EXTENDIBILITY AND STABLE EXTENDIBILITY OF NORMAL BUNDLES ASSOCIATED TO IMMERSIONS OF REAL PROJECTIVE SPACES

Dedicated to the Memory of Professor Katsuo Kawakubo

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1. Introduction

The extension problem is one of the fundamental problems in topology. We consider the problem for vector bundles over real projective spaces.

Let $F$ be the real field $\mathbb{R}$, the complex field $\mathbb{C}$ or the quaternion field $\mathbb{H}$. Let $X$ be a space and $A$ be a subspace. A $t$-dimensional $F$-vector bundle $\zeta$ over $A$ is called extendible (respectively stably extendible) to $X$, if there is a $t$-dimensional $F$-vector bundle over $X$ whose restriction to $A$ is equivalent (respectively stably equivalent) to $\zeta$ as $F$-vector bundles, that is, if $\zeta$ is equivalent (respectively stably equivalent) to $i^*\alpha$ for a $t$-dimensional $F$-vector bundle $\alpha$ over $X$, where $i: A \rightarrow X$ is the inclusion (cf. [13] and [5]).

As is seen in [7, Theorem 6.4] and [11, Theorem 2.2], the extendibility (or the stable extendibility) is closely related to the span, i.e., the maximum number of linearly independent cross-sections of an $F$-vector bundle, and one can see in the proof of Theorem C of this paper how the stable extendibility is related to the immersion problem.

Let $R^n$ be the $n$-dimensional Euclidean space and $FP^n$ be the $n$-dimensional $F$-projective space. Concerning stably extendible $F$-vector bundles for $F = \mathbb{R}$ and $\mathbb{C}$, R.L.E. Schwarzenberger obtained the following results (cf. [2], [3], [7], [12] and [13]).

**Theorem** (Schwarzenberger). Let $F = \mathbb{R}$ or $\mathbb{C}$. If a $k$-dimensional $F$-vector bundle $\zeta$ over $FP^n$ is stably extendible to $FP^m$ for every $m > n$, then $\zeta$ is stably equivalent to a sum of $k$ $F$-line bundles.

In the original results of Schwarzenberger, the $F$-vector bundles are assumed to be extendible, but his results are also valid for the stably extendible $F$-vector bundles.

Recently, M. Imaoka and K. Kuwana have proved in [5] that if a $k$-dimensional

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$H$-vector bundle $\zeta$ over $H\mathbb{P}^n$ is stably extendible to $H\mathbb{P}^m$ for every $m > n$ and its top non-zero Pontrjagin class is not zero mod 2, then $\zeta$ is stably equivalent to a sum of $k$ $H$-line bundles provided $k \leq n$.

We study the question: Determine the necessary and sufficient condition that a $R$-vector bundle over $R\mathbb{P}^n$ is stably extendible to $R\mathbb{P}^m$ for every $m > n$. We have obtained the results for the tangent bundle $\tau = \tau(R\mathbb{P}^n)$ of $R\mathbb{P}^n$ (cf. [7] and [9]), for the normal bundle $\nu$ associated to an immersion of $R\mathbb{P}^n$ in $R^{2n+1}$ (cf. [10]) and for the complexification $c\nu$ of $\nu$ (cf. [10]) as follows:

1) $\tau$ is stably extendible to $R\mathbb{P}^m$ for every $m > n$ if and only if $n = 1, 3$ or 7.
2) $\nu$ is stably extendible to $R\mathbb{P}^m$ for every $m > n$ if and only if $1 \leq n \leq 8$.
3) $c\nu$ is stably extendible to $R\mathbb{P}^m$ for every $m > n$ if and only if $1 \leq n \leq 9$.

The purpose of this paper is to improve 2) and 3) for the normal bundle $\nu$ associated to an immersion of $R\mathbb{P}^n$ in $R^{2n+k}$ where $k$ is any positive integer and for the complexification $c\nu$ of $\nu$.

Let $\phi(n)$ be the number of integers $s$ such that $0 < s \leq n$ and $s \equiv 0, 1, 2$ or 4 mod 8. Then we have

**Theorem A.** Let $\nu$ be the normal bundle associated to an immersion of $R\mathbb{P}^n$ in $R^{2n+k}$, where $k > 0$. Then $\nu$ is stably extendible to $R\mathbb{P}^m$ for every $m > n$ if and only if $k \geq 2\phi(n) - n - 1$.

**Theorem B.** Let $\nu$ be the normal bundle associated to an immersion of $R\mathbb{P}^n$ in $R^{2n+k}$, and let $n + 1 \leq k \leq n + 12$. Then the following three conditions are equivalent:

1) $\nu$ is extendible to $R\mathbb{P}^m$ for every $m > n$.
2) $\nu$ is stably extendible to $R\mathbb{P}^m$ for every $m > n$.
3) $1 \leq n \leq 8$.

These are improvements of Theorem A in [10].

Let $[\chi]$ denote the integral part of a real number $\chi$. Then for the complexification of the normal bundle, we have

**Theorem C.** Let $c\nu$ be the complexification of the normal bundle $\nu$ associated to an immersion of $R\mathbb{P}^n$ in $R^{2n+k}$, where $k > 0$. Then the following hold.

(i) For $n \geq 6$, $c\nu$ is stably extendible to $R\mathbb{P}^m$ for every $m > n$ if and only if $k \geq 2\phi(n/2) - n - 1$.
(ii) For $1 \leq n \leq 5$, $c\nu$ is stably extendible to $R\mathbb{P}^m$ for every $m > n$.

The following is an improvement of Theorem 4.4 in [10].

**Theorem D.** Let $c\nu$ be the complexification of the normal bundle $\nu$ associated to an immersion of $R\mathbb{P}^n$ in $R^{2n+k}$, and let $n \leq k \leq n + 8$. Then the following three
conditions are equivalent:
(1) $\nu$ is extendible to $\mathbb{R}P^m$ for every $m > n$.
(2) $\nu$ is stably extendible to $\mathbb{R}P^m$ for every $m > n$.
(3) $1 \leq n \leq 9$.

This note is arranged as follows. In Section 2 we study relations between extendibility and stable extendibility. In Section 3 we prove Theorem A. We prove Theorem B and give some examples in Section 4. In Section 5 we prove Theorem C. We prove Theorem D and give some examples in Section 6.

2. Extendibility and stable extendibility

In the following, we use the same letter for a vector bundle and its equivalence class, and use an integer $k$ for a $k$-dimensional trivial bundle.

Let $d$ denote $\dim F$, where $F = \mathbb{R}, \mathcal{C}$ or $H$. The following fact is known (cf. [4, Theorem 1.5, p.100]).

\begin{equation}
\langle x \rangle \leq \dim(\mathbb{R}P^m) \text{ for every } m > n.
\end{equation}

If $\alpha$ and $\beta$ are two $t$-dimensional $F$-vector bundles over an $n$-dimensional CW-complex $X$ such that $\langle (n+2)/d - 1 \rangle \leq t$ and $\alpha \oplus k = \beta \oplus k$ for some $k$-dimensional trivial $F$-bundle $k$ over $X$, then $\alpha = \beta$, where $\oplus$ denotes the Whitney sum and $\langle x \rangle$ denotes the smallest integer $m$ with $x \leq m$.

**Theorem 2.2.** Let $X$ be a subcomplex of a finite dimensional CW-complex $Y$ and let $\zeta$ be an $\mathbb{R}$-vector bundle over $X$ such that $\dim \zeta > \dim X$. Then $\zeta$ is extendible to $Y$ if and only if $\zeta$ is stably extendible to $Y$.

In case $\dim \zeta = \dim X$, this does not hold in general.

Proof. The “only if” part is clear. Suppose that $\zeta$ is stably equivalent to $i^*(\alpha)$ for some $\mathbb{R}$-vector bundle $\alpha$ over $Y$, where $i: X \to Y$ is the inclusion. In case $\dim \zeta > \dim X$, $\zeta$ is equivalent to $i^*(\alpha)$ by (2.1).

A counter example is given by the $n$-sphere $S^n$ in the $(n+1)$-sphere $S^{n+1}$ and the tangent bundle $\tau = \tau(S^n)$ of $S^n$ for $n \neq 1, 3, 7$. In fact, $\tau \oplus 1$ is the $(n+2)$-dimensional trivial bundle over $S^n$ and so $\tau \oplus 1 = i^*(n) \oplus 1$, where $i: S^n \to S^{n+1}$ is the inclusion and $n$ denotes the $n$-dimensional trivial $\mathbb{R}$-vector bundle over $S^{n+1}$. Hence $\tau$ is stably extendible to $S^{n+1}$. On the other hand, if there is an $n$-dimensional $\mathbb{R}$-vector bundle $\alpha$ over $S^{n+1}$ such that $\tau = i^*(\alpha)$, $\tau$ is trivial, since $i: S^n \to S^{n+1}$ is homotopic to a constant map. Hence $n = 1, 3$ or 7. So $\tau$ is not extendible to $S^{n+1}$ for $n \neq 1, 3, 7$.

The following is proved in the way similar to the former part of the proof of Theorem 2.2.
**Theorem 2.3.** Let $X$ be a subcomplex of a finite dimensional CW-complex $Y$ and let $\zeta$ be a $C$-vector bundle over $X$ such that $\dim \zeta \geq \frac{1}{2} \dim X$. Then $\zeta$ is extendible to $Y$ if and only if $\zeta$ is stably extendible to $Y$.

**Corollary 2.4.** Let $M$ be a submanifold of a finite dimensional differentiable manifold $N$ and $c\tau(M)$ be the complexification of the tangent bundle $\tau(M)$ of $M$. Then $c\tau(M)$ is extendible to $N$ if and only if $c\tau(M)$ is stably extendible to $N$.

3. Proof of Theorem A

Let $\xi_n$ be the canonical line bundle over $RP^n$.

**Lemma 3.1.** Let $\nu$ be the normal bundle associated to an immersion of $RP^n$ in $R^{n+k}$, where $k > 0$. Then the equality

$$\nu = (a^{2^{\phi(n)}} - n - 1)\xi_n + n + k + 1 - a^{2^{\phi(n)}}$$

holds in $KO(RP^n)$, where $a$ is any integer.

Proof. Let $\tau = \tau(RP^n)$ be the tangent bundle of $RP^n$. Then we have $\tau \oplus \nu = n + k$ and $\tau \oplus 1 = (n + 1)\xi_n$. Hence

$$\nu = n + k + 1 - (n + 1)\xi_n = (a^{2^{\phi(n)}} - n - 1)\xi_n + n + k + 1 - a^{2^{\phi(n)}}$$

in $KO(RP^n)$ for any integer $a$, since $\xi_n - 1$ is of order $2^{\phi(n)}$ (cf. [1, Theorem 7.4]).

**Theorem 3.2.** Let $\nu$ be the normal bundle associated to an immersion of $RP^n$ in $R^{n+k}$, where $k > 0$. Then $\nu$ is stably extendible to $RP^m$ for every $m > n$ if $k \geq 2^{\phi(n)} - n - 1$, and if $k > n$, in addition, $\nu$ is extendible to $RP^m$ for every $m > n$.

Proof. By Lemma 3.1, we have $\nu = A\xi_n + B$, where $A = 2^{\phi(n)} - n - 1$ and $B = n + k + 1 - 2^{\phi(n)}$. Clearly $A \geq 0$, and $B \geq 0$ by the assumption. For $m > n$, $i^*(A\xi_m \oplus B) = A\xi_n \oplus B$, where $i: RP^n \to RP^m$ is the standard inclusion. Hence $\nu$ is stably extendible to $RP^m$ for every $m > n$, since $\nu$ is stably equivalent to $A\xi_n \oplus B$. If $k > n$, in addition, $\dim RP^m = n < k = \dim \nu = A + B$, and so we obtain $\nu = A\xi_n \oplus B$ by (2.1). Thus $\nu$ is extendible to $RP^m$ for every $m > n$.

The following result ([9, Theorem 4.1]) is the “stably extendible version” of Theorem 6.2 in [7].

(3.3). Let $\zeta$ be a $t$-dimensional $R$-vector bundle over $RP^n$. Assume that there is a positive integer $l$ such that $\zeta$ is stably equivalent to $(t + l)\xi_n$ and $t + l < 2^{\phi(n)}$. Then
n < t + l and \( \zeta \) is not stably extendible to \( \mathbb{R}P^{t+l} \).

Using (3.3), we have obtained the following in [10, Theorem 2.4] (cf. [11, Proposition 6.4(iii)(b)]).

(3.4) The normal bundle associated to an immersion of \( \mathbb{R}P^n \) in \( \mathbb{R}P^{n+k} \) is not stably extendible to \( \mathbb{R}P^{n+k+1} \), if \( 0 < k < 2^{\phi(n)} - n - 1 \).

**Theorem 3.5.** Let \( \nu \) be the normal bundle associated to an immersion of \( \mathbb{R}P^n \) in \( \mathbb{R}P^{n+k} \). Then \( \nu \) is not stably extendible to \( \mathbb{R}P^m \) for \( m = \min\{2^{\phi(n)} - n - 1, n + k + 1\} \), if \( 0 < k < 2^{\phi(n)} - n - 1 \).

Proof. Put \( \zeta = \nu, t = k \) and \( l = 2^{\phi(n)} - n - k - 1 \) in (3.3). Then clearly \( t + l < 2^{\phi(n)} \), and \( l > 0 \) by the assumption. So \( \nu \) is not stably extendible to \( \mathbb{R}P^m \) for \( m = 2^{\phi(n)} - n - 1 \). By (3.4), \( \nu \) is not stably extendible to \( \mathbb{R}P^m \) for \( m = n + k + 1 \).

Putting \( n = 9 \) in Theorem 3.5, we have

**Corollary 3.6.** If \( 1 \leq k \leq 21 \), the normal bundle associated to an immersion of \( \mathbb{R}P^9 \) in \( \mathbb{R}P^{n+k} \) is not stably extendible to \( \mathbb{R}P^m \) for \( m = \min\{22, k + 10\} \).

Proof of Theorem A. The “if” part follows from Theorem 3.2 and the “only if” part follows from Theorem 3.5.

4. Proof of Theorem B

Let \( \xi_n \) be the canonical line bundle over \( \mathbb{R}P^n \).

**Theorem 4.1.** Let \( \nu = \nu(f_n) \) be the normal bundle associated to an immersion \( f_n : \mathbb{R}P^n \rightarrow \mathbb{R}P^{n+k} \), where \( k > 0 \). Then, for \( 1 \leq n \leq 10 \), we have the equalities

\[
\nu(f_1) = k, \quad \nu(f_2) = \xi_2 + k - 1, \quad \nu(f_3) = k, \\
\nu(f_4) = 3\xi_1 + k - 3, \quad \nu(f_5) = 2\xi_5 + k - 2, \quad \nu(f_6) = \xi_6 + k - 1, \\
\nu(f_7) = k, \quad \nu(f_8) = 7\xi_8 + k - 7, \quad \nu(f_9) = 22\xi_9 + k - 22 \\
\text{and} \quad \nu(f_{10}) = \xi_{10} + 53 - 53
\]

in \( KO(\mathbb{R}P^n) \).

If \( 1 \leq n \leq 8 \) and \( k > n \) or if \( n \geq 9 \) and \( k \geq 2^{\phi(n)} - n - 1 \), the equalities hold in the set of equivalence classes of \( \mathbb{R} \)-vector bundles over \( \mathbb{R}P^n \).

Proof. By Lemma 3.1, we have

\[
\nu = n + k + 1 - (n + 1)\xi_n = (a2^{\phi(n)} - n - 1)\xi_n + n + k + 1 - a2^{\phi(n)}
\]
in $KO(RP^n)$ for any integer $a$. So we have the former part by putting $a = 1$.

The latter part is a consequence of the former part by (2.1), since $\nu = A\xi_n + B$ for non-negative integers $A$ and $B$ such that $\dim RP^n = n < k = \dim \nu = A + B$, if $1 \leq n \leq 8$ and $k > n$ or if $n \geq 9$ and $k \geq 2^{\phi(n)} - n - 1$.

**Corollary 4.2.** If $1 \leq n \leq 8$ and $k > n$ or if $n \geq 9$ and $k \geq 2^{\phi(n)} - n - 1$, $\nu(f_n)$ is extendible to $RP^m$ for every $m > n$.

**Proof.** Since $\xi_n$ and the trivial $R$-bundles over $RP^n$ are extendible to $RP^m$ for every $m > n$, the result follows from the latter part of Theorem 4.1.

Theorem B is a consequence of the following

**Theorem 4.3.** Let $\nu$ be the normal bundle associated to an immersion of $RP^n$ in $R^{n+k}$. Then we have

(i) $\nu$ is stably extendible to $RP^m$ for every $m > n$ if $1 \leq n \leq 8$ and $k \geq n$, and $\nu$ is extendible to $RP^m$ for every $m > n$ if $1 \leq n \leq 8$ and $k > n$.

(ii) $\nu$ is not stably extendible to $RP^{n+k+1}$ if $n \geq 9$ and $1 \leq k \leq n + 12$.

**Proof.** The former part of Theorem 4.1 implies the former part of (i). In fact, if $k \geq n$, the $R$-vector bundles $k, \xi_2 \oplus (k - 1), k, 3\xi_4 \oplus (k - 3), 2\xi_5 \oplus (k - 2), \xi_6 \oplus (k - 1), k$ and $7\xi_8 \oplus (k - 7)$ over $RP^n$, where $1 \leq n \leq 8$ respectively, are extendible to $RP^m$ for every $m > n$, and they are stably equivalent to $\nu(f_n)$ respectively.

The latter part of (i) follows from the former part of (i) by Theorem 2.2.

(ii) is a consequence of (3.4), because $0 < k < 2^{\phi(n)} - n - 1$ if $n \geq 9$ and $1 \leq k \leq n + 12$.

In [6, Theorem 1], the following (4.4) is proved (cf. [11, Corollary 2.3 (2)]).

(4.4). Let $\zeta$ be a $t$-dimensional $R$-vector bundle over $RP^n$. If $n < t$, $\zeta$ is extendible to $RP^m$ for every $m$ with $n < m \leq t$.

The next example is due to (4.4) and Corollary 3.6.

**Example 4.5.** The normal bundle associated to an immersion of $RP^9$ in $R^{30}$ is extendible to $RP^{21}$, but is not stably extendible to $RP^{22}$.

5. **Proof of Theorem C**

**Lemma 5.1.** Let $cv$ be the complexification of the normal bundle $\nu$ associated to an immersion of $RP^n$ in $R^{n+k}$, where $k > 0$. Then the equality

$$cv = (b2^{[n/2]} - n - 1)\xi_n + n + k + 1 - b2^{[n/2]}$$
holds in $K(\mathbb{P}^n)$, where $b$ is any integer.

Proof. Complexifying the equality in Lemma 3.1 and considering that $c_2^{n} - 1$ is of order $2^{\lfloor n/2 \rfloor}$, we have the equality above, since $|n/2| \leq \phi(n)$. \hfill \Box

**Theorem 5.2.** Let $c_\nu$ be the complexification of the normal bundle $\nu$ associated to an immersion of $\mathbb{P}^n$ in $\mathbb{R}^{2^k}$, where $k > 0$. Then $c_\nu$ is stably extendible to $\mathbb{P}^m$ for every $m > n$ if $k \geq 2^{\phi(n)} - n - 1$, or if $k \geq 2^{\lfloor n/2 \rfloor} - n - 1 \geq 0$. And if $2k \geq n$, in addition, $c_\nu$ is extendible to $\mathbb{P}^m$ for every $m > n$.

Proof. To prove the first part, by Lemma 5.1, we have $c_\nu = A^*c_2^{n} + B$, where $A = 2^{\phi(n)} - n - 1$ and $B = n + k + 1 - 2^{\phi(n)}$, since we may take $b = 1$ if $n \equiv 6, 7$ or 0 mod 8 and $b = 2$ otherwise. Clearly $A \geq 0$, and $B \geq 0$ by the assumption. For $m > n$, $i^*(A^*c_2^{m} + B) = A^*c_2^{n} + B$, where $i : \mathbb{P}^n \to \mathbb{P}^m$ is the standard inclusion. Hence $c_\nu$ is stably extendible to $\mathbb{P}^m$ for every $m > n$, since $c_\nu$ is stably equivalent to $A^*c_2^{n} + B$.

To prove the second part, taking $b = 1$ in Lemma 5.1, we have $c_\nu = Ac_2^{n} + B$, where $A = 2^{\lfloor n/2 \rfloor} - n - 1$ and $B = n + k + 1 - 2^{\lfloor n/2 \rfloor}$. By the assumption $A \geq 0$ and $B \geq 0$. So $c_\nu$ is stably extendible to $\mathbb{P}^m$ for every $m > n$, in the way similar to the proof above.

If $2k \geq n$, in addition, $\langle (\dim \mathbb{R}^m)/2 \rangle = \langle n/2 \rangle \leq k = \dim c_\nu = A + B$, and so we obtain $c_\nu = A^*c_2^{n} + B$ by (2.1). Thus $c_\nu$ is extendible to $\mathbb{P}^m$ for every $m > n$. \hfill \Box

We recall the following result ([9, Theorem 2.1]) which is the "stably extendible version" of Theorem 4.2 for $d = 1$ in [8].

(5.3). Let $\zeta$ be a $t$-dimensional $C$-vector bundle over $\mathbb{P}^n$. Assume that there is a positive integer $l$ such that $\zeta$ is stably equivalent to $(t + l)c_2^{n}$ and $t + l < 2^{\lfloor n/2 \rfloor}$. Then $\lfloor n/2 \rfloor < t + l$ and $\zeta$ is not stably extendible to $\mathbb{P}^m$ for every $m$ with $t + l \leq \lfloor m/2 \rfloor$.

**Theorem 5.4.** Let $c_\nu$ be the complexification of the normal bundle $\nu$ associated to an immersion of $\mathbb{P}^n$ in $\mathbb{R}^{2^k}$, where $k > 0$. Then $c_\nu$ is not stably extendible to $\mathbb{P}^m$ for every $m$ with $2^{\lfloor n/2 \rfloor} + 2n - 2 \leq m$, if $k < 2^{\lfloor n/2 \rfloor} - n - 1$.

Proof. Put $\zeta = c_\nu$, $t = k$ and $l = 2^{\lfloor n/2 \rfloor} - n - 1$ in (5.3). Then clearly $t + l < 2^{\lfloor n/2 \rfloor}$, and $l > 0$ by the assumption. So $c_\nu$ is not stably extendible to $\mathbb{P}^m$ for every $m$ with $2^{\lfloor n/2 \rfloor} - n - 1 \leq \lfloor m/2 \rfloor$. \hfill \Box

Proof of Theorem C. (i) For $n \geq 6$, the "only if" part follows from Theorem 5.4, and the "if" part follows from Theorem 5.2, since $2^{\lfloor n/2 \rfloor} - n - 1 \geq 0$ if $n \geq 6$.

(ii) As is well-known, $\mathbb{R}^1 \subseteq \mathbb{R}^2$, $\mathbb{R}^2 \subseteq \mathbb{R}^4$, $\mathbb{R}^2 \subseteq \mathbb{R}^4$, $\mathbb{R}^4 \subseteq \mathbb{R}^7$ and $\mathbb{R}^5 \subseteq \mathbb{R}^7$, where we denote by $\mathbb{R}^n \subseteq \mathbb{R}^N$ the existence of an immersion of $\mathbb{R}^n$ in $\mathbb{R}^N$, and these immersions are best possible, that is, there do not exist immersions of $\mathbb{R}^n$ in
Hence we have $k \geq 2^{\phi(n)} - n - 1$ for $1 \leq n \leq 5$. So the result follows also from Theorem 5.2.

6. Proof of Theorem D

**Theorem 6.1.** Let $cv = cv(f_n)$ be the complexification of the normal bundle $\nu = \nu(f_n)$ associated to an immersion $f_n: \mathbb{R}^n \to \mathbb{R}^{n+k}$, where $k \geq n$. Then we have the Whitney sum decompositions as follows:

- $cv(f_1) = k$, $cv(f_2) = c\xi_2 \oplus (k - 1)$, $cv(f_3) = k$,
- $cv(f_4) = 3c\xi_4 \oplus (k - 3)$, $cv(f_5) = 2c\xi_5 \oplus (k - 2)$, $cv(f_6) = c\xi_6 \oplus (k - 1)$,
- $cv(f_7) = k$, $cv(f_8) = 7c\xi_8 \oplus (k - 7)$ and $cv(f_9) = 6c\xi_9 \oplus (k - 6)$.

Proof. Complexifying the equalities in the former part of Theorem 4.1, we have the equalities above for $1 \leq n \leq 8$ using (2.1). So it suffices to prove the equality for $n = 9$. By the former part of Theorem 4.1, $\nu(f_9) = 22\xi_9 + k - 22$, and so $cv(f_9) = 22c\xi_9 + k - 22$. According to [1, Theorem 7.3], $c\xi_9 - 1$ is of order 16. Hence $16c\xi_9 - 16 = 0$ in $K(\mathbb{R}P^9)$, and so $cv(f_9) = 6c\xi_9 + k - 6$. Therefore, $cv(f_9) = 6c\xi_9 \oplus (k - 6)$ by (2.1).

**Corollary 6.2.** If $1 \leq k \leq 20$, the complexification $cv(f_{10})$ of the normal bundle $\nu(f_{10})$ associated to an immersion $f_{10}: \mathbb{R}P^{10} \to \mathbb{R}^{10+k}$ is not stably extendible to $\mathbb{R}P^{42}$.

Proof. By the former part of Theorem 4.1, $\nu(f_{10}) = 53\xi_{10} + k - 53$, and so $cv(f_{10}) = 53c\xi_{10} + k - 53 = 21c\xi_{10} + k - 21$, since $c\xi_{10} - 1$ is of order 32. Hence we have the result from (5.3) by putting $n = 10$, $\zeta = cv(f_{10})$, $t = k$ and $l = 21 - k$, since $l > 0$ for $1 \leq k \leq 20$ and since $t + l = 21 < 2^{10/2} = 32$.

Define $l(n) = 2^{n/2} - n - k - 1$. Then we have

**Lemma 6.3.** $l(n) > 0$ for any $k$ and $n$ such that $10 \leq n \leq k \leq n + 8$, and $k + l(n) < 2^{n/2}$, for any $k$ and $n$.

Proof. For $10 \leq n \leq 17$, the inequalities hold clearly. For $n \geq 18$, we prove the inequalities by induction.

Theorem D is a consequence of the following

**Theorem 6.4.** Let $cv$ be the complexification of the normal bundle $\nu$ associated to an immersion of $\mathbb{R}P^n$ in $\mathbb{R}^{n+k}$. Then we have

(i) $cv$ is extendible to $\mathbb{R}P^n$ for every $m > n$ if $1 \leq n \leq 9$ and $k \geq n$. 

Proof. Since $c\xi_n$ and the trivial $C$-vector bundles over $RP^n$ are extendible to $RP^m$ for every $m > n$, Theorem 6.1 implies (i).

By Lemma 5.1, we have

$$c\nu = \{b2^{[n/2]} - (n + 1)\}c\xi_n + n + 1 - b2^{[n/2]}$$

for any integer $b$. (ii) follows from (5.3), Lemma 6.3 and the equality above by putting $\zeta = c\nu$, $t = k$ and $l = 2^{[n/2]} - n - k - 1$.

In [6, Theorem 2], the following (6.5) is proved (cf. [11, Corollary 2.3 (2)]).

(6.5). Let $\zeta$ be a $t$-dimensional $C$-vector bundle over $RP^n$. If $n < 2t + 1$, $\zeta$ is extendible to $RP^m$ for every $m$ with $n < m \leq 2t + 1$.

The next example is due to (6.5) and Corollary 6.2 for $k = 20$.

**Example 6.6.** The complexification of the normal bundle associated to an immersion of $RP^{10}$ in $R^{30}$ is extendible to $RP^{41}$, but is not stably extendible to $RP^{42}$.

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