ON LIFTS OF IRREDUCIBLE 2-BRAUER CHARACTERS
OF SOLVABLE GROUPS

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1. Introduction

Let $G$ be a finite group. Write $|G| = p^k$, where $p$ is a prime number and $(k, p) = 1$, and let $Q_k = Q(e^{2\pi i / k})$, the field generated by $e^{2\pi i / k}$ over the field $Q$ of rationals. Recall that an ordinary character $\chi$ of $G$ is said to be $p$-rational if $\chi(x) \in Q_k$ for every $x \in G$.

Now, let $\varphi$ be an irreducible 2-Brauer character of $G$ and denote by $n$ (resp. $m$), the number of ordinary (resp. 2-rational) irreducible characters $\xi$ of $G$ such that the restriction $\xi_{G'_{2'}}$ of $\xi$ to the subset $G_{2'}$ of $2'$-elements of $G$, is equal to $\varphi$.

Let $V$ be a simple $FG$-module affording $\varphi$, where $F$ is an algebraically closed field of characteristic 2, and let $Q$ be a vertex of $V$.

If $Q$ is cyclic and $|Q| > 1$, the module $V$ belongs to a 2-block $B$ of $G$ having $Q$ as a defect group (see Theorem VII.15.1 in [2]). By the theory of blocks with cyclic defect groups (see, for instance, Theorem 68.1 in [1]), we have $n = |Q|$.

More generally, assume now that $G$ has a normal subgroup $N$ such that $Q \nsubseteq N$ and that the quotient group $QN/N$ is cyclic. Then, in case $G$ is solvable, the main result of this paper (Theorem 1) asserts that $n \geq |QN/N|$ and that $m \geq 2$.

It is worth mentioning that the statement concerning $m$ above is really an exclusive feature of the prime 2. In fact, it has been shown by I.M. Isaacs that if $\phi$ is an irreducible $p$-Brauer character of a $p$-solvable group $H$, where $p$ is odd, then there exists a unique irreducible $p$-rational character $\theta$ of $H$ such that $\theta_{H, p'} = \phi$ (see Theorem X.2.3 in [2]).

2. Background

Although the main result of this paper (Theorem 1) concerns ordinary characters and 2-Brauer characters of solvable groups, its proof relies heavily on Isaacs’ theory of partial characters developed in [6, 7]. In this section, we review few concepts of that theory needed for our purpose.

Let $\pi$ be an arbitrary set of primes and assume throughout this section that $G$ is a finite $\pi$-separable group. Recall that the $\pi'$-partial characters of $G$ are just the restrictions $\chi^0$ of ordinary characters $\chi$ of $G$ to the set of $\pi'$-elements of $G$. Further-
more, \( \chi^0 \) is said to be irreducible if it cannot be written as a sum of two \( \pi' \)-partial characters. The set of irreducible \( \pi' \)-partial characters of \( G \) is denoted by \( \mathcal{I}_{\pi'}(G) \). For any \( \xi \in \text{Irr}(G) \), there are uniquely determined nonnegative integers \( d_{\xi \psi} \), such that \( \xi^0 = \sum_{\psi} d_{\xi \psi} \psi \), where \( \psi \) runs through \( \mathcal{I}_{\pi'}(G) \).

In case \( \pi = \{ p \} \), it follows from the Fong-Swan theorem that the \( \pi' \)-partial characters of \( G \) are exactly the Brauer characters (at \( p \)) and consequently \( \mathcal{I}_{\pi'}(G) = \text{IBr}(G) \).

Next, assume that \( K \) is a subgroup of \( G \) and that \( \psi \) is a \( \pi' \)-partial character of \( G \). Then, it is obvious that the restriction \( \psi_K \) is a \( \pi' \)-partial character of \( K \). For \( \varphi \in \mathcal{I}_{\pi'}(K) \), we denote by \( \mathcal{I}_{\pi'}(G \mid \varphi) \), the set of all \( \omega \in \mathcal{I}_{\pi'}(G) \) such that \( \varphi \) is a constituent of \( \omega_K \). Induction \( \tau^G \) of a \( \pi' \)-partial character \( \tau \) of \( K \) can also be defined by using the usual formula of induced characters and applying it only to \( \pi' \)-elements. It is easy to see that \( \tau^G \) is a \( \pi' \)-partial character of \( G \).

