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ACCIDENTAL SURFACES AND EXCEPTIONAL SURGERIES

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1. Introduction

The main subject of this paper is an *accidental surface* in the exterior of a knot in the 3-sphere S^3, which is defined as follows. Let S be a closed essential (i.e., incompressible and not ∂-parallel) surface in a knot exterior. We call S *accidental* if it contains a non-trivial loop which is isotopic into the peripheral torus of the knot. There are some motivations to study the accidental surface from the topological or the geometrical viewpoint. For example, it is known that accidental surfaces in a hyperbolic knot complement have a particular geometric behavior [11]. See [7] for more detail.

First, we will consider the *accidental slope* for accidental surfaces. A slope on the peripheral torus of a knot is determined by the isotopy from a non-trivial loop on an accidental surface into the torus. It is shown in [7, Theorem 1] that the slope is independent of the choice of the non-trivial loop on the surface. Hence we call this slope the *accidental slope* for the accidental surface. In contrast, the accidental slope is not determined uniquely for a knot. In fact, an example of a knot admitting two accidental surfaces with accidental slopes 0 and ∞ was given in [7, Figure 1]. We know that any accidental slope is integral or meridional [1, Lemma 2.5.3], and the example shows that there is a knot with integral and meridional accidental slopes. Hence, it is natural to ask how many *integral* accidental slopes exist for a knot. In this paper, we give a bound of the minimal intersection number of accidental slopes.

Theorem 3.2. *Let S_1 and S_2 be accidental surfaces with accidental slopes s_1 and s_2 in the exterior of a knot in S^3. Then*

$$\Delta(s_1, s_2) \leq \min\{-\chi_1, -\chi_2\},$$

where χ_i denotes the Euler characteristic of S_i for $i = 1, 2$ and Δ the minimal geometric intersection number of the slopes.

Next, we will consider the number of *accidental annuli* for an accidental surface. An isotopy from a non-trivial loop on an accidental surface into the peripheral torus

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of a knot gives an annulus, which we call an *accidental annulus*. The authors showed in [7, Theorem 1] that an accidental surface with integral accidental slope admits only one accidental annulus. On the other hand, an accidental surface with meridional accidental slope can admit plural accidental annuli. For example the accidental surface obtained by tubing a tangle decomposing sphere admits plural accidental annuli. In this paper, we prove

Theorem 4.2. *Let S be an accidental surface with meridional accidental slope in the exterior of a knot in S^3. Then there are at most $2g - 1$ mutually non-parallel accidental annuli for S, where g denotes the genus of the knot.*

An accidental annulus as in the theorem above gives a compressing disk for the surface in S^3 intersecting the knot at a single point. Such a disk is called a *meridionally compressing disk*. The theorem above implies that there are at most $2g - 1$ mutually non-parallel meridionally compressing disks for any closed essential surface.

As an application of the above results, we consider *exceptional surgeries* on knots with an accidental surface. The Hyperbolic Dehn Surgery Theorem [11] due to Thurston says that all but finitely many Dehn surgeries on a hyperbolic knot give hyperbolic 3-manifolds. Concerning the problem of when the exceptional cases occur, a large number of works have accomplished. See [3] for a survey.

Theorem 5.1. *Let K be a hyperbolic knot in S^3 which admits S an accidental surface with accidental slope s in the exterior of K, and let $K(r)$ denote the manifold obtained by Dehn surgery on a knot K along slope r.*
(A) *If $s = \infty$ and S admits at least three mutually non-parallel accidental annuli, then $K(r)$ is hyperbolic for every slope $r \neq \infty$.*
(B) *If s is integral and $K(r)$ is non-Haken for an integral slope r, then $|r| \leq 4g - 1$, where g denotes the genus of K.*

When a knot complement contains a closed essential surface with meridionally compressing disks, the number of such disks has influence on the topology of the surgered manifolds. In fact, Short [10] proved that if a knot complement contains a closed essential surface with at least *two* meridionally compressing disks, then all non-trivial Dehn surgeries yield Haken manifolds. Theorem 5.1 (A) is an extention of this result. We remark that there exists a knot whose exterior contains a closed essential surface admitting n meridionally compressing disks for any positive integer n. See [14] for example.

It was shown in [1] that if $\pi_1(K(r))$ is cyclic then r is integral. Therefore we obtain as an immediate corollary of Theorem 5.1 (B) that if $\pi_1(K(r))$ is cyclic then $|r| \leq 4g - 1$. This gives a partial affirmative answer to the following conjecture raised by Goda and Teragaito [5]. If $K(r)$ is a lens space, then K is fibered and

$2g+8 \leq |r| \leq 4g-1$.

This paper is organized as follows. In Section 2, we prepare some tools which we use in almost all proofs. We will be concerned with the accidental slope and the accidental annulus in Section 3 and 4 respectively. The results on exceptional surgeries will be presented in Section 5.

2. Preparation

Throughout this paper, K denotes a knot in S^3 and M denotes a compact, orientable and irreducible 3-manifold with an incompressible torus as the boundary.

Our notations are as follows. For a space X, $\mathrm{Int}(X)$, ∂X, $|X|$ and $\chi(X)$ denote the interior, the boundary, the number of connected components and the Euler characteristic of X, respectively. For a subset Y of X, $\mathrm{Ext}(Y)$ denotes the exterior of Y in X, that is, the closure of $X - \mathrm{N}(Y)$, where $\mathrm{N}(Y)$ denotes the regular neighborhood of Y in X. For a graph G, $\epsilon(G)$ denotes the number of edges of G and $\nu(G)$ the number of vertices of G.

First, we introduce the *strongly essential surface*, which is closely related with the accidental surface. Let S be a surface with non-empty boundary properly embedded in M. We say that S is *essential* if it is incompressible, ∂-incompressible and not ∂-parallel. Let us define that S is *strongly essential* if it is essential and at least one component of $\mathrm{Ext}(S)$ is ∂-irreducible. The *boundary slope* of S is defined as the slope represented by a component of ∂S.

One can construct a strongly essential surface from an accidental surface by an annulus compression. The boundary slope of the resultant surface is equal to the accidental slope of the prescribed one. Since the converse operation also can be done [7, Theorem 2], the existence of an accidental surface with the accidental slope r in $\mathrm{Ext}(K)$ is equivalent to the existence of a strongly essential surface with the boundary slope r in $\mathrm{Ext}(K)$.

The following lemma will work anywhere in this paper.

Lemma 2.1. *Let F be an essential surface, and S a separating* (*respectively non-separating*) *strongly essential surface in M. Then, F and S can be homotoped such that there are at most $-2\chi(F)$* (*resp. $-\chi(F)$*) *arc components of $F \cap S$.*

Proof. Suppose that the number of arc components of $F \cap S$ is minimal up to isotopy.

CLAIM 1. The closure of any open disk region obtained by cutting F along $F \cap S$ gives a ∂-reducing disk for $\mathrm{Ext}(S)$.

Proof. Suppose that there exists an open disk region obtained by cutting F along $F \cap S$ such that its closure D is not a ∂-reducing disk for $\mathrm{Ext}(S)$. Then there exists a disk D' in $\partial\,\mathrm{Ext}(S)$ such that $\partial D = \partial D'$. Note that $\partial\,\mathrm{Ext}(S)$ consists of two copies S_1, S_2 of S and the union of annuli A appearing as the closure of $\partial\,\mathrm{Ext}(K) - N(\partial S; \partial\,\mathrm{Ext}(K))$. We consider $D' \cap (\partial S_1 \cup \partial S_2)$ and an outermost disk D'' in D'. If $D'' \subset S_i$, then by the essentiality of F and irreducibility of M, we can remove an arc component of $F \cap S$ by an isotopy of F. If $D'' \subset A$, then by an isotopy of S along D'', we can also remove an arc component of $F \cap S$. In either cases, we have a contradiction for the minimality of the number of arc components of $F \cap S$. □

Construct a closed surface F' by contracting each component of ∂F to a point. Note that $\chi(F')$ is equal to $\chi + |\partial F|$. By regarding points corresponding to ∂F as vertices and arc components of $F \cap S$ as edges, a graph G on F' is obtained. The number of vertices of G is equal to $|\partial F|$ and the number of edges of G is equal to the number of arc components of $F \cap S$.

If S is non-separating in M, then by the above claim, there is no open disk regions of $F' - G$. Hence we have $\chi(F') \leq \nu(G) - \epsilon(G)$, and $\epsilon(G) \leq |\partial F| - (\chi(F) + |\partial F|) = -\chi(F)$.

If S is separating in M, then G satisfies the next two conditions:

(1) Regions of $F' - G$ can be colored like a checkerboard.

(2) No regions are open disks for one of the two colors.

CLAIM 2. $\epsilon(G)$ is not greater than $2(\nu(G) - \chi(F'))$.

Proof. Without loss of generality, we assume that no black regions are open disks. If there exists a non-simply-connected white region, then one can add new edges which are parallel to the edges bounding the region so that the new graph also satisfies the conditions. Moreover, if there exists a white region which is a disk but not a bigon, then one can remove one of edges surrounding the region, and change other edges with parallel edges, so that the new graph also satisfies the conditions. Consequently, one can obtain a graph G' on F' satisfying the conditions 1 and 2 such that no black regions are open disks, all white regions are bigons and $\epsilon(G) \leq \epsilon(G')$. In this case, $\chi(F')$ is less than or equal to the Euler characteristic of the closure of the white regions. This implies that $\chi(F') \leq \nu(G) - \epsilon(G')/2 \leq \nu(G) - \epsilon(G)/2$. Thus, $\epsilon(G) \leq 2(\nu(G) - \chi(F'))$. □

By Claim 2, we have $\epsilon(G) \leq 2\{|\partial F| - (\chi(F) + |\partial F|)\} = -2\chi(F)$. □

3. Accidental slopes

In this section, we are concerned with accidental slopes, equivalently, the boundary slopes of strongly essential surfaces.

Lemma 3.1. *Let F be an essential surface and S a strongly essential surface in M. Let s denote the boundary slope of S, f the boundary slope of F. Then,* $\Delta(s, f) \leq -2\chi(F)/|\partial F|$.

Proof. Suppose that the number of arc components of $F \cap S$ is minimal up to isotopy. Then the number of the arc components of $F \cap S$ is equal to $|\partial S|\,|\partial F|\Delta(s, f)/2$. If S is separating in M then $|\partial S| \geq 2$, and if S is non-separating in M then $|\partial S| \geq 1$. In both cases, $\Delta(s, f) \leq -2\chi(F)/|\partial F|$ by Lemma 2.1. $\square$

With this lemma, we prove the next theorem.

Theorem 3.2. *Let S_1 and S_2 be accidental surfaces with accidental slopes s_1 and s_2 in the exterior of a knot in S^3. Then* $\Delta(s_1, s_2) \leq \min\{-\chi_1, -\chi_2\}$.

Proof. Let us construct strongly essential surfaces in $\mathrm{Ext}(K)$ from the given accidental surface S_i ($i = 1, 2$) by an annulus compression. When the annulus compression gives disconnected surfaces, we take the connected component having larger Euler characteristic. Let F_i be the surface so obtained from S_i for $i = 1, 2$. Note that $\chi_i = \chi(S_i) \leq 2\chi(F_i)/|\partial F_i|$. Then, by Lemma 3.1, $\Delta(s_1, s_2) \leq 2\min\{-\chi(F_1)/|\partial F_1|, -\chi(F_2)/|\partial F_2|\} \leq \min\{-\chi_1, -\chi_2\}$. $\square$

Concerning the intersection number of the boundary slopes of essential surfaces, Torisu [12] showed the following. Let F_i be essential surfaces with boundary slopes f_i for i =1, 2 in an acylindrical 3-manifold. Then $\Delta(f_1, f_2) < 36(2g_1 - 1)(2g_2 - 1)$, where g_i denotes the genus of F_i for i = 1, 2.

By using Lemma 3.1, we obtain some bounds on accidental slopes.

