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REMARKS ON TAUBERIAN THEOREM OF EXPONENTIAL TYPE AND FENCHEL-LEGENDRE TRANSFORM

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1. Introduction

Let $U(x)$, $x \geq 0$ be a nondecreasing right-continuous function such that $U(0) = 0$. The asymptotics of U and its Laplace-Stieltjes transform $\omega(s) = \int_0^\infty e^{-sx}\,dU(x)$ are closely linked and results in which we pass from $U(x)$ to $\omega(s)$ are called Abelian theorems and ones in converse direction are called Tauberian, and they play a very important role in probability theory. A most well-known result on this subject is Karamata's theorem (cf. Chapter 1 of [1]). Also the cases when $\omega(s)$ and $U(x)$ vary exponentially are treated by many authors (e.g. [2], [3], [4], [8], [9]. See also Chapter 4 of [1]). Among them [2] studied the relationship between the limit of $(1/\lambda)\log U(1/\phi(\lambda))$ as $\lambda \to \infty$ and that of the Laplace-Stieltjes transform modified as

$$\frac{1}{\lambda}\log\int_0^\infty e^{-\lambda\phi(\lambda)x}\,dU(x) \tag{1.1}$$

with regularly varying $\phi(\lambda) = \lambda^\alpha L(\lambda)$. Now notice here that (1.1) can be rewritten as

$$\frac{1}{\lambda}\log\int_0^\infty e^{-\lambda x}\,dU_\lambda(x),$$

where $U_\lambda(x) = U(x/\phi(\lambda))$. Thus we arrive at the following question, which may be regarded as a problem of large deviation. Consider a family of functions $\{U_\lambda(x)\}_{\lambda\geq 1}$ instead of a fixed $U(x)$. Then what is the relationship between the limiting behaviour of $(1/\lambda)\log U_\lambda(x)$ and that of

$$\varphi_\lambda(s) := \frac{1}{\lambda}\log\int_0^\infty e^{-\lambda s x}\,dU_\lambda(x)\,?$$

This problem was studied by one of the authors (see [5]) and the present paper is its continuation: Our aim here is to check if each of the assumptions of the previous paper is essential or not and to remove unnecessary conditions. To state our results we start with reviewing the main result of the previous paper: Throughout the paper $U_\lambda\colon [0,\infty) \to \mathbb{R}$ are nondecreasing, right-continuous functions with $U_\lambda(0) = 0$ such that $\int_0^\infty e^{-\lambda s x}\,dU_\lambda(x) < \infty$, for all $s > 0$, $\lambda \geq 1$. Then the main result of [5] (see also its correction) can be rewritten as follows.

Theorem A ([5]). *Suppose that the following limit exists for every* $s > 0$:

$$\varphi(s) := \lim_{\lambda\to\infty} \frac{1}{\lambda} \log \int_0^\infty e^{-\lambda s x}\, dU_\lambda(x). \tag{1.2}$$

If $\varphi(s)$, $s > 0$ *is a continuously differentiable, strictly convex function satisfying* $\lim_{s\to 0} \varphi'(s) = -\infty$, $\lim_{s\to\infty} \varphi'(s) = 0$, *then*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log U_\lambda(x) = \inf_{s>0}\{\varphi(s) + sx\}, \qquad \text{for every} \quad x > 0. \tag{1.3}$$

Here we remark that [5] did not discuss much on the assumptions on the strict convexity of φ or on the continuity of φ', since it proved the above theorem in order to study the asymptotic behaviour of multiple convolutions and there φ, which was given explicitly in advance, was a smooth function. However, sometimes it is difficult to check these conditions in the general cases where φ is not given explicitly, and therefore, it would be convenient if we could remove such conditions. For example, if this were possible, the case of the 'limit on oscillations' (see [6]) could easily be reduced to Theorem A and hence could be improved greatly.

The main results of the present paper are as follows: We shall see that the assumption in Theorem A on the strict convexity of φ is inessential and may be removed (Theorem 2.3). On the other hand, however, the continuity of the derivative φ' is indispensable: If φ' is discontinuous, then (1.3) does not hold in general. (For a counter example, see Example 2.2). In such cases we shall study what can be said about the upper and the lower bounds, which are in a sense best possible (Theorem 2.4).

2. Main Results

We start with an Abelian theorem. We omit the proof since there is no need to modify that of Theorem 1 of [5].

Theorem 2.1. *Let* $f(x)\colon (0,\infty) \mapsto [-\infty,\infty)$ *be a nondecreasing function, and suppose*

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log \int_0^\infty e^{-\lambda s x}\, dU_\lambda(x) < \infty, \qquad \text{for all} \quad s > 0. \tag{2.1}$$

If

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log U_\lambda(x) = f(x), \qquad \text{at all continuity points of} \quad f(x), \tag{2.2}$$

then

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\int_0^\infty e^{-\lambda sx}\,dU_\lambda(x)=\sup_{x>0}\{f(x)-sx\},\qquad \textit{for all}\quad s>0. \tag{2.3}$$

We now turn to the Tauberian part. Our question is whether (2.3) implies (2.2) or not. Obviously the answer is 'no' without further assumptions because of the properties of the Fenchel-Legendre transform. Facts on Fenchel-Legendre transform which we quote in this paper are

Theorem 2.2. *Let $f(x)$ be a nondecreasing function on $(0,\infty)$ and define*

$$\varphi(s):=\sup_{x>0}\{f(x)-sx\},\quad (s>0), \tag{2.4}$$

and

$$\varphi^*(x):=\inf_{s>0}\{\varphi(s)+sx\},\quad (x>0). \tag{2.5}$$

Then
(i) $f(x)\le\varphi^*(x)$, *for every* $x>0$.
(ii) *If* $f(x)$ *is concave, then* $f(x)=\varphi^*(x)$ *for every* $x>0$.
(iii) φ^* *is the concave hull of* f.
(iv) *Suppose* φ *is continuously differentiable and satisfies* $\lim_{s\to0}\varphi'(s)=-\infty$, *and* $\lim_{s\to\infty}\varphi'(s)=0$. *Then* $\varphi^*(x)$ *is strictly concave on* $(0,\infty)$ *and* $f(x)=\varphi^*(x)$ *for every* $x>0$.

Since the proof of (i) is easy and that of (iv) can be reduced to Theorem 26.3 of [10], we omit them. We prove (ii) and (iii) in the next section.

In Theorem 2.2 (iv), the assumption that φ has continuous derivative is essential and in fact if φ is not continuously differentiable, then $\varphi^*(x)=\inf_{s>0}\{\varphi(s)+sx\}$ does not necessarily coincide with f (see Example 2.2). If we keep these facts in mind, we shall see that the assumptions of the following theorem, which is an extension of Theorem A, are reasonable.

Theorem 2.3. *Suppose that the following limit exists for every* $s>0$:

$$\varphi(s):=\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\int_0^\infty e^{-\lambda sx}\,dU_\lambda(x). \tag{2.6}$$

If $\varphi(s)$, $s>0$ *is a continuously differentiable function satisfying* $\lim_{s\to0}\varphi'(s)=-\infty$ *and* $\lim_{s\to\infty}\varphi'(s)=0$, *then*

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log U_\lambda(x)=\varphi^*(x),\qquad \textit{for every}\quad x>0, \tag{2.7}$$

where $\varphi^*(x) = \inf_{s>0}\{\varphi(s) + sx\}$.

We next study what can be said when φ does not have continuous derivative.

Theorem 2.4. *Suppose that* (2.6) *exists for every* $s > 0$ *and let* φ^* *be as in* Theorem 2.3. *Then* φ^* *is a nondecreasing concave function, and for every* $x > 0$ *and* x_0 $(0 < x_0 \le x)$ *such that* φ^* *is strictly concave at* x_0, *it holds that*

$$\varphi^*(x_0) \le \liminf_{\lambda\to\infty} \frac{1}{\lambda} \log U_\lambda(x) \le \limsup_{\lambda\to\infty} \frac{1}{\lambda} \log U_\lambda(x) \le \varphi^*(x). \tag{2.8}$$

REMARK. Let $\{\mu_\lambda(dx)\}_\lambda$ be a family of Radon measures on $(0,\infty)$, and let $\omega_\lambda(s) = \int_0^\infty e^{sx} d\mu_\lambda(x)$. Then, we can similarly show the relationship between the asymptotic behaviour of $\log \omega_\lambda(\lambda s)$ and that of $\log \mu_\lambda(x,\infty)$.

We postpone the proofs of the above theorems until the next section and conclude this section with examples.

EXAMPLE 2.1. Let

$$f_1(x) = \begin{cases} \sqrt{x}, & \text{if } 0 < x < 1 \text{ or } 4 \le x, \\ 1, & \text{if } 1 \le x < 4, \end{cases}$$

and

$$f_2(x) = \begin{cases} \sqrt{x}, & \text{if } 0 < x < 1 \text{ or } 4 \le x, \\ x/3 + 2/3, & \text{if } 1 \le x < 4. \end{cases}$$

Notice that f_2 is the concave hull of f_1. Then it holds

$$\sup_{x>0}\{f_1(x) - sx\} = \sup_{x>0}\{f_2(x) - sx\} = \varphi(s),$$

where

$$\varphi(s) = \begin{cases} 1/(4s), & \text{if } 0 < s < 1/4 \text{ or } 1/2 \le s, \\ 2 - 4s, & \text{if } 1/4 \le s < 1/3, \\ 1 - s, & \text{if } 1/3 \le s < 1/2. \end{cases}$$

This example illustrates that $\varphi := \sup_{x>0}\{f(x) - sx\}$ does not determine f uniquely without further conditions on φ.