In [9], a vertex of \( \psi \in \mathcal{I}_{\pi'}(G) \) is defined to be a Hall \( \pi \)-subgroup of some subgroup \( J \) of \( G \) for which there exists \( \alpha \in \mathcal{I}_{\pi'}(J) \) such that \( \alpha^G = \psi \) and \( \alpha(1) \) is a \( \pi' \)-number. It turns out that the set of vertices of \( \psi \) is not empty and that it forms a single conjugacy class of \( \pi \)-subgroups of \( G \) (see Theorem B in [9]). If \( \psi \) is an irreducible \( \pi' \)-Brauer character of a \( p \)-solvable group, then it is not hard to see that the vertices of \( \psi \) defined above (when \( \pi = \{ p \} \)), are exactly the vertices of the simple module (in characteristic \( p \)) affording \( \psi \).

It is clear from the definitions that for every \( \psi \in \mathcal{I}_{\pi'}(G) \), there exists \( \chi \in \text{Irr}(G) \) such that \( \chi^0 = \psi \). However, \( \chi \) is not unique in general. Nevertheless, in [6], Isaacs has canonically defined a set \( B_{\pi'}(G) \) of irreducible characters of \( G \) such that the map \( \chi \mapsto \chi^0 \) is a bijection of \( B_{\pi'}(G) \) onto \( \mathcal{I}_{\pi'}(G) \).

Let now \( N \triangleleft G \) and \( \mu \in B_{\pi'}(N) \). Two characters \( \chi_1, \chi_2 \in \text{Irr}(G \mid \mu) \) are said to be linked if there exists \( \psi \in \mathcal{I}_{\pi'}(G) \) such that \( d_{\chi_1 \psi} \neq 0 \) and \( d_{\chi_2 \psi} \neq 0 \). The equivalence classes defined by the transitive extension of this linking relation are called relative \( \pi \)-blocks of \( G \) with respect to \( (N, \mu) \), and the set of all these relative \( \pi \)-blocks is denoted by \( \mathcal{B}_{\pi'}(G \mid \mu) \) (see Section 3 in [11]). In case \( (N, \mu) = (\{ 1 \}, 1_{\{ 1 \}}) \), where \( 1_{\{ 1 \}} \) is the trivial character of \( \{ 1 \} \), the relative \( \pi \)-blocks of \( G \) with respect to \( (N, \mu) \) are just the \( \pi \)-blocks defined by M. Slattery [12].

3. The main theorem

We start this section by stating the main theorem of this paper.

**Theorem 1.** Let \( G \) be a finite solvable group and let \( F \) be an algebraically closed field of characteristic \( 2 \). Let \( V \) be a simple \( FG \)-module with vertex \( Q \) and let \( \varphi \) be the irreducible Brauer character afforded by \( V \). Suppose that there exists a normal subgroup \( N \) of \( G \) such that \( Q \not\subseteq N \) and \( QN/N \) is cyclic. Then

(i) \( G \) has at least \( |QN/N| \) ordinary irreducible characters \( \chi \) such that the restriction \( \chi_G^N \) of \( \chi \) to the subset \( G_2' \) of \( 2' \)-elements of \( G \), is equal to \( \varphi \).

(ii) \( G \) has at least two \( 2 \)-rational irreducible characters \( \xi \) such that \( \xi_{G_2'} = \varphi \).
In order to prove this theorem, we need few preliminary results. For the sake of generality, all but the last of these results are proved in the general setting of finite $\pi$-separable groups, where $\pi$ is an arbitrary set of prime numbers. (Note that a solvable group is necessarily $\pi$-separable.)