Proposition 3.3. *Let S be an accidental surface with integral accidental slope s in* $\mathrm{Ext}(K)$. *Then,*

(i) $|s| \leq 4g - 2$,
(ii) $|s| \leq 2(c - b)$ *and*
(iii) $|s| \leq 2c - 6$,

where g, c and b denote the genus, the crossing number and the braid index of K respectively.

Proof. (i) is an immediate consequence of Lemma 3.1 by considering a minimal genus Seifert surface and the strongly essential surface obtained from S by an annulus compression. (ii) is obtained by (i) together with the result that the minimum number of Seifert circles equals to the braid index of the knot [13]. (iii) follows from (ii) together with the fact that 2-bridge knots are small [6]. $\square$

Since the last statement holds for the boundary slope of a strongly essential surface, Proposition 3.3 (iii) gives a partial affirmative answer to the following conjecture raised in [8]. Let K be a knot in S^3, D a diagram of K with c_D crossings, w_D the writhe of D, and S non-meridional essential surface with boundary slope s. Then $|s| \leq c_D + |w_D|$. In particular, $|s| \leq 2c$, where c denotes the crossing number of K.

4. Accidental annuli

Here, let us consider the number of accidental annuli for an accidental surfaces. As we stated in Introduction, an accidental surface with integral accidental slope has only one accidental annulus. In other words, the next holds.

Corollary 4.1. *Let K be a knot in S^3 and F a strongly essential surface in* $\mathrm{Ext}(K)$. *If F has an integral slope, then $|\partial F| \leq 2$.* □

This corollary shows a kind of simplicity of the 3-sphere. Indeed, if the ambient space is not 3-sphere, we can construct a knot and a strongly essential surface F with $|\partial F| > 2$ and an integral slope in its exterior.

Next, we consider the number of accidental annuli for an accidental surface with the meridional accidental slope. Equivalently, we consider the number of boundary components of a meridional strongly essential surface.

Theorem 4.2. *Let S be an accidental surface with meridional accidental slope in the exterior of a knot in S^3. Then there are at most $2g - 1$ mutually non-parallel accidental annuli for S, where g denotes the genus of the knot.*

Proof. Suppose that there exist n mutually non-parallel accidental annuli for S. We use Lemma 2.1 in which we put F a minimal genus Seifert surface and S' a separating strongly essential surface obtained from S by n annulus compressions. Then F and S' can be isotoped so that there are at most $-2(1 - 2g(F)) = 4g(F) - 2$ arc components of $F \cap S'$. On the other hand, the number of the arc components of $F \cap S'$ is equal to $2n$ if we take $|F \cap S'|$ minimal up to isotopy. Hence we have $n \leq 2g(K) - 1$. □

As we remarked in Introduction, an accidental annulus for an accidental surface with the meridional accidental slope gives a meridionally compressing disk for the surface.

5. Exceptional surgeries

Throughout this section, let $K(r)$ denote the 3-manifold obtained by Dehn surgery on K along r. That is, $K(r)$ denotes the 3-manifold obtained by attaching a solid torus V to $\mathrm{Ext}(K)$ so that a simple closed curve with slope r bounds a meridian disk of V.

Theorem 5.1. *Let K be a hyperbolic knot in S^3 and S an accidental surface with accidental slope s in the exterior of K.*
(A) *If $s = \infty$ and S admits at least three accidental annuli, then $K(r)$ is hyperbolic for every slope $r \neq \infty$.*
(B) *If s is integral and $K(r)$ is non-Haken for an integral slope r, then $|r| \leq 4g - 1$, where g denotes the genus of K.*

Proof. (A) We know that $K(r)$ is Haken for any $r \neq \infty$ [10]. Therefore, we only need to show that $K(r)$ is irreducible and atoroidal by [9]. By annulus compressions, we construct a meridional strongly essential surface S' with $|\partial S'| \geq 6$ from S in Theorem 5.1 (A).

CLAIM 3. Any closed essential surface of genus less than $(|\partial S'| + 4)/8$ in $K(r)$ can be isotoped into $\mathrm{Ext}(K)$ for $r \neq \infty$.

Proof. Suppose that there exists a closed essential surface F in $K(r)$ which can not be isotoped into $\mathrm{Ext}(K)$. Let $|F \cap V|$ be minimal up to isotopy of F in $K(r)$. Then $F' = F \cap \mathrm{Ext}(K)$ is essential in $\mathrm{Ext}(K)$. By Lemma 2.1, we have

$$|\partial S'|\,|\partial F'|\Delta(s, f)/2 \leq -2(2 - 2g(F) - |\partial F'|) = -4 + 4g(F) + 2|\partial F'|,$$

where s and f denote the boundary slopes of S' and F' respectively. Thus,

$$|\partial F'|(|\partial S'|\Delta(s, f) - 4) + 8 \leq 8g(F).$$

Since $|\partial F'| \geq 1$, $|\partial S'| \geq 6$ and $\Delta(s, f) \geq 1$,

$$(|\partial S'| - 4) + 8 = |\partial S'| + 4 \leq 8g(F). \qquad \square$$

Hence any closed essential surface of genus less than $(6 + 4)/8$ can be isotoped into $\mathrm{Ext}(K)$. But this contradicts that K is hyperbolic.

(B) By annulus compressions, we construct a strongly essential surface S' with integral boundary slope s from S in Theorem 5.1 (B). We note that $|\partial S'| = 1$ or 2 by Corollary 4.1.

CLAIM 4. Suppose that $\Delta(s, r) \geq 2$. Then any closed essential surface of genus less than $(\Delta(s, r) + 2)/4$ in $K(r)$ can be isotoped into $\mathrm{Ext}(K)$.

Proof. Suppose that there exists a closed essential surface F in $K(r)$ which can not be isotoped into $\mathrm{Ext}(K)$. Let $|F \cap V|$ be minimal up to isotopy of F in $K(r)$. Then

$F' = F \cap \mathrm{Ext}(K)$ is essential in $\mathrm{Ext}(K)$.

If $|\partial S'| = 1$, then we have $|\partial F'| \geq 2$ since $\Delta(s, r) \geq 2$. By Lemma 2.1,

$$|\partial S'|\,|\partial F'|\Delta(s, f)/2 \leq -(2 - 2g(F) - |\partial F'|) = -2 + 2g(F) + |\partial F|,$$
$$|\partial F'|(|\partial S'|\Delta(s, f) - 2) + 4 \leq 4g(F).$$

Since $|\partial F'| \geq 2$, $|\partial S'| = 1$ and $\Delta(s, f) \geq 2$,

$$2(\Delta(s, f) - 2) + 4 = 2\Delta(s, f) \leq 4g(F).$$

Hence we have $\Delta(s, f)/2 \leq g(F)$ if $|\partial S'| = 1$.

Next suppose $|\partial S'| = 2$. Since $|\partial F'| \geq 1$ and $\Delta(s, f) \geq 2$, a similar argument shows that

$$(2\Delta(s, f) - 4) + 8 = 2\Delta(s, f) + 4 \leq 8g(F).$$

Hence we have $(\Delta(s, f) + 2)/4 \leq g(F)$ if $|\partial S'| = 2$. In either cases, $(\Delta(s, f) + 2)/4 \leq g(F)$ holds. □

Claim 4 implies that $K(r)$ is irreducible if $\Delta(s, r) \geq 2$. In addition, S remains incompressible in $K(r)$ [1, 7], and so $K(r)$ is Haken if $\Delta(s, r) \geq 2$. On the other hand, we have $|s| \leq 4g - 2$ by Proposition 3.3 (i). Hence we conclude $|r| \leq 4g - 1$ if r is integral. □

If an annulus compression along an accidental annulus yields disconnected surface, then we obtain a strongly essential Seifert surface, which is called a *totally knotted Seifert surface*. We have the following in this case.

Corollary 5.2. *If a knot K has a totally knotted Seifert surface, then $K(r)$ is irreducible for every $r \neq \infty$.*

Proof. By virtue of Claim 4, $K(r)$ is Haken if $|r| = \Delta(r, 0) \geq 2$. Since it was shown in [2, 4] that $K(0)$, $K(\pm 1)$ are irreducible, the corollary holds. □

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