EXAMPLE 2.2 (continued). Let $U_\lambda^{(k)}(x) = \exp\{\lambda f_k(x)\} - 1$, $(k = 1,\ 2)$. Then by Theorem 2.1 both $U_\lambda^{(1)}$ and $U_\lambda^{(2)}$ satisfy

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \int_0^\infty e^{-\lambda sx}\, dU_\lambda^{(k)}(x) = \varphi(s), \quad (k = 1,\ 2) \quad \text{for all} \quad s > 0.$$

By an easy calculus we have

$$\varphi^*(x)\left(:=\inf_{s>0}\{sx+\varphi(s)\}\right)=f_2(x).$$

However, on the other hand, we easily see

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log U_\lambda^{(k)}(x)=f_k(x)\quad(k=1,\ 2)\qquad\text{for every}\qquad x>0.$$

Thus (2.6) does not necessarily imply (2.7) and this is an example which shows that the assumption that $\varphi\in C^1$ is essential in Theorem 2.3.

EXAMPLE 2.3 (continued). Notice that f_2 is strictly concave on $(0,1)\cup(4,\infty)$, and for $1\le x\le 4$, the supremum x_0 ($x_0\le x$) of the strictly concave points of φ^* is 1. Thus we can choose $f_1(x)$ for the extreme left side of (2.8). Therefore, in this case the assertion of Theorem 2.4 can be rewritten as follows: If

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\int_0^\infty e^{-\lambda sx}\,dU_\lambda(x)=\varphi(s),\qquad\text{for all}\qquad s>0,$$

then

$$f_1(x)\le\liminf_{\lambda\to\infty}\frac{1}{\lambda}\log U_\lambda(x)\le\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log U_\lambda(x)\le f_2(x),\qquad\text{for every}\qquad x>0. \tag{2.9}$$

Since the lower bound is attained by $U_\lambda^{(1)}$ and the upper by $U_\lambda^{(2)}$ as we have seen in Example 2.2, we cannot improve the above inequality.

3. Proofs

In this section, we give proofs of Theorems 2.2 (ii), (iii), 2.3, and 2.4.

Proof of Theorem 2.2. (ii) In the special case where $D_+f(x)\ne 0$ for every $x>0$, for a given $x>0$, put $s_0=D_+f(x)$. Then x attains $\varphi(s_0)$ in (2.4), i.e., $\varphi(s_0)=f(x)-s_0x$. Thus, we have

$$\varphi^*(x)\le\varphi(s_0)+s_0x=f(x),$$

and combining with Theorem 2.2 (i), we have $f(x)=\varphi^*(x)$. In the general case when $D_+f(x)$ may vanish, approximate $f(x)$ by $f_\varepsilon(x)=f(x)+\varepsilon(1-e^{-x})$, ($\varepsilon>0$).
(iii) We can easily see that φ^* is a nondecreasing concave function on $(0,\infty)$ (see [5]), and from (i), $\varphi^*(x)\ge f(x)$ for every $x>0$. Next, let $g(x)$ be a concave function on $(0,\infty)$, such that $f(x)\le g(x)$ for every $x>0$, and show that

$\varphi^*(x) \le g(x)$ for every $x > 0$. Now, define $\tilde{\varphi}(s) = \sup_{x>0}\{g(x) - sx\}$, and $\tilde{\varphi}^*(x)$ in the obvious manner. Then $\varphi(s) \le \tilde{\varphi}(s)$ for every $s > 0$, which implies $\varphi^*(x) \le \tilde{\varphi}^*(x)$. From (ii), we see $\tilde{\varphi}^*(x) = g(x)$, which proves the assertion. □

Proof of Theorems 2.3 and 2.4. Let x_0 be a strictly concave point of $\varphi^*(x)$, and let

$$a = \liminf_{\lambda\to\infty} \frac{1}{\lambda} \log U_\lambda(x_0).$$

We can choose a subsequence $\{\lambda_j\} \subset \{\lambda\}$ such that

$$\lim_{\lambda_j\to\infty} \frac{1}{\lambda_j} \log U_{\lambda_j}(x_0) = a.$$

Choosing a subsequence if necessary, we may and do assume that for some nondecreasing function $f(x)$,

$$\lim_{\lambda_j\to\infty} \frac{1}{\lambda_j} \log U_{\lambda_j}(x) = f(x),$$

at all continuity points of $f(x)$. Then from Theorem 2.1, we have

$$\varphi(s) = \lim_{\lambda_j\to\infty} \frac{1}{\lambda_j} \log \int_0^\infty e^{-\lambda_j sx}\, dU_{\lambda_j}(x) = \sup_{x>0}\{f(x) - sx\},$$

and from Theorem 2.2 (iii), we see

$$\varphi^*(x_0) = f(x_0) \ (= a).$$

Thus, for any $x \ge x_0$,

$$f(x) \ge f(x_0) = \varphi^*(x_0),$$

which gives the lower bound of (2.8), while (i) gives the upper one. Combining Theorem 2.2 (iv), we obtain Theorem 2.3 as a corollary of Theorem 2.4. □

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