Before stating our first preliminary result, recall that a character-triple is a triple $(H, M, \alpha)$, where $M$ is a normal subgroup of the group $H$ and $\alpha$ is an $H$-invariant irreducible character of $M$. By definition (see Definition 11.23 in [5]), if the triple $(H, M, \alpha)$ is isomorphic to $(H', M', \alpha')$, then there exists an isomorphism $\tau: H/M \rightarrow H'/M'$. If $M \subseteq L \subseteq H$ and $L'$ is the subgroup of $H'$ containing $M'$ such that $\tau(L/M) = L'/M'$, then also by the definition of character-triple isomorphism, we have a bijection $\sigma_L: \text{Irr}(L \mid \alpha) \rightarrow \text{Irr}(L' \mid \alpha')$. Let $\sigma$ be the union of the maps $\sigma_L$. Then, the pair $(\tau, \sigma)$ is the corresponding isomorphism from $(H, M, \alpha)$ to $(H', M', \alpha')$.

**Lemma 2.** Let $\pi$ be a set of primes and let $H$ be a $\pi$-separable group. Let $M \triangleleft H$ and let $\alpha$ be an $H$-invariant $\pi'$-special character of $M$. Then, there exist a central extension $H'$ of $H = H/M$ by a $\pi'$-subgroup $M'$ of $H'$, a linear character $\alpha'$ of $M'$ and bijections $\Psi$ of $\text{Irr}(H \mid \alpha)$ onto $\text{Irr}(H' \mid \alpha')$ and $\Psi^0$ of $\text{Irr}_\pi(H \mid \alpha^0)$ onto $\text{Irr}_\pi(H' \mid \alpha')$ such that

(a) For any $\theta \in \text{Irr}_\pi(H \mid \alpha^0)$, if $\xi$ is any character in $\text{Irr}(H \mid \alpha)$ such that $\xi^0 = \theta$, we have $\Psi^0(\theta) = \Psi(\xi)^0$.

(b) For any $\chi \in \text{Irr}(H \mid \alpha)$ and any $\theta \in \text{Irr}_\pi(H \mid \alpha^0)$, we have $d_{\chi, \theta} = d_{\Psi(\chi), \Psi^0(\theta)}$.

(c) The correspondence $B \mapsto \Psi(B)$ is a bijection of $\text{B}_\pi(H \mid \alpha)$ onto the set of $(Slattery)$ $\pi$-blocks of $H'$ over $\alpha'$.

(d) If $\theta \in \text{Irr}_\pi(H \mid \alpha^0)$, then $\theta$ has a vertex $Q$ such that $QM/M$ is isomorphic to some vertex $Q'$ of $\Psi^0(\theta)$.

**Proof.** This lemma without (d), is the invariant case of Theorem 3.1 in [10]. Recall, by the proof of that theorem, that the triple $(H', M', \alpha')$ is chosen to be isomorphic to $(H, M, \alpha)$. In other words, there exists a character-triple isomorphism $(\tau, \sigma): (H, M, \alpha) \rightarrow (H', M', \alpha')$. The bijection $\Psi$ is just the map $\sigma_H$ introduced just before the lemma. All we need now is to show (d).

Let $\theta \in \text{Irr}_\pi(H \mid \alpha^0)$. Then, there exists $\xi \in \text{B}_\pi(H)$ such that $\theta = \xi^0$. Since the irreducible constituents of $\xi_M$ are all in $\text{B}_\pi(M)$ (Corollary 7.5 in [6]) and $\theta$ lies over $\alpha^0$, it follows that $\xi$ lies over $\alpha$. Now, as $\alpha$ is $\pi'$-special, Lemma 1.2 in [13] says that there exists a nucleus $(K, \rho)$ of $\xi$ such that $M \subseteq K$ and $\rho \in \text{Irr}(K \mid \alpha)$ (see Section 4 of [6], for the definition of the nucleus). In particular, we have $\xi = \rho^H$ and hence $\theta = \xi^0 = (\rho^0)^H$. Moreover, since $\xi \in \text{B}_\pi(H)$, the character $\rho$ is $\pi'$-special. Therefore, a Hall $\pi$-subgroup $Q$ of $K$ is a vertex for $\theta$.

Next, by Lemma 11.35 in [5], we have

$$\Psi(\xi) = \Psi(\rho^H) = \sigma_H(\rho^H) = (\sigma_K(\rho))^H.$$
As $Ψ^0(θ) = Ψ(ξ)^0$ by (a), we get that $Ψ^0(θ) = (σ_{K}(ρ))^0H'$. Let $K'$ be the subgroup of $H'$ containing $M'$ such that $π(K/M) = K'/M'$. Now, since $Ψ^0(θ) ∈ I_π^+(H')$, it follows that $σ_{K}(ρ)^0 ∈ I_π^+(K')$.

By Lemma 11.24 in [5], we have $ρ(1)α(1) = σ_{K}(ρ)(1)α(1)$. Since $ρ(1)$, $α(1)$ and $α'(1)$ are all $π'$-numbers, we conclude that $σ_{K}(ρ)(1)$ is a $π'$-number. Hence, a Hall $π'$-subgroup $Q'$ of $K'$ is a vertex of $Ψ^0(θ)$.

Now, $QM/M$ is a Hall $π'$-subgroup of $K/M$ and $Q'M/M'$ is a Hall $π'$-subgroup of $K'/M'$. As $K/M ≅ K'/M'$, we obtain $QM/M ≅ Q'M/M' ≅ Q'/Q' ∩ M'$. Furthermore, since $Q'$ is a $π$-group and $M'$ is a $π'$-group, we get $Q' ∩ M' = 1$ and it follows that $QM/M ≅ Q'$. This proves (d) and completes the proof of the lemma.

We can now improve Theorem 3.1 of [11].

**Theorem 3.** Let $N$ be a normal subgroup of a $π$-separable group $G$ and let $μ ∈ Bπ+(N)$ with $T = I_π(μ)$. Then, there exist a central extension $U$ of $T = T/N$ by a $π'$-subgroup $Z$ of $U$, a linear character $ν$ of $Z$ and bijections $Γ$ of $Irr(G | μ)$ onto $Irr(U | ν)$ and $Γ^0$ of $I_π^+(G | μ^0)$ onto $I_π^+(U | ν)$ such that the following hold.

(a) For any $χ ∈ Irr(G | μ)$ and any $φ ∈ I_π^+(G | μ^0)$, we have $d_{χ φ} = d_{Γ(χ)}Γ^0(φ)$.

(b) The correspondence $B ↦ Γ(B)$ is a bijection of $Blπ(G | μ)$ onto the set of (Slattery) $π$-blocks of $U$ over $ν$.

(c) If $φ ∈ I_π^+(G | μ^0)$, then $φ$ has a vertex $Q$ such that $QN/N$ is isomorphic to some vertex $P$ of $Γ^0(φ)$.

Proof. Let $(W, γ)$ be a nucleus for $μ$ and let $S = N_T((W, γ))$, the stabilizer of $(W, γ)$ in $T$. First, we note that this theorem without (c), is Theorem 3.1 in [11]. Recall, by its proof, that Theorem 3.1 of [11] is obtained by first applying Theorem 3.2 in [11] to the group $G$, the normal subgroup $N$ and the character $μ ∈ Bπ+(N)$, and then applying the invariant form of Theorem 3.1 in [10] (this is Lemma 2 above without (d)) to the group $S$, the normal subgroup $W$ and the $S$-invariant $π'$-special character $γ$ of $W$.

To complete the proof, we need to show (c). Let $φ ∈ I_π^+(G | μ^0)$. By Theorem 3.2 (b) in [11], there exists a partial character $θ ∈ I_π^+(S | γ^0)$ such that $φ = θ^G$. Now, $Γ^0(φ)$ is the element of $I_π^+(U | ν)$ corresponding to $θ$ via the bijection $Ψ^0$ of Lemma 2.

By Lemma 2 (d), $θ$ has a vertex $Q$ such that $QW/W$ is isomorphic to some vertex $P$ of $Γ^0(φ) = Ψ^0(θ)$.

Now, by Lemma 3.6 (a) of [11], we have $S ∩ N = W$. Therefore, we get $Q ∩ N = Q ∩ S ∩ N = Q ∩ W$. Since $QN/N ≅ Q/Q ∩ N$ and $QW/W ≅ Q/Q ∩ W$, it follows that $QN/N ≅ QW/W ≅ P$. Finally, note that the subgroup $Q$ is also a vertex of $φ$ as $φ = θ^G$. This proves (c) and finishes the proof of the theorem.
Let \((H,M,\alpha)\) be a character-triple, where \(H\) is a \(\pi\)-separable group and \(\alpha\) is \(\pi'\)-special, and let \(B\) be a relative \(\pi\)-block of \(H\) with respect to \((M,\alpha)\).

Let \(R\) be the normal subgroup of \(H\) containing \(M\) such that \(R/M = \text{O}_{\pi'}(H/M)\). If \(\zeta \in \text{Irr}(H \mid \alpha)\), then by Lemma 2.3 in [5], there exists a \(\pi'\)-special character \(\delta\) of \(R\) such that \(\delta\) is a constituent of \(\zeta_R\). By Lemma 3.2 in [10], if \(\beta \in \text{B}_{\pi'}(H)\) satisfies \(d_{\zeta\beta} \neq 0\), we have \(\beta \in \text{Irr}(H \mid \delta)\). Therefore, the constituents of \(\beta_R\) are precisely the constituents of \(\zeta_R\) by Clifford’s theorem (Theorem 6.2 in [5]). It follows that if \(\zeta' \in \text{Irr}(H \mid \alpha)\) also satisfies \(d_{\zeta'\beta} \neq 0\), then \(\zeta'\) also lies over the \(H\)-orbit of \(\delta\). This implies that the characters of \(B\) all lie over the \(H\)-orbit of some \(\pi'\)-special character \(\eta\) of \(R\). So, there exists a relative \(\pi\)-block \(B_0\) of \(H\) with respect to \((R,\eta)\) such that \(B \subseteq B_0\). Assume now that \(\xi \in B\) and \(\xi_0 \in B_0\) satisfy \(d_{\xi\omega} \neq 0\) and \(d_{\xi_0\omega} \neq 0\) for some \(\omega \in \text{Irr}(H)\). Then, as \(\eta\) lies over \(\alpha\), the character \(\xi_0\) lies over \(\alpha\) and it follows that \(\xi_0 \in B\). Consequently, \(B = B_0\) and thus we may view \(B\) as a relative \(\pi\)-block of \(H\) with respect to \((R,\eta)\).

We now have the following “Fong reduction” type result.

**Lemma 4.** Let \(N \triangleleft G\), where \(G\) is \(\pi\)-separable and let \(\mu \in B_{\pi'}(N)\). If \(B \in \text{Bl}_{\pi'}(G \mid \mu)\), then there exist a subgroup \(A\) of \(G\), a normal subgroup \(E\) of \(A\) satisfying \(O_{\pi'}(A/E) = 1\) and an \(A\)-invariant \(\pi'\)-special character \(\beta\) of \(E\) such that induction defines a bijection of \(\text{Irr}(A \mid \beta)\) onto \(B\). Furthermore, if \(D\) is a Hall \(\pi\)-subgroup of \(A\), then \(D\) is a defect group of \(B\) and \(DN/N\) is isomorphic to \(DE/E\).

*Proof.* Let \((W,\gamma)\) be a nucleus for \(\mu\) and let \(S = N_T((W,\gamma))\), where \(T = \text{I}_G(\mu)\). (Note that \(\gamma\) is \(\pi'\)-special as \(\mu \in B_{\pi'}(N)\)). By Theorem 3.2 of [11], there exists a relative \(\pi\)-block \(B_0 \in \text{Bl}_{\pi}(S \mid \gamma)\) such that the induction map \(\alpha \mapsto \alpha^G\) defines a bijection of \(B_0\) onto \(B\). Let now \(P\) be any defect group of \(B_0\). Then, \(P\) is also a defect group of \(B\) (see Section 4 in [11]). Since \(P \subseteq S\), we have \(P \cap N = P \cap W\), by Proposition 4.1 in [11] and it follows that \(PN/N \cong PW/W\).

Now, to complete the proof, we may therefore assume that \(\mu\) is a \(G\)-invariant \(\pi'\)-special character.

Let \(R\) be the normal subgroup of \(G\) containing \(N\) such that \(R/N = \text{O}_{\pi'}(G/N)\). By the discussion preceding the lemma, there exists a \(\pi'\)-special character \(\eta\) of \(R\) lying over \(\mu\) such that \(B \in \text{Bl}_{\pi}(G \mid \eta)\). Next, let \(J = \text{I}_G(\eta)\). Then, by Lemma 3.4 in [10], there is a unique relative \(\pi\)-block \(\mathcal{B}\) of \(J\) with respect to \((R,\eta)\) such that the induction map \(\chi \mapsto \chi^G\) is a bijection of \(\mathcal{B}\) onto \(B\).

**Case 1.** Assume \(J = G\). Then, the character \(\eta\) is \(G\)-invariant and so by Lemma 2, there exist a central extension \(G'\) of \(\mathcal{G} = G/R\) by a \(\pi'\)-subgroup \(R'\) of \(G'\), a linear character \(\eta'\) of \(R'\) and a bijection \(\Psi\) of \(\text{Irr}(G \mid \eta)\) onto \(\text{Irr}(G' \mid \eta')\) such that the correspondence \(b \mapsto \Psi(b)\) is a bijection of \(\text{Bl}_{\pi}(G \mid \eta)\) onto the set of (Slattery) \(\pi\)-blocks of \(G'\) over \(\eta'\).

Since \(G/R \cong G'/R'\) and \(O_{\pi'}(G/R) = 1\), we get that \(O_{\pi'}(G'/R') = 1\), and hence...
By Theorem 2.8 in [12], \( G' \) has a single \( \pi \)-block over \( \eta' \). It follows that \( \text{Irr}(G \mid \eta) \) consists of the single relative \( \pi \)-block \( B \). We can then take \( A = G \), \( E = R \) and \( \beta = \eta \).

By the definition of defect groups (see Section 4 in [10]), if \( D \) is a Hall \( \pi \)-subgroup of \( G \), then \( D \) is a defect group for \( B \). Next, we show that \( DN/N \cong DR/R \). Since \( DN/N \cong D/D \cap N \) and \( DR/R \cong D/D \cap N \), it suffices to show that \( D \cap N \) is also a Hall \( \pi \)-subgroup of \( R \). Moreover, if \( D \) is a Hall \( \pi \)-subgroup of \( A \), then \( D \) is a defect group of \( B \) and \( DR/R \cong DE/E \).

Now, since induction of characters defines a bijection of \( \hat{B} \) onto \( B \), the induction map \( \theta \mapsto \theta^G \) is a bijection of \( \text{Irr}(A \mid \beta) \) onto \( B \). Next, by the definition of defect groups (Section 4 of [10]), the subgroup \( D \) is a defect group of \( B \). Furthermore, by Lemma 4.1 in [10], \( D \cap N \) is a Hall \( \pi \)-subgroup of \( N \). Now, just as in case 1, it follows that \( DN/N \cong DR/R \), and consequently \( DN/N \cong DE/E \). This completes the proof of the lemma.

**Lemma 5.** Let \( \theta \in B_2^\ast(G) \), where \( G \) is solvable and let \( Q \) be a vertex of \( \varphi = \theta^0 \). Suppose that \( N \) is a normal subgroup of \( G \) such that \( Q \not\subseteq N \) and \( QN/N \) is cyclic, and let \( \mu \) be an irreducible constituent of \( \theta_N \). Then, \( \mu \in B_2^\ast(N) \) and if \( \theta \in B \in B_2^\ast(G \mid \mu) \), we have \( QN/N \cong DN/N \), for any defect group \( D \) of \( B \). Furthermore, \( B = \{ \chi \in \text{Irr}(G \mid \mu) : \chi^0 = \varphi \} \) and the number of elements of \( B \) is exactly \( |QN/N| \).

**Proof.** Since \( \theta \in \text{Irr}(G \mid \mu) \), the character \( \mu \) lies in \( B_2^\ast(N) \) by Corollary 7.5 in [6]. Consequently, we have \( \mu^0 \in I_2^\ast(N) \) and \( \varphi \in I_2^\ast(G \mid \mu^0) \).

By Theorem 3, there exist a solvable group \( U \), a \( 2' \)-subgroup \( Z \subseteq Z(U) \), a linear character \( \nu \) of \( Z \) and bijections \( \Gamma \) of \( \text{Irr}(G \mid \mu) \) onto \( \text{Irr}(U \mid \nu) \) and \( \Gamma^0 \) of \( I_2^\ast(G \mid \mu^0) \) onto \( I_2^\ast(U \mid \nu) \). Furthermore, \( \omega = \Gamma^0(\varphi) \) has a vertex \( P \) such that \( P \cong QN/N \).

Let \( b = \Gamma(B) \). By Theorem 3 (b), \( b \) is a 2-block of \( U \) over \( \nu \). Moreover, \( \omega \) is an irreducible Brauer character associated with \( b \). As \( QN/N \) is cyclic, the vertex \( P \) is cyclic and it follows by Theorem VII.15.1 in [2], that \( b \) has \( P \) as a defect group.

Let now \( D \) be any defect group for \( B \). Then, by Theorem 4.2 in [11], we have \( D/D \cap N \cong P \). Since \( QN/N \cong P \), it follows that \( QN/N \cong DN/N \).

Next, by the theory of blocks with cyclic defect groups (Theorem 68.1 in [1]), \( \omega \) is the unique irreducible Brauer character associated with \( b \) and there are exactly \( |P| = |QN/N| \) ordinary irreducible characters \( \lambda \) in \( b \). Furthermore, every character \( \lambda \) lies over \( \nu \) and satisfies \( \lambda^0 = \omega \). In particular, we have \( b = \{ \eta \in \text{Irr}(U \mid \nu) : \eta^0 = \omega \} \).
It follows from Theorem 3 (a) that \( B = \Gamma^{-1}(b) = \{ \chi \in \text{Irr}(G \mid \mu) : \chi^0 = \varphi \} \), and consequently, the number of elements of \( B \) is equal to the number \( |QN/N| \) of elements in \( b \). The proof of the lemma is now complete.

We are almost ready to start the proof of Theorem 1. All we need now are few general facts about \( p \)-rational characters.

Let \( G \) be any finite group and for an integer \( h \geq 1 \), denote by \( Q_h \) the field \( \Q(e^{2\pi i/h}) \) generated by \( e^{2\pi i/h} \) over the field \( \Q \) of rationals. Now, fix a prime \( p \) and write \( |G| = l = p^a k \), where \((k, p) = 1\). Let \( \mathcal{C} \) denote the Galois group \( \text{Gal}(\Q_l/\Q_k) \). If \( \chi \) is a character (resp. irreducible character) of \( G \) and if \( \sigma \in \mathcal{C} \), the function \( \chi^\sigma \) defined by \( \chi^\sigma(x) = \chi(x)\sigma \) is also a character (resp. irreducible character) of \( G \). It is clear from the definition of \( p \)-rational characters given in the introduction of this paper, that \( \chi \) is \( p \)-rational if and only if \( \chi^\sigma = \chi \) for all \( \sigma \in \mathcal{C} \).

Next, let \( H \) be a subgroup of \( G \). Then, \( Q_{|H|} \subseteq Q_l \) and it follows that \( \theta^\sigma \) is defined for every character \( \theta \) of \( H \) and every \( \sigma \in \mathcal{C} \).

We can now prove our main result.

**Proof of Theorem 1.** Let \( \theta \) be the element of \( B_2(G) \) such that \( \theta^0 = \varphi \), and fix an irreducible constituent \( \mu \) of \( \theta_N \). By Lemma 5, \( \mu \in B_2(N) \) and if \( \theta \in B \in B_2(G \mid \mu) \), we have, \( B = \{ \chi \in \text{Irr}(G \mid \mu) : \chi^0 = \varphi \} \), and the number of elements of \( B \) is \( |QN/N| \). This suffices to prove (i). Next, we prove (ii).

By Lemma 4, there exist subgroups \( E \triangleleft A \subseteq G \) satisfying \( O_2^-(A/E) = 1 \) and an \( A \)-invariant \( 2' \)-special character \( \beta \) of \( E \) such that induction defines a bijection of \( \text{Irr}(A \mid \beta) \) onto \( B \). Furthermore, if \( D \) is a Sylow \( 2 \)-subgroup of \( A \), then \( D \) is a defect group of \( B \) and \( DE/E \cong DN/N \). By Lemma 5, we have \( DN/N \cong QN/N \). Therefore, \( DE/E \cong QN/N \) and it follows that \( DE/E \) is a cyclic \( 2 \)-group. Moreover, as \( Q \not\cong N \), we have that \( |DE/E| > 1 \).

