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## ON LOCAL RIGHT PURE SEMISIMPLE RINGS OF LENGTH TWO OR THREE

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### 1. Introduction

We investigate in this paper non-commutative local rings  $R$  of the smallest length that are potential counter-examples to the pure semisimplicity conjecture.

Throughout the paper  $R$  is an associative ring with an identity element. We call  $R$  *local*, if the Jacobson radical  $J(R)$  of  $R$  is a two-sided maximal ideal. We denote by  $\text{mod}(R)$  the category of finitely generated right  $R$ -modules. Given a right  $R$ -module  $X_R$  of finite length we denote by  $l(X_R)$  the length of  $X_R$ .

We recall that a ring  $R$  is said to be of *finite representation type*, if  $R$  is artinian and the number of the isomorphism classes of finitely generated indecomposable right (and left)  $R$ -modules is finite. Following [24] we call a ring  $R$  *right pure semisimple*, if every right  $R$ -module is a direct sum of finitely generated modules.

It is well known that a ring  $R$  is of finite representation type if and only if  $R$  is right pure semisimple and  $R$  is left pure semisimple (see [2], [11], [18], [22]–[24]). It is still an open question, called the *pure semisimplicity conjecture*, if a right pure semisimple ring  $R$  is of finite representation type (see [2] and [24], [25], [28]). In [13] the question is answered in affirmative for rings  $R$  satisfying a polynomial identity and for self-injective rings  $R$  (see also [7], [19] and [31]). The reader is referred to [42] and to the author's expository papers [30] and [32] for a basic background and historical comments on the pure semisimplicity conjecture.

It was shown by the author in [28] and [33] that there is a chance to find a counter-example  $R$  to the pure semisimplicity conjecture and  $R$  might be hereditary with two simple non-isomorphic modules. The existence of a counter-example depends on a generalized Artin problem on division ring extensions.

In the present paper we are mainly interested in the existence of counter-examples  $R$  to pure semisimplicity conjecture that are local of the smallest length, that is, of length  $l(R_R)$  two or three. This continues our study started in [28], [35] and [33].

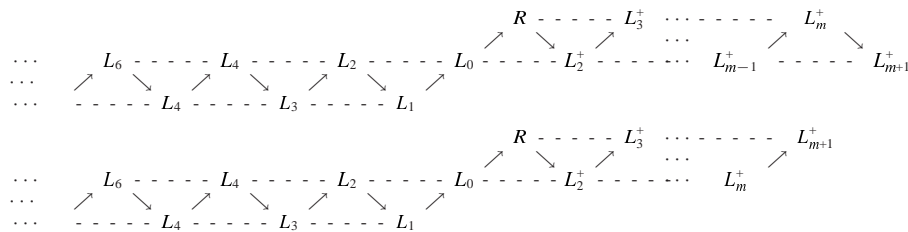
It is shown in Lemma 3.1 that every such a local ring  $R$  has  $J(R)^2 = 0$ . Therefore we study representation-infinite right pure semisimple local rings  $R$  with  $J(R)^2 = 0$  such that the Auslander-Reiten quiver  $\Gamma(\text{mod } R)$  is of the form  $\cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow$

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$\cdots \rightarrow \bullet \rightarrow \bullet$  by applying our recent results on right pure semisimple hereditary rings and generalized Artin problem on division ring extensions obtained in [33] and [36].

Assume that  $R$  is a local ring such that the square of the Jacobson radical  $J = J(R)$  of  $R$  is zero. Then  $F = R/J$  is a division ring and  $J$  is an  $F$ - $F$ -bimodule. By applying the results of [33] and [36] we show in Theorem 3.4 that the Auslander-Reiten quiver  $\Gamma(\text{mod } R)$  is connected of the form  $\cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \bullet$  if and only if the infinite dimension-sequence  $\mathbf{d}_{-\infty}({}_F J_F)$  of the  $F$ - $F$ -bimodule  ${}_F J_F$  (see Section 2) belongs to the set  $\mathcal{DS}_{\text{pss}}$  (of cardinality  $2^{\aleph_0}$ ) of infinite pure semisimple dimension-sequences  $v = (\dots, v_{-s}, v_{-s+1}, \dots, v_{-2}, v_{-1}, v_0, \infty)$  with  $v_j \in \mathbb{N}$  constructed in [33] (see also Section 2). In this case, we show that the Auslander-Reiten translation quiver  $\Gamma(\text{mod } R)$  of the category  $\text{mod}(R)$  is connected and has any of the forms (see (3.5) and (3.6))



Moreover, the infinite Jacobson radical  $\text{rad}^\infty(\text{mod } R) = \bigcap_{m=1}^\infty \text{rad}^m(\text{mod } R)$  of the category  $\text{mod}(R)$  is non-zero and it is generated by all  $R$ -module homomorphisms from the ring  $R$  to  $L_m$ , for  $m = 0, 1, 2, \dots$ . The square  $(\text{rad}^\infty(\text{mod } R))^2$  of  $\text{rad}^\infty(\text{mod } R)$  is zero.

For the notion of the Auslander-Reiten translation quiver the reader is referred to [3] and [27].

In particular, Theorem 3.4 shows how potential local counter-examples  $R$  to the pure semisimplicity conjecture of length  $l(R_R)$  two or three should look like, if the Auslander-Reiten quiver of  $\text{mod}(R)$  is connected of the following form

$$\cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \bullet.$$

The main results of the paper are presented in Section 3, where also two open problems are formulated. In Section 2 we collect preliminary facts and notation we need in this paper.

Throughout this paper we use freely the terminology and notation introduced in [28] and [33]. The reader is referred to [3] and [27] for a background and terminology on representation theory of finite dimensional algebras and artinian rings.

By the Auslander-Reiten quiver of the category  $\text{mod}(R)$  we mean the oriented graph  $\Gamma(\text{mod}(R))$  whose vertices are the isomorphism classes  $[X]$  of indecomposable modules  $X$  in  $\text{mod}(R)$  and there exists an arrow  $[X] \rightarrow [Y]$  in  $\Gamma(\text{mod}(R))$  if and only if there exists an irreducible morphism  $X \rightarrow Y$  in  $\text{mod}(R)$  (see [3], [27]). Usually we identify the isomorphism class  $[X]$  in  $\Gamma(\text{mod } R)$  with the module  $X$  in  $\text{mod}(R)$ . Sometimes we view  $\Gamma(\text{mod } R)$  as a translation quiver (see [3, Section VII.4] and [27,

11.49]). In this case we draw a dashed edge between indecomposable modules  $X$  and  $Z$  in  $\Gamma(\text{mod } R)$  if there exists an almost split sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ .

The Jacobson radical  $\text{rad}_R = \text{rad}(\text{mod } R)$  of the category  $\text{mod}(R)$  is the two-sided ideal of the category  $\text{mod}(R)$  such that  $\text{rad}_R(X, Y)$  consists of all non-invertible elements of the group  $\text{Hom}_R(X, Y)$  for each pair of indecomposable modules  $X$  and  $Y$  in  $\text{mod}(R)$  (see [3] and [27]). The two-sided ideal

$$\text{rad}^\infty(\text{mod } R) = \bigcap_{j=0}^\infty \text{rad}^j(\text{mod } R)$$

of the category  $\text{mod}(R)$  is called the infinite Jacobson radical of  $\text{mod}(R)$ . The reader is referred to [32] and [37] for some applications of  $\text{rad}^\infty(\text{mod } R)$  in the representation theory of artinