fat tails will be presented below.

Since the process $\{X_n\}_{n\geq 0}$ is obtained by time change of $\{S_t\}_{t\geq 0}$ and the speed of $\{S_t\}_{t\geq 0}$ slows down, intuition says that the degree of recurrence of $\{X_n\}_{n\geq 0}$ decreases compared with $\{S_t\}_{t\geq 0}$. In conclusion, we can prove that if $\{S_t\}_{t\geq 0}$ is transient, then $\{X_n\}_{n\geq 0}$ is transient, too. This means that the barriers have no effect at all to help the default stock process not to default in the long term. Also if $\{S_t\}_{t\geq 0}$ is recurrent, then $\{X_n\}_{n\geq 0}$ is recurrent, provided that $\{\rho^{\pm}(x)\}$ satisfies some weak conditions. Moreover, we show that there exists $\{\rho^{\pm}(x)\}$ such that $\{X_n\}_{n\geq 0}$ is null recurrent even though $\{S_t\}_{t\geq 0}$ is positive recurrent. As for the fat tail, we obtain the following results. Here, for simplicity, we consider the diffusion process $\{S_t\}_{t\geq 0}$ in nature scale (see Sect. 3).

1. if $\int_{\mathbf{R}} |x| m(dx) < \infty$ and $\{X_n\}_{n\geq 0}$ is positive recurrent, then the tail of $\mu(\cdot)$ is fatter than $m(\cdot)$, that is,

$$\int_0^\infty x\mu(dx) = \int_{-\infty}^0 |x|\mu(dx) = \infty.$$

2. assume that

$$c_1|x|^{-\alpha} \leq m(x) \leq c_2|x|^{-\alpha}, \text{ for any } |x| \geq M,$$

and $\{\rho^{\pm}(x)\}$ satisfies $\rho^+(x) \geq x + c^+|x|^s$, $\rho^-(x) \leq x - c^-|x|^t$ whenever $|x| \geq M$, where $\alpha, c_1, c_2, M, c^{\pm}, s, t$ are all positive constants. If $s \wedge t \in (0, 1 \wedge \alpha/2)$, then the tail of $\mu(\cdot)$ is fatter than $m(\cdot)$, that is, for any $\gamma \in (2(s \wedge t) - 1, \alpha - 1)$

$$\int_0^\infty x^\gamma \mu(dx) = \int_{-\infty}^0 |x|^\gamma \mu(dx) = \infty, \quad \int_{\mathbf{R}} |x|^\gamma m(dx) < \infty.$$

The content of this paper is organized as follows. In Sect. 2 we introduce some definitions and the setting which we need later. In Sect. 3 we present the main Theorems and some remarks. In Sect. 4 we give the proofs of the main Theorems.

2. The setting and definitions

Throughout this paper, assume that $\rho^+(x), \rho^-(x)$ are both not dependent on time, continuous and $0 < \rho^-(x) < x < \rho^+(x) < \infty$ for any $x \in (0, \infty)$. We use $\rho^+(x)$ (resp. $\rho^-(x)$) to denote the upper (resp. lower) bound at the state x. Suppose that $\{S_t\}_{t\geq 0}$ is a time homogeneous diffusion on $(0,\infty)$, and is generated by a generator

$$L \equiv \frac{1}{2}\sigma(x)^2\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x},$$

which $\sigma(x)$ and $b(x)$ are continuous and $\sigma(x) > 0$ for all $x \in (0,\infty)$. Here notice that we don't give any boundary conditions to $\{S_t\}_{t\geq 0}$, since the boundary conditions are irrelevant to the definition of our Markov chain below. Denote $s(x)$ by

$$s(x) \equiv \int_{x_0}^{x} e^{-I(y,x_0)}dy, \text{ where } I(y,x_0) \equiv \int_{x_0}^{y}\frac{2b(z)}{\sigma(z)^2}dz.$$

This $s(x)$ is called 'scale function' of the diffusion $\{S_t\}_{t\geq 0}$ and satisfies $Ls(x) = 0$ and is a strictly increasing function. Formally, the generator L takes the following simple form;

$$L \equiv \frac{d}{d\Psi(x)}\frac{d}{ds(x)},$$

where

$$\Psi(x) \equiv \int_{x_0}^{x}\frac{2}{\sigma(y)^2}e^{I(y,x_0)}dy.$$

Now we construct the time homogeneous Markov chain $\{X_n\}_{n\geq 0}$ on $(0,\infty)$ as follows.

1. $X_0 \equiv S_0 \equiv x$ and $X_1 \equiv S_{1\wedge\tau}$, where

$$\tau^{\pm} \equiv \inf\{t \geq 0 : S_t = \rho^{\pm}(x)\}, \quad \tau \equiv \tau^{+}\wedge\tau^{-}.$$

2. $\{X_n\}_{n=0}^{\infty}$ has a stationary transition probability

$$p(x,dy) \equiv P_x(S_{1\wedge\tau} \in dy) = p_c(x,dy) + p_d(x,dy),$$

where

$$\begin{aligned} p_c(x,dy) &\equiv P_x(S_1 \in dy, \tau > 1), \\ p_d(x,dy) &\equiv p^{+}(x)\delta_{\{\rho^{+}(x)\}}(dy) + p^{-}(x)\delta_{\{\rho^{-}(x)\}}(dy), \\ p^{\pm}(x) &\equiv P_x(\tau^{\pm} \leq \tau^{\mp}, \tau^{\pm} \leq 1). \end{aligned}$$

We call the above $\{X_n\}_{n\geq 0}$ "Markov chain induced from $\{S_t\}_{t\geq 0}$ and the barrier $\{\rho^{\pm}(x)\}$". From the assumptions that $\rho^{+}(x), \rho^{-}(x), \sigma(x), b(x)$ are all continuous, we see that p_c has a positive continuous kernel $p_c(x,y)$ on $(0,\infty)\times(\rho^{-}(x),\rho^{+}(x))$ and $p^{\pm}$ are continuous.

Since the state space of $\{X_n\}_{n\geq 0}$ is $(0,\infty)$, it is not trivial to define the irreducibility for the Markov chain. We follow the idea of Revuz [9]. For $c \in (0,1)$, let $U_c(x,\cdot)$ be the resolvent kernel of $\{X_n\}_{n\geq 0}$, that is,

$$U_c(x,A) \equiv \sum_{n=0}^{\infty}(1-c)^n P_x(X_{n+1} \in A),\ A \in \mathbf{B}((0,\infty)),\ x \in (0,\infty).$$

Suppose $\nu(\cdot)$ is a Radon measure on $(0,\infty)$. Then $\{X_n\}_{n\geq 0}$ is called *ν-irreducible* if $\nu(\cdot)$ is absolutely continuous with respect to $U_c(x,\cdot)$ for all $x \in \mathbf{R}$ and $c \in (0,1)$. In our case, we can take $\nu(\cdot)$ as Lebesgue measure. Now the recurrence is defined as follows.

DEFINITION 2.1. A *ν-irreducible* Markov chain $\{X_n\}_{n\geq 0}$ is called *recurrent in the sense of Harris* if and only if, there exists a σ-finite invariant measure $\mu(\cdot)$ such that $\mu(A) > 0$, $A \in \mathbf{B}((0,\infty))$ implies

$$P_x\left(\sum_{n=1}^{\infty}\chi_A(X_n) = \infty\right) = 1, \tag{1}$$

for all $x \in (0,\infty)$.

REMARK. In this framework, an invariant measure $\mu(\cdot)$ is unique under which $\nu(\cdot)$ is absolutely continuous with respect to $\mu(\cdot)$, provided with the irreducibility.

Under this definition, it is trivial that if $\mu(A) > 0$, then

$$P_a(\tau_A < \infty) = 1, \text{ for any } a \in (0,\infty),\ A \in \mathbf{B}((0,\infty)), \tag{2}$$

where $\tau_A \equiv \inf\{n > 0 : X_n \in A\}$. In general, it is not true that (2) implies (1).

The definition of positive recurrence of $\{X_n\}_{n\geq 0}$ is given by

DEFINITION 2.2. A recurrent Markov chain $\{X_n\}_{n\geq 0}$ in the sense of Harris is called *positive recurrent* if and only if the invariant measure $\mu(\cdot)$ is a probability measure. Otherwise $\{X_n\}_{n\geq 0}$ is said to be *null recurrent*.

Another definition of positive recurrence for X_n can be given as $E_x\tau_A < \infty$ for any $x \in (0,\infty)$ and any open subset A of $(0,\infty)$. In general, the necessary and sufficient condition was given by Meyn and Tweedie [10]. From Theorem 4.1 of [10], A Markov chain $\{X_n\}_{n\geq 0}$ is positive recurrent if and only if a petite set A exists with $P_x(\tau_A < \infty) = 1$ for all $x \in (0,\infty)$ and $\sup_{x\in A} E_x\tau_A < \infty$. Moreover, if A is a petite set, then $\mu(A) < \infty$.

REMARK. Meyn and Tweedie also showed; For a ν-*irreducible* Markov chain $\{X_n\}_{n\geq 0}$ with Feller property which means $Pf(x) \equiv E_x f(X_1)$ is continuous for every bounded continuous function $f(x)$, if the support of ν has non-empty interior, then every compact set of $(0,\infty)$ is petite. Since our Markov chain $\{X_n\}_{n\geq 0}$ posseses Feller property and the support of ν is $(0,\infty)$, this gives that every closed bounded interval is petite.

Finally, we set

$$\gamma_+(\mu) \equiv \sup\left\{\gamma : \int_1^\infty x^\gamma \mu(dx) < \infty\right\},$$
$$\gamma_-(\mu) \equiv \inf\left\{\gamma : \int_0^1 x^\gamma \mu(dx) < \infty\right\},$$

and introduce

DEFINITION 2.3. The recurrent Markov chain $\{X_n\}_{n\geq 0}$ induced from a recurrent time homogeneous diffusion $\{S_t\}_{t\geq 0}$ and the barrier $\{\rho^\pm(x)\}$ is called to have *a fat tail at* $x = \infty$ if $\gamma_+(\mu) < \gamma_+(\Psi)$. Analogously it is called *a fat tail at* $x = 0$ if $\gamma_-(\mu) > \gamma_-(\Psi)$.

REMARK. In the empirical studies [1] [2] [6], the fat tail property is argued by the positive recurrent process. In this paper, we extend the argument to including the null recurrent process.

For simplicity, we say $\{X_n\}_{n\geq 0}$ is 'recurrent' instead of 'recurrent in the sense of Harris' and is 'positive recurrent' instead of 'positive recurrent in the sense of Harris' in the sequel. Notice that the definition of the recurrence of $\{S_t\}_{t\geq 0}$ is different from the definition of the recurrence of $\{X_n\}_{n\geq 0}$.

3. The main Theorems

Our main results are the followings.

Theorem 3.1. 1. *If* $-\infty < s(0), s(\infty) = \infty$, *then*

$$P_a\left(\lim_{n\to\infty} X_n = 0\right) = 1, \text{ for any } a \in (0,\infty).$$

2. *If* $-\infty = s(0), s(\infty) < \infty$, *then*

$$P_a\left(\lim_{n\to\infty} X_n = \infty\right) = 1, \text{ for any } a \in (0,\infty).$$

3. *If* $-\infty < s(0)$, $s(\infty) < \infty$, *then*

$$0 < q_a(0) \equiv P_a\left(\lim_{n\to\infty} X_n = 0\right) < 1, \quad 0 < q_a(\infty) \equiv P_a\left(\lim_{n\to\infty} X_n = \infty\right) < 1,$$

where

$$q_a(0) = \frac{s(\infty) - s(a)}{s(\infty) - s(0)}, \quad q_a(\infty) = \frac{s(a) - s(0)}{s(\infty) - s(0)}, \quad \textit{for any } a \in (0,\infty).$$

REMARK. This result shows that if $\{S_t\}_{t\geq 0}$ is transient, that is, $s(0) > -\infty$ or $s(\infty) < \infty$, then $\{X_n\}_{n\geq 0}$ and $\{S_t\}_{t\geq 0}$ have the same asymptotic behavior as n and t tending to infinity. It also shows that if $0, \infty$ are the regular boundaries, that is, $s(0) > -\infty$ and $s(\infty) < \infty$, $\Psi(0) > -\infty$ and $\Psi(\infty) < \infty$, then X_n converges to the boundaries.

On the other hand, if $\{S_t\}_{t\geq 0}$ is recurrent, namely

$$\lim_{x\to 0} s(x) = -\infty \text{ and } \lim_{x\to\infty} s(x) = \infty,$$

then we get the following theorem.

Theorem 3.2. *Suppose that* $\{S_t\}_{t\geq 0}$ *is recurrent. If* $\{\rho^{\pm}(x)\}$ *satisfies*

$$\limsup_{x\to 0} \rho^{+}(x) < \liminf_{x\to\infty} \rho^{-}(x), \tag{1}$$

then $\{X_n\}_{n\geq 0}$ *is recurrent.*

REMARK. 1. It is trivial that if $\rho^{+}(x) \equiv \infty$, $\rho^{-}(x) \equiv 0$ for all $x \in (0,\infty)$, then $X_n \equiv S_n$. This means that $\{X_n\}_{n\geq 0}$ is recurrent. But here note that $\rho^{+}(x) \equiv \infty$, $\rho^{-}(x) \equiv 0$ for all $x \in (0,\infty)$ do not satisfy the conditions of Theorem 3.2, which implies the conditions of Theorem 3.2 to be not always necessary.

2. As we mentioned in the introduction, $\{X_n\}_{n\geq 0}$ is obtained intuitively by time change of $\{S_t\}_{t\geq 0}$. Thus we conjecture that it is impossible to get the transient $\{X_n\}_{n\geq 0}$ when $\{S_t\}_{t\geq 0}$ is recurrent.

Intuitively, it may be conjectured that $\{X_n\}_{n\geq 0}$ will be positive recurrent when $\{S_t\}_{t\geq 0}$ is positive recurrent for any $\{\rho^{\pm}(x)\}$. The answer is negative. To see this easily, we use the transformation of changing $s(x)$ to x and deform the generator L of $\{S_t\}_{t\geq 0}$ into

$$L = \frac{\partial^2}{m(x)\partial x^2}.$$

Notice there is no loss of generality when we consider the diffusion $\{S_t\}_{t\geq 0}$ in natural scale in the sequel, that is, $s(x) \equiv x$.