Next, write \( |G| = 2^r k \) and \( |A| = h = 2^s m \), where both \( k \) and \( m \) are odd integers. Assume that a character \( \zeta \in \text{Irr}(A \mid \beta) \) is \( 2 \)-rational. Then, the values of \( \zeta \) are in \( Q_m \). Since \( m \) divides \( k \), the values of \( \zeta \) lie in \( Q_k \) and it follows that the values of the character \( \zeta^G \) all lie in \( Q_k \). In other words, \( \zeta^G \) is \( 2 \)-rational. Therefore, to show (ii), it suffices to find two \( 2 \)-rational characters in \( \text{Irr}(A \mid \beta) \).

Recall that \( D \) is a Sylow \( 2 \)-subgroup of \( A \). Then, \( DE/E \) is a Sylow \( 2 \)-subgroup of \( A/E \). Since \( O_2^-(A/E) = 1 \) and \( DE/E \) is cyclic, it follows from Theorem 6.3.3 in [4], that \( DE/E \triangleleft A/E \). Now, let \( R \) be the subgroup of \( A \) such that \( R/E \) is the unique subgroup of \( DE/E \) of index 2. As \( DE/E \triangleleft A/E \), it is clear that \( R/E \triangleleft A/E \), and so \( R \triangleleft A \).

Since \( \beta \) is \( A \)-invariant, Corollary 4.8 in [3] implies that there exists a \( 2' \)-special character \( \zeta_0 \in \text{Irr}(A \mid \beta) \). Set \( (\zeta_0)^0 = \omega \). Next, fix an irreducible constituent \( \eta \) of \( (\zeta_0)_R \) and write \( S = \{ \lambda \in \text{Irr}(A \mid \eta) : \lambda^0 = \omega \} \). As \( \zeta_0 \) is \( 2' \)-special, \( D \) is a vertex of \( \omega \). Moreover, since \( DR/R \) is cyclic of order 2, Lemma 5 says that \( S \) contains exactly 2
elements, the character $\zeta_0$, of course, and another character $\zeta_1$.

As $\zeta_0 \in \text{Irr}(A \mid \beta)$ and $\beta$ is $A$-invariant, we have $\eta \in \text{Irr}(R \mid \beta)$, and consequently $S \subseteq \text{Irr}(A \mid \beta)$. So now, to complete the proof, it suffices to show that the characters $\zeta_0$ and $\zeta_1$ are 2-rational. First, note that $\zeta_0$ is 2-rational by Lemma 3.1 in [8]. Next, we prove that $\zeta_1$ is 2-rational.

Write $(\zeta_1)_R = \sum_{i=1}^{n} \eta_i$, where $\eta_i \in \text{Irr}(R)$ and $\eta_1 = \eta$. Now, let $\sigma \in \text{Gal}(Q_4/Q_m)$. Then, for each $i$, $\eta_i^\sigma$ is well defined (see the remarks preceding the proof) and $(\zeta_1^\sigma)_R = \sum_{i=1}^{n} \eta_i^\sigma$. Since $\zeta_0$ is 2'-special, then so is $\eta$ by Lemma 2.2 in [6]. Hence, $\eta$ is 2-rational by Lemma 3.1 in [8]. In other words, the values of $\eta$ are in $Q_4$, where $l$ is the order of a Hall 2'-subgroup of $R$. As $l$ divides $m$, we have $Q_4 \subseteq Q_m$ and it follows that $\eta_l = \eta$. This shows that $\zeta_1^\sigma \in \text{Irr}(A \mid \eta)$. Next, we have $(\zeta_1^\sigma)^0 = \omega$. So, clearly $(\zeta_1^\sigma)^0 = \omega$ and we conclude that $\zeta_1^\sigma \in S$. Now, as $S = \{\zeta_0, \zeta_1\}$ and $\zeta_0$ is 2-rational, we have $\zeta_1^\sigma = \zeta_1$, necessarily. Hence, $\zeta_1$ is 2-rational, as wanted.

References


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