Before see the answer of the conjecture above, set the constants below

$$\alpha,\ c,\ c_1,\ c_2,\ c^{\pm},\ d_{\pm},\ e_{\pm},\ s,\ t,\ M,$$

are all positive henceforth. Suppose that

$$c_1|x|^{-\alpha} \le m(x) \le c_2|x|^{-\alpha},\ \text{ for } |x| \ge M, \tag{2}$$

and $g_+(x) \equiv \rho^+(x) - x$; $g_-(x) \equiv x - \rho^-(x)$. Now $\{X_n\}_{n\ge0}$ is distinguished from the null recurrence or positive recurrence by the magnitude of $\{\rho^{\pm}(x)\}$ as follows.

Theorem 3.3. 1. *Assume that* $\{\rho^{\pm}(x)\}$ *satisfies the conditions of* Theorem 3.2. *If there exists* $s \in (0, 1/2)$ *such that one of the following conditions holds*;

$$\limsup_{x\to\infty} x^{-s} g_+(x) = 0\ ,\quad \limsup_{x\to\infty} x^{-s} g_-(x) = 0,$$
$$\limsup_{x\to-\infty} |x|^{-s} g_+(x) = 0\ ,\quad \limsup_{x\to-\infty} |x|^{-s} g_-(x) = 0,$$

then $\{X_n\}_{n\ge0}$ *is null recurrent.*
2. *Assume* (2) *holds for* $\alpha > 1$. *If there exists* $s \in (1/2, \infty)$ *such that*

$$\liminf_{x\to\infty} x^{-s} g_{\pm}(x) > 0, \quad \liminf_{x\to-\infty} |x|^{-s} g_{\pm}(x) > 0,$$

then $\{X_n\}_{n\ge0}$ *is positive recurrent.*

REMARK. 1. Assume that $\{S_t\}_{t\ge0}$ is recurrent. If $\{\rho^{\pm}(x)\}$ satisfies the conditions of Theorem 3.2 and

$$\limsup_{x\to\infty} |\sigma(x)s^{'}(x)| < \infty,\ \text{ or } \limsup_{x\to0} |\sigma(x)s^{'}(x)| < \infty,$$

then $\{X_n\}_{n\ge0}$ is null recurrent. This implies that if $\{X_n\}_{n\ge0}$ is induced from Black-Scholes model taking $dS_t = S_t dB_t$, then $\{X_n\}_{n\ge0}$ is null recurrent. We omit the proof because it is similar to Theorem 3.3.
2. We conjecture that $\{X_n\}_{n\ge0}$ is positive recurrent only if $\{S_t\}_{t\ge0}$ is positive recurrent.

About checking the fat tail, we have the following Theorems.

Theorem 3.4. *If* $\{X_n\}_{n\ge0}$ *is positive recurrent with the unique invariant probability measure* $\mu(\cdot)$, *then for any fixed* $\{\rho^{\pm}(x)\}$

$$\int_0^{\infty} x\mu(dx) = \int_{-\infty}^0 |x|\mu(dx) = \infty,$$

hence

$$\gamma_+(\mu) \le 1, \quad \gamma_-(\mu) \le 1.$$

Theorem 3.5. *For $\alpha \ge 2$, assume $m(x) \le cx^{-\alpha}$ whenever $x \ge M$. Suppose that $\{X_n\}_{n\ge 0}$ is positive recurrent and $\{\rho^{\pm}(x)\}$ satisfies $\rho^+(x) \ge d_+x$, $\rho^-(x) \le d_-x$ whenever $x \ge M$. If*

$$0 < d_- < 1 < d_+,$$

then $\gamma_+(\mu) = 1$.

Theorem 3.6. *Assume that $c_1x^{-\alpha} \le m(x) \le c_2x^{-\alpha}$ whenever $x \ge M$ and $\{\rho^{\pm}(x)\}$ satisfies $\limsup_{x\to-\infty} \rho^+(x) < \liminf_{x\to\infty} \rho^-(x)$ such that $\rho^+(x) \ge x + c^+x^s$; $\rho^-(x) \le x - c^-x^t$ whenever $x \ge M$. If $s \wedge t \in (0, 1)$, then*

$$\gamma_+(\mu) = (2s \wedge 2t \wedge \alpha) - 1.$$

To sum up, we conclude the results as follows.

1. If $\{S_t\}_{t\ge 0}$ is transient, then $\{X_n\}_{n\ge 0}$ is transient, too. This means that if $\{S_t\}_{t\ge 0}$ is the default stock process, then the barriers have no effect at all to help the stock process not to default in the long term.
2. If $\{S_t\}_{t\ge 0}$ is recurrent and $\{\rho^{\pm}(x)\}$ satisfies the conditions of Theorem 3.2, then $\{X_n\}_{n\ge 0}$ is recurrent, too.
3. If $\{S_t\}_{t\ge 0}$ is positive recurrent, then $\{X_n\}_{n\ge 0}$ is null recurrent or positive recurrent depending on the magnitude of $\{\rho^{\pm}(x)\}$.
4. For $\alpha > 2$, assume that $m(x) \le c|x|^{-\alpha}$ whenever $|x| \ge M$ and $\{\rho^{\pm}(x)\}$ satisfies

$$\begin{aligned} \rho^+(x) &\ge d_+x\ , \quad \rho^-(x) \le d_-x, \quad \text{whenever } x \ge M,\\ \rho^+(x) &\ge e_+x\ , \quad \rho^-(x) \le e_-x, \quad \text{whenever } x \le -M. \end{aligned}$$

If $d_+, e_- \in (1, \infty)$, $d_-, e_+ \in (0, 1)$, then we obtain the fat tail at $\pm\infty$, Namely,

$$1 = \gamma_{\pm}(\mu) < \gamma_{\pm}(m) = \alpha - 1.$$

5. Assume that (2) holds and $\{\rho^{\pm}(x)\}$ satisfies

$$\rho^+(x) \ge x + c^+|x|^s, \rho^-(x) \le x - c^-|x|^t, \quad \text{whenever } |x| \ge M.$$

If $s \wedge t \in (0, 1 \wedge \alpha/2)$, then we obtain the fat tail at $\pm\infty$. Namely,

$$\gamma_{\pm}(\mu) = (2s \wedge 2t) - 1 < \gamma_{\pm}(m) = \alpha - 1.$$

Although the state space of $\{X_n\}_{n\geq 0}$ in results 4. 5. above are **R**, it is easy to obtain the original Markov chain $\{X_n\}_{n\geq 0}$ with the state space $(0,\infty)$ by transforming this X_n into $s^{-1}(X_n)$.

4. The proofs of main Theorems

In order to prove the main theorems, we need the following lemmas.

Lemma 4.1. *Suppose that $\{X_n\}_{n\geq 0}$ is a time homogeneous Markov chain with state space (c,d) and K is a fixed compact subset of (c,d). If there exists a positive number n_K such that*

$$\sup_{x\in K} p_{n_K}(x,K) \equiv \alpha_K < 1,$$

then $P_x(X_m \in K^c, i.o.) = 1$ for any $x \in (c,d)$.

Proof. It is easy to see that

$$\begin{aligned} P_a(X_n \in K, \forall n \geq m) &= E_a\{P_{X_m}(X_n \in K, \forall n \geq 0) : X_m \in K\} \\ &= \int_K P_y(X_n \in K, \forall n \geq 0) P_a(X_m \in dy) \\ &\leq \int_K P_y(X_{j\,n_K} \in K, \forall j \geq 0) P_a(X_m \in dy). \end{aligned}$$

The last part of the above calculation allows us to compute further. To do this calculation, we note that for any $r \in \mathbf{N}$,

$$\begin{aligned} &\int_K P_y(X_{j\,n_K} \in K, \forall j \geq 0) P_a(X_m \in dy) \\ &\leq \int_K P_y(X_{j\,n_K} \in K, \forall j \in [0,r]) P_a(X_m \in dy) \\ &= \int_K \left\{ \int_K \dots \int_K p_{n_K}(x, dz_1) p_{n_K}(z_1, dz_2) \dots p_{n_K}(z_{r-1}, dz_r) \right\} P_a(X_m \in dy) \\ &\leq \alpha_K^r P_a(X_m \in K). \end{aligned}$$

This shows

$$\int_K P_y(X_{j\,n_K} \in K, \forall j \geq 0) P_a(X_m \in dy) = 0.$$

Therefore

$$P_a\left(\bigcup_{m=1}^{\infty} \{X_n \in K, \forall n \geq m\} \right) = 0.$$

Now $P_x(X_m \in K^c, i.o.) = 1$ follows by taking complement on the left hand side of the above formula. This completes the proof. $\square$

Lemma 4.2. *Let $\{X_n\}_{n\geq 0}$ be recurrent and $\mu(\cdot)$ be its invariant measure. For $A \in \mathbf{B}(\mathbf{R})$ with $\mu(A) > 0$, set $\tau_A \equiv \{n > 0 : X_n \in A\}$. Then*

$$\int_{\mathbf{R}} g(x)\mu(dx) = \int_A E_x \left\{ \sum_{i=0}^{\tau_A - 1} g(X_i) \right\} \mu(dx), \tag{1}$$

where $g(x)$ is any given non-negative Borel function. In particular, if

$$\int_{\mathbf{R}} g(x)\mu(dx) < \infty; \quad 0 < \int_{A^c} g(x)\mu(dx), \tag{2}$$

then there exists $c \in A^c$ such that

$$E_c \left\{ \sum_{i=0}^{\tau_A - 1} g(X_i) \right\} < \infty. \tag{3}$$

Proof. Since (1) was proved by [11], we see only for the latter of the statement. Since $\int_{\mathbf{R}} g(x)\mu(dx) < \infty$, we have

$$\int_A E_x \left\{ \sum_{i=0}^{\tau_A - 1} g(X_i) \right\} \mu(dx) < \infty.$$

But since

$$\begin{aligned}\int_A E_x \left\{ \sum_{i=0}^{\tau_A - 1} g(X_i) \right\} \mu(dx) &= \int_A g(x)\mu(dx) + \int_A \mu(dx) \left\{ \int_{A^c} E_y \left\{ \sum_{i=0}^{\tau_A - 1} g(X_i) \right\} p(x, dy) \right\} \\ &= \int_A g(x)\mu(dx) + \int_{A^c} E_y \left\{ \sum_{i=0}^{\tau_A - 1} g(X_i) \right\} \sigma(dy)\end{aligned}$$

where $\sigma(dy) \equiv \int_A p(x, dy)\mu(dx)$. Therefore

$$\int_{A^c} g(x)\mu(dx) = \int_{A^c} E_y \left\{ \sum_{i=0}^{\tau_A - 1} g(X_i) \right\} \sigma(dy) < \infty. \tag{4}$$

Since (4) and (2), it implies $\sigma(A^c) > 0$ which shows (3). $\square$

Lemma 4.3. *Suppose that there exists a non-negative function $\psi(x)$ which is twice differentiable for $x > M$, $\psi(x) \equiv 0$ for all $x \leq M$ and*

$$\lim_{x\to\infty} \psi(x) = \infty, \quad L\psi(x) \leq C - \theta\psi(x), \quad \textit{for all } x > M,$$

where C, θ *are constants,* $\theta > 0$*. If* $\{\rho^{\pm}(x)\}$ *satisfies*

$$\limsup_{x\to\infty}\left\{\frac{\psi(\rho^+(x))}{\psi(x)}Q^+(1,x)+\frac{\psi(\rho^-(x))}{\psi(x)}Q^-(1,x)\right\}<1-e^{-\theta},$$

then there exist positive constants $T, \gamma \in (0,1)$ *such that*

$$E_x\psi(X_1) < T + \gamma\psi(x), \ \textit{for all } x \in \mathbf{R},$$

where

$$Q^{\pm}(1,x) \equiv E_x\{1-e^{-(1-\tau^{\pm})\theta} : \tau^{\pm} \le \tau^{\mp}, \tau^{\pm} \le 1\}.$$

Proof. Using Itô's formula, we get

$$\psi(S_t) = \psi(x) + M_t + \int_0^t (L\psi)(S_u)du,$$

where M_t is a martingale. Replacing t by $t \wedge \tau$ in the equality above, we obtain

$$\psi(S_{t\wedge\tau}) = \psi(x) + M_{t\wedge\tau} + \int_0^t (L\psi)(S_{u\wedge\tau})du - (t-\tau)\chi_{\{\tau\le t\}}(L\psi)(S_\tau).$$

Since $L\psi(x) \le C - \theta\psi(x)$, we get

$$\begin{aligned}\frac{\partial}{\partial t}E_x\psi(S_{t\wedge\tau}) &= E_x(L\psi)(S_{t\wedge\tau}) - E_x\chi_{\{\tau<t\}}(L\psi)(S_\tau)\\ &= E_xL\psi(S_t)\chi_{\{\tau\ge t\}}\\ &\le CP_x(\tau\ge t) - \theta E_x\psi(S_{t\wedge\tau})\chi_{\{\tau\ge t\}}\\ &= CP_x(\tau\ge t) - \theta E_x\psi(S_{t\wedge\tau}) + \theta E_x\psi(S_{t\wedge\tau})\chi_{\{\tau\le t\}}\\ &= C - \theta E_x\psi(S_{t\wedge\tau}) + E_x\{\theta\psi(S_\tau) - C\}\chi_{\{\tau\le t\}}, \nu - a.e.\, t.\end{aligned}$$

Integrating both sides with respect to t, this implies

$$\begin{aligned}E_x\psi(S_{t\wedge\tau}) &\le \frac{C(1-e^{-\theta t})}{\theta} + e^{-\theta t}\psi(x) + E_x\frac{1-e^{\theta(\tau-t)}}{\theta}\{\theta\psi(S_\tau)-C\}\chi_{\{\tau\le t\}}.\\ &= H + e^{-\theta t}\psi(x) + \psi(\rho^+(x))Q^+(t,x) + \psi(\rho^-(x))Q^-(t,x),\end{aligned}$$

where

$$H \equiv \frac{C}{\theta}\{(1-e^{-\theta t}) - Q^+(t,x) - Q^-(t,x)\}.$$

On the other hand, since

$$\limsup_{x\to\infty}\left\{\frac{\psi(\rho^+(x))}{\psi(x)}Q^+(1,x)+\frac{\psi(\rho^-(x))}{\psi(x)}Q^-(1,x)\right\}<1-e^{-\theta},$$

we see

$$E_x\psi(X_1) \le T + \gamma\psi(x), \text{ for } x \in \mathbf{R},$$

where $\gamma \in (0,1)$ and

$$T \equiv \sup_{x\in(-\infty,J]} \{E_x\psi(X_1) - \gamma\psi(x)\},$$

J is a constant greater than M. □

Lemma 4.4.

$$Q^+(1,x) \le (1-e^{-\theta})\phi^+(x), \quad Q^-(1,x) \le (1-e^{-\theta})\phi^-(x),$$

where

$$\phi^+(x) \equiv \frac{x-\rho^-(x)}{\rho^+(x)-\rho^-(x)}, \quad \phi^-(x) \equiv \frac{\rho^+(x)-x}{\rho^+(x)-\rho^-(x)}.$$

Proof. From the definition, it is obvious that

$$\begin{aligned} Q^{\pm}(1,x) &\le (1-e^{-\theta})P_x(\tau^{\pm} \le \tau^{\mp}, \tau^{\pm} \le 1) \\ &\le (1-e^{-\theta})P_x(\tau^{\pm} \le \tau^{\mp}) \\ &= (1-e^{-\theta})\phi^{\pm}(x). \end{aligned}$$ □

Lemma 4.5. *For a given* $\beta \in (0,1)$, *define*

(5) $$\psi(x) \equiv \begin{cases} x^\beta & \textit{for } x > M, \\ 0 & \textit{for } x \le M. \end{cases}$$

1. *If there exist two barriers* $\{\rho_1^{\pm}(x)\}$, $\{\rho_2^{\pm}(x)\}$ *such that*

$$\rho_1^+(x) \le \rho_2^+(x), \quad \rho_1^-(x) \ge \rho_2^-(x), \ \textit{whenever } x \ge M,$$

then for x *satisfying* $\rho_2^-(x) \ge M$,

$$E_x\{1\wedge\tau_1\} \le E_x\{1\wedge\tau_2\}, \quad E_x\psi(X_1^{(2)}) \le E_x\psi(X_1^{(1)}),$$

where $\{X_n^{(1)}\}_{n\ge0}$*(resp.* $\{X_n^{(2)}\}_{n\ge0}$*) is induced from* $\{S_t\}_{t\ge0}$, $\{\rho_1^{\pm}(x)\}$*(resp.* $\{\rho_2^{\pm}(x)\}$*) and*

$$\tau_1 \equiv \inf\{t \ge 0 : S_t = \rho_1^-(x) \textit{ or } \rho_1^+(x)\},$$

$$\tau_2 \equiv \inf\{t \geq 0 : S_t = \rho_2^-(x) \text{ or } \rho_2^+(x)\}.$$

2. *For any* $x \in \mathbf{R}$, *we have* $E_x\{1\wedge\tau\} \geq E_x\tau - E_x\tau^2$. *Moreover, assume that* $c_1x^{-\alpha} \leq m(x) \leq c_2x^{-\alpha}$ *whenever* $x \geq M$ *and* $\{\rho^{\pm}(x)\}$ *satisfies*

$$\rho^+(x) \equiv x + c^+x^s, \quad \rho^-(x) \equiv x - c^-x^t, \text{ whenever } x \geq M. \tag{6}$$

If $s, t \in (0, 1)$, *then*

$$c_1x^{2(s\wedge t)-\alpha} \leq E_x\tau \leq c_2x^{2(s\wedge t)-\alpha}, \quad c_1x^{4(s\wedge t)-2\alpha} \leq E_x\tau^2 \leq c_2x^{4(s\wedge t)-2\alpha},$$

whenever $x \geq F$, *where* $F > M$ *is a proper positive constant.*

3. *If* $m(x) \leq c_2x^{-\alpha}$ *whenever* $x \geq M$, *then for* x *satisfying* $\rho^-(x) \geq M$,

$$E_x\psi(X_1) \leq \psi(x) - \frac{\beta(1-\beta)}{c_2}\rho^-(x)^{\beta+\alpha-2}E_x\{1 \wedge \tau\}.$$

Proof. For x satisfying $\rho^-(x) \geq M$, since $P_x(\tau_1 \leq \tau_2) = 1$ and $\{\psi(S_{t\wedge\tau_2})\}_{t\geq 0}$ is a supermartingale with $S_0 \equiv X_0 \equiv x$, the statement 1 is trivial. To prove the statement 2, without loss of generality, we assume $s = t$. Since for $\tau \geq 0$, $1 \wedge \tau \geq \tau - \tau^2$, it is obvious $E_x\{1 \wedge \tau\} \geq E_x\tau - E_x\tau^2$. Further, since $s \in (0, 1)$, $c_1x^{-\alpha} \leq m(x) \leq c_2x^{-\alpha}$ whenever $x \geq M$ and

$$\begin{aligned} w(x) &\equiv E_x\tau \\ &= \frac{\rho^+(x) - x}{\rho^+(x) - \rho^-(x)}\int_{\rho^-(x)}^{x}(y - \rho^-(x))m(y)dy \\ &\quad + \frac{x - \rho^-(x)}{\rho^+(x) - \rho^-(x)}\int_{x}^{\rho^+(x)}(\rho^+(x) - y)m(y)dy, \end{aligned} \tag{7}$$

$$\begin{aligned} E_x\tau^2 &= \frac{2(\rho^+(x) - x)}{\rho^+(x) - \rho^-(x)}\int_{\rho^-(x)}^{x}(y - \rho^-(x))w(y)m(y)dy \\ &\quad + \frac{2(x - \rho^-(x))}{\rho^+(x) - \rho^-(x)}\int_{x}^{\rho^+(x)}(\rho^+(x) - y)w(y)m(y)dy, \end{aligned} \tag{8}$$

the statement 2 follows easily by substituting (6) into (7), (8). Finally, since $m(x) \leq c_2|x|^{-\alpha}$ whenever $x \geq M$, we get

$$E_x\psi(X_1) = E_x\psi(S_{1\wedge\tau}) = \psi(x) - \frac{\beta(1-\beta)}{c_2}E_x\int_0^{1\wedge\tau} S_u^{\beta+\alpha-2}du,$$

for x satisfying $\rho^-(x) \geq M$, which shows the statement 3. □

Proof of Theorem 3.1. Since $\{s(X_n)\}_{n\geq 0}$ is a martingale and $s(x)$ is an increasing function, by the Martingale Convergent Theorem and Lemma 4.1, the statements

1, 2 and 3 are trivial. Finally, by

$$E_a s(X_\infty) = s(a), \quad q_a(0) + q_a(\infty) = 1,$$

the last part of the statement 3 is clear. $\square$

Proof of Theorem 3.2. We divide the proof into two parts. The first part is to claim that there exists a bounded closed interval K such that $P_x(\tau_K < \infty) = 1$ for $x \in (0, \infty)$. To see this, we need to construct the non-negative function $\eta(x)$ from $s(x)$. Let $\alpha \equiv \limsup_{x\to 0} \rho^+(x)$, $\beta \equiv \liminf_{x\to\infty} \rho^-(x)$. Because of $\liminf_{x\to\infty} \rho^-(x) > \limsup_{x\to 0} \rho^+(x)$, we obtain that there exists an $\epsilon > 0$ such that $\beta - \epsilon > \alpha + \epsilon$. Set $F \equiv [\alpha + \epsilon, \beta - \epsilon]$. Further for this ϵ, there exist c_0 and d_0 such that

$$0 < \rho^+(x) < \alpha + \epsilon \text{ for } x \le c_0; \quad \beta - \epsilon < \rho^-(x) < \infty \quad \text{for } \forall x \ge d_0.$$

Take $K \equiv [c_0, d_0]$ and define

$$\eta(x) \equiv \begin{cases} s(x) & \text{if } x \in (\beta - \epsilon, \infty), \\ -s(x) & \text{if } x \in (0, \alpha + \epsilon), \\ 0 & \text{if } x \in F. \end{cases}$$

Since

$$\eta(S_{1\wedge\tau}) = \eta(x) + \int_0^{1\wedge\tau} L\eta(S_u)du + M_{1\wedge\tau}, \text{ whenever } x \in K^c,$$

where $\{M_{t\wedge\tau}\}_{t\ge 0}$ is a martingale with $M_0 \equiv 0$, $E_x\eta(X_1) = \eta(x)$ for any $x \in K^c$. Set $\tau_K \equiv \inf\{n \ge 0 : X_n \in K\}$ and $\tilde{X}_n \equiv X_{n\wedge\tau_K}$. It is evident that $E_x\eta(\tilde{X}_1) = \eta(x)$ for all $x \in (0, \infty)$. This gives that $\{\eta(\tilde{X}_n)\}_{n\ge 0}$ is a non-negative martingale. It turns out that there exists a random variable $Z < \infty$ such that $\lim_{n\to\infty} \eta(\tilde{X}_n) = Z$. Now suppose $P_a(\tau_K = \infty) > 0$, so we have for each n

$$X_n(\omega) = \tilde{X}_n(\omega), \text{ on } \{\tau_K = \infty\}.$$

Since it is easy to see that our $\{X_n\}_{n\ge 0}$ satisfies the condition of Lemma 4.1 for any compact subset K of $(0, \infty)$, we get

$$P_a(\{X_n\}_{n\ge 0} \text{ is unbounded}) = 1.$$

Since $\lim_{x\to\infty} \eta(x) = \infty$ and $\lim_{x\to 0} \eta(x) = \infty$, we obtain

$$P_a(\{\eta(X_n)\}_{n\ge 0} \text{ is unbounded}) = 1.$$

However

$$\lim_{n\to\infty} \eta(X_n) = Z < \infty, \text{ on } \{\tau_K = \infty\},$$

which is a contradiction. In consequence, $P_x(\tau_K < \infty) = 1$ for $x \in (0, \infty)$.

The second part of the proof is to claim that there exists a unique σ-finite invariant measure $\mu(\cdot)$. To show this, set $G \equiv I + U_0$, we apply the Revuz's result; for a ν-*irreducible* Markov chain $\{X_n\}_{n\geq 0}$, if there do not exist increasing sets $\{E_n\}_{n=0}^{\infty}$ with $\lim_{n\to\infty} E_n = (0, \infty)$ such that the functions $G(\cdot, E_n), n \geq 0$ are bounded, then there exists a unique σ-finite invariant measure $\mu(\cdot)$ such that $\nu(\cdot)$ is absolutely continuous with respect to $\mu(\cdot)$ and $\mu(A) > 0$, $A \in \mathbf{B}((0, \infty))$ implies

$$P_x\left(\sum_{n=1}^{\infty} \chi_A(X_n) = \infty\right) = 1,$$

for all $x \in (0, \infty)$. Thus we will claim that there do not exist increasing sets $\{E_n\}_{n=0}^{\infty}$ with $\lim_{n\to\infty} E_n = (0, \infty)$ such that the functions $G(\cdot, E_n), n \geq 0$ are bounded. To see this, it suffices to claim that there does not exist a Borel function $f(x)$, $0 < f(x) \leq 1$ such that $Gf(x) < \infty$ for all $x \in (0, \infty)$. We will show this by contradiction. Suppose that there exists a Borel function $f(x)$, $0 < f(x) \leq 1$ such that $Gf(x) < \infty$ for all $x \in (0, \infty)$. Since

$$P_x(X_1 \in dy) = p_c(x, dy) + p_d(x, dy),$$

we see

$$Gf(x) \geq G(Pf)(x) \geq G(p_c f)(x).$$

Let $\hat{f}(x) \equiv (p_c f)(x) = \int_0^{\infty} p_c(x, y) f(y) dy$. This implies that $\hat{f}(x)$ is a continuous function and $0 < \hat{f}(x) \leq 1$, $G\hat{f}(x) < \infty$ for all $x \in (0, \infty)$. Furthermore, we have $\hat{f}(x) \geq r\chi_K(x)$, where $r \equiv \inf_{x\in K} \hat{f}(x)$. On the other hand, let

$$\tau_K^{(k)} \equiv \inf\{n > \tau_K^{(k-1)} : X_n \in K\}, \tau_K^{(1)} \equiv \tau_K.$$

By the strong Markov property, we have

$$P_x(\tau_K^{(k)} < \infty) = 1, \text{ for any } k \geq 1.$$

Further it is not hard to see

$$\sum_{k=1}^{\infty} \chi_{\{\tau_K^{(k)} < \infty\}} = \sum_{m=0}^{\infty} \chi_{\{X_m \in K\}}.$$

This shows $\sum_{m=0}^{\infty} P_x(X_m \in K) = \infty$. Consequently, $G\chi_K(x) = \infty$, for all $x \in (0, \infty)$. But this gives $Gf(x) = \infty$, for all $x \in (0, \infty)$. This contradicts the assumption. Therefore the second part of the proof follows. □

Proof of Theorem 3.3. Because the argument of this proof is similar to Theorem 3.6, we give the concise proof of this theorem in the remark of Theorem 3.6. The outline of this proof is the following. For the first part, we claim that

$$\int_0^\infty x^\gamma \mu(dx) = \infty, \text{ for any } \gamma > 2s - 1$$

where $s < 1/2$. For the second part, we claim

1. there exist positive constants r and L with $\rho^-(L) > M$, $\rho^+(-L) < -M$ such that

$$E_x \hat{\psi}(X_1) \leq \hat{\psi}(x) - r, \text{ for } |x| \geq L, \tag{9}$$

where $\hat{\psi}(x) \equiv \psi(|x|)$, $\psi(x)$ is the same as (5) for any $x \in \mathbf{R}$.

2.

$$\sup_{x \in E} E_x \tau_E < \infty, \tag{10}$$

where $E \equiv [-L, L]$ and $\tau_E \equiv \inf\{n > 0 : X_n \in E\}$. Notice that E is a petite set. □

Proof of Theorem 3.4. Suppose

$$\int_0^\infty x \mu(dx) < \infty.$$

Since $\sup_n E_\mu(X_n \vee 0) = \int_0^\infty x\mu(dx) < \infty$ and $\{(X_n \vee 0)\}_{n \geq 0}$ is a submartingale under P_μ, we obtain

$$P_\mu \left(\lim_{n \to \infty} (X_n \vee 0) < \infty \right) = 1.$$

However, since the support of the invariant measure $\mu(\cdot)$ is $\mathbf{R}$, we have

$$P_\mu \left(\limsup_{x \to \infty} X_n = \infty \right) = 1,$$

which is a contradiction. □

REMARK. It is clear that $\{S_t\}_{t \geq 0}$ is a local martingale but not a martingale under P_m when

$$\int_{-\infty}^0 |x| m(x) dx < \infty \text{ or } \int_0^\infty x m(x) dx < \infty.$$

However, $\{X_n\}_{n \geq 0}$ is a martingale since for fixed $n \geq 0$, X_n is a bounded random variable.

Proof of Theorem 3.5. Assume that $\psi(x)$ below is the same as (5). It is clear that for each $x \geq M$,

$$\phi^+(x) = \frac{1-d_-}{d_+ - d_-}, \quad \phi^-(x) = \frac{d_+ - 1}{d_+ - d_-}; \quad \frac{\psi(\rho^+(x))}{\psi(x)} = d_+^\beta, \quad \frac{\psi(\rho^-(x))}{\psi(x)} = d_-^\beta.$$

Thus by Lemma 4.4, we obtain for $x \geq M$,

$$\begin{aligned} \frac{\psi(\rho^+(x))}{\psi(x)} Q^+(1,x) + \frac{\psi(\rho^-(x))}{\psi(x)} Q^-(1,x) &\leq \left(\phi^+ d_+^\beta + \phi^- d_-^\beta\right)\left(1 - e^{-\theta}\right) \\ &= \left(\frac{1-d_-}{d_+ - d_-} d_+^\beta + \frac{d_+ - 1}{d_+ - d_-} d_-^\beta\right)\left(1 - e^{-\theta}\right). \\ &< 1 - e^{-\theta}. \end{aligned}$$

Then by Lemma 4.3, we obtain

$$C \equiv \sup_n E_x \psi(X_n) < \infty.$$

Let

$$\pi_n(A) \equiv \frac{1}{n} \sum_{k=0}^{n-1} P_x(X_k \in A), \text{ for any } A \in \mathbf{B}(\mathbf{R}).$$

Then the ergode theorem tells us that for almost everywhere $x \in \mathbf{R}$ with respect to $\mu(\cdot)$, $\{\pi_n(\cdot)\}_{n\geq 1}$ converges weakly to the invariant probability measure $\mu(\cdot)$. However, the above estimate shows

$$\int_{\mathbf{R}} \psi(y)\pi_n(dy) \leq C,$$

which implies

$$\int_0^\infty x^\beta \mu(dx) < \infty. \qquad \square$$

Proof of Theorem 3.6. By Lemma 4.5, without loss of generality, we assume that $s \in (0,1)$ such that

$$\rho^+(x) = x + x^s; \quad \rho^-(x) = x - x^s, \text{ whenever } x \geq M.$$

Since $\{\rho^\pm(x)\}$ satisfies the conditions of Theorem 3.2, it implies that the invariant measure $\mu(\cdot)$ of $\{X_n\}_{n\geq 0}$ exists. And we have

$$E_x\psi(X_1) = \psi(x) + E_x \int_0^{1\wedge\tau} (L\psi)(S_u)du,$$

for x satisfying $\rho^-(x) > M$, where $\psi(x)$ is the same as (5). To claim $\gamma_+(\mu) \geq (2s \wedge \alpha) - 1$, we divide the proof of this claim into two steps.

STEP 1. for $s \in (0, \alpha/2)$, by Lemma 4.5, there exist positive constants l and $N > (M \vee 1)$ such that

$$E_x\psi(X_1) \leq \psi(x) - lx^{\beta+2s-2}, \tag{11}$$

whenever x satisfying $\rho^-(x) \geq N$, where $\beta \in (0,1)$. Set

$$A \equiv (-\infty, N], \quad \tau_A \equiv \inf\{n > 0 : X_n \in A\}, \quad K \equiv \sup_{x\in A} |\psi(x) - E_x\psi(X_1)|.$$

It is clear that for any $\gamma < 2s - 1$, there exists $\beta \in (0,1)$ such that $\gamma \leq \beta + 2s - 2$ and by Lemma 4.2,

$$\begin{aligned}
l\int_{\mathbf{R}} (x^+)^\gamma \mu(dx) &= l\int_A \left\{ E_x \sum_{i=0}^{\tau_A - 1} (X_i^+)^\gamma \right\} \mu(dx) \\
&= l\int_A (x^+)^\gamma \mu(dx) + l\int_A \mu(dx) \left\{ \int_{A^c} E_y \left\{ \sum_{i=0}^{\tau_A-1} (X_i^+)^\gamma \right\} p(x,dy) \right\} \\
&\leq l\int_A (x^+)^\gamma \mu(dx) + l\int_A \mu(dx) \left\{ \int_{A^c} E_y \left\{ \sum_{i=0}^{\tau_A-1} (X_i^+)^{\beta+2s-2} \right\} p(x,dy) \right\} \\
&\leq l\int_A (x^+)^\gamma \mu(dx) + l\int_A \mu(dx) \left\{ \int_{A^c} \{\psi(y) - E_y\psi(X_{\tau_A})\}\, p(x,dy) \right\} \\
&\leq l\int_A (x^+)^\gamma \mu(dx) + l\int_A \mu(dx) \left\{ \int_{A^c} \psi(y) p(x,dy) \right\} \\
&\leq l\int_A (x^+)^\gamma \mu(dx) + l\int_A E_x\psi(X_1)\mu(dx) \\
&\leq l\int_0^N x^\gamma \mu(dx) + l\int_M^N \psi(x)\mu(dx) + lK < \infty.
\end{aligned}$$

Here notice that $\mu([0,N]) < \infty, \mu([M,N]) < \infty$ because $[0,N],[M,N]$ are petite sets. This gives immediately

$$\int_0^\infty x^\gamma \mu(dx) < \infty, \ \text{ for } \gamma < 2s - 1.$$

STEP 2. for $s \geq \alpha/2$, by Lemma 4.5 and (11), we have

$$E_x\psi(X_1) = E_x\psi(X_1^{(2)}) \leq E_x\psi(X_1^{(1)}) \leq \psi(x) - x^{\beta+2s_1-2},$$

for x satisfying $\rho^-(x) \geq N$, where $s_1 \in (0, \alpha/2)$ and

$$\rho_1^+(x) \equiv x + x^{s_1}\ , \ \rho_1^-(x) \equiv x - x^{s_1}, \ \text{ whenever } x \geq M,$$

$$\rho^+(x) \equiv \rho_2^+(x) \equiv x + x^s, \quad \rho^-(x) \equiv \rho_2^-(x) \equiv x - x^s, \text{ whenever } x \geq M.$$

As in the proof of Step 1, we get for $s \geq \alpha/2$,

$$\int_0^\infty x^\gamma \mu(dx) < \infty, \text{ whenever } \gamma < 2s_1 - 1.$$

Let s_1 approach to $\alpha/2$, we obtain $\gamma_+(\mu) \geq \alpha - 1$ for $s \in [\alpha/2, 1)$. This completes that

$$\gamma_+(\mu) \geq (2s \wedge \alpha) - 1.$$

On the other hand, to claim $\gamma_+(\mu) \leq (2s \wedge \alpha) - 1$, suppose that

$$\int_0^\infty x^\gamma \mu(dx) < \infty, \text{ for some } \gamma > 2s - 1. \tag{12}$$

The following is to claim that the assumption is wrong. Let

$$\epsilon \equiv \gamma - (2s - 1).$$

Take β such that $\beta + \epsilon \in (1, 1 + \epsilon)$. Define

$$\xi(x) \equiv \begin{cases} x^{\beta+\epsilon} & \text{for } x > M, \\ 0 & \text{for } x \leq M. \end{cases}$$

By Lemma 4.5, there exist positive constants $H > (M \vee 1)$ and r such that

$$\begin{aligned} E_x \xi(X_1) &= \xi(x) + E_x \int_0^{1\wedge\tau} L\xi(S_u)du \\ &\leq \xi(x) + r x^{\beta+\epsilon+2s-2}, \end{aligned} \tag{13}$$

for any x satisfying $\rho^-(x) \geq H$. Let

$$B \equiv (-\infty, H], \quad \tau_B \equiv \inf\{n > 0 : X_n \in B\}.$$

By Lemma 4.2, we have

$$\begin{aligned} \int_B E_x \left\{ \sum_{i=0}^{\tau_B - 1} (X_i^+)^{\beta+\epsilon+2s-2} \right\} \mu(dx) &= \int_{\mathbf{R}} (x^+)^{\beta+\epsilon+2s-2} \mu(dx) \\ &\leq \int_{\mathbf{R}} (x^+)^\gamma \mu(dx) \\ &= \int_0^\infty x^\gamma \mu(dx) < \infty, \end{aligned}$$

Moreover, it is easy to see

$$\begin{aligned}
&\int_B E_x\left\{\sum_{i=0}^{\tau_B-1}(X_i^+)^{\beta+\epsilon+2s-2}\right\}\mu(dx)\\
&=\int_B (x^+)^{\beta+\epsilon+2s-2}p(x,B)\mu(dx)+\int_B E_x\left\{\sum_{i=0}^{\tau_B-1}(X_i^+)^{\beta+\epsilon+2s-2}:X_1\in B^c\right\}\mu(dx)\\
&=\int_B (x^+)^{\beta+\epsilon+2s-2}\mu(dx)+\int_B \mu(dx)\left\{\int_{B^c} E_y\left\{\sum_{i=0}^{\tau_B-1}(X_i^+)^{\beta+\epsilon+2s-2}\right\}p(x,dy)\right\}\\
&=\int_0^H x^{\beta+\epsilon+2s-2}\mu(dx)+\int_{B^c} E_y\left\{\sum_{i=0}^{\tau_B-1}(X_i^+)^{\beta+\epsilon+2s-2}\right\}\hat{\sigma}(dy),
\end{aligned}$$

where $\hat{\sigma}(dy)\equiv\int_B p(x,dy)\mu(dx)$. Since (2) holds for $g(x)\equiv(x^+)^{\beta+\epsilon+2s-2}$ in Lemma 4.2, we get that there exists $x_0\in B^c=(H,\infty)$ such that

$$E_{x_0}\left\{\sum_{i=0}^{\tau_B-1}(X_i^+)^{\beta+\epsilon+2s-2}\right\}<\infty.$$

But since $\{X^+_{n\wedge\tau_B}\}_{n\geq 0}$ is a submartingale with $X_0\equiv x_0$ and

$$\begin{aligned}
E_{x_0}\xi(X_{k\wedge\tau_B}) &= E_{x_0}\xi(X^+_{k\wedge\tau_B})\\
&\leq \xi(x_0)+rE_{x_0}\left\{\sum_{i=0}^{k\wedge(\tau_B-1)}(X_i^+)^{\beta+\epsilon+2s-2}\right\},\quad\text{for each } k\geq 0,
\end{aligned}$$

this shows

$$\begin{aligned}
E_{x_0}\left\{\sup_{0\leq k\leq\tau_B-1}\xi(X_k)\right\} &= E_{x_0}\left\{\sup_{0\leq k\leq\tau_B-1}\xi(X_{k\wedge\tau_B})\right\}\\
&\leq E_{x_0}\left\{\sup_{0\leq k\leq\infty}\xi(X_{k\wedge\tau_B})\right\}\\
&\leq C\sup_{0\leq k\leq\infty}E_{x_0}\xi(X_{k\wedge\tau_B})\\
&\leq C\sup_{0\leq k\leq\infty}\left\{\xi(x_0)+rE_{x_0}\left\{\sum_{i=0}^{k\wedge(\tau_B-1)}(X_i^+)^{\beta+\epsilon+2s-2}\right\}\right\}\\
&\leq C\left\{\xi(x_0)+rE_{x_0}\left\{\sum_{i=0}^{\tau_B-1}(X_i^+)^{\beta+\epsilon+2s-2}\right\}\right\}<\infty, \qquad (14)
\end{aligned}$$

where

$$C \equiv \left(\frac{\beta + \epsilon}{\beta + \epsilon - 1} \right)^{\beta+\epsilon}.$$

Combine

$$E_{x_0}\{\xi(X_j) : \tau_B > j\} \leq E_{x_0}\left\{ \sup_{0 \leq k \leq \tau_B - 1} \xi(X_k) : \tau_B > j \right\},$$

with (14), we get

$$\lim_{j\to\infty} E_{x_0}\{\xi(X_j) : \tau_B > j\} = 0.$$

However,

$$\begin{aligned} \xi(x_0) &\leq E_{x_0}\xi(X_{j\wedge\tau_B}) \\ &= E_{x_0}\{\xi(X_{\tau_B}) : \tau_B \leq j\} + E_{x_0}\{\xi(X_j) : \tau_B > j\}, \end{aligned}$$

Let j approach to infinity, we obtain

$$\xi(x_0) \leq E_{x_0}\xi(X_{\tau_B}) \leq \xi(H) < \xi(x_0),$$

which is a contradiction. This completes that $\gamma_+(\mu) \leq 2s - 1$ for $s \in (0, 1)$. Similarly to (13) we can evaluate $E_x\xi(X_1)$ as follows also when $s \geq \alpha$;

$$\begin{aligned} E_x\xi(X_1) &= \xi(x) + E_x \int_0^{1\wedge\tau} L\xi(S_u)du \\ &\leq \xi(x) + rx^{\beta+\epsilon+\alpha-2}, \end{aligned}$$

for x satisfying $\rho^-(x) \geq H$. Therefore, by the same proceeding above, we obtain $\gamma_+(\mu) \geq \alpha - 1$ for $s \in [\alpha/2, 1)$. In consequence, it completes that

$$\gamma_+(\mu) \leq (2s \wedge \alpha) - 1.$$

This completes the proof. □

REMARK. For the first part of the proof in Theorem 3.3, without loss of generality, we can assume

$$g_+(x) \equiv x^s, \text{ whenever } x \geq M, \text{ for some } s < \frac{1}{2}.$$

By Theorem 3.2, there exists an invariant measure $\mu(\cdot)$ of $\{X_n\}_{n\geq 0}$. Moreover, from (12), it is clear that $\mu(\cdot)$ is not a probability measure.

For the second part of the proof, assume that $\{g_{\pm}(x)\}$ satisfies

$$g_{+}(x) = g_{-}(x) \equiv |x|^s, \text{ whenever } |x| \geq M, \text{ for some } s > \frac{1}{2}.$$

By Lemma 4.2, Lemma 4.5 and (11), we get (9) easily. Further, it is not hard to see that

$$\hat{\psi}(X_n) \equiv \hat{\psi}(x) + M_n + \sum_{i=0}^{n-1}(P - I)(\hat{\psi}(X_i)),$$

where $\{M_n\}_{n\geq 0}$ is a martingale with $M_0 \equiv 0$ and

$$(P - I)\hat{\psi}(x) \equiv E_x\hat{\psi}(X_1) - \hat{\psi}(x).$$

Thus we obtain (10) which shows

$$\mu(\mathbf{R}) = \int_{-L}^{L} E_x\{\tau_E\}\mu(dx) \leq \mu(E)\left\{\sup_{x\in E} E_x\tau_E\right\} < \infty.$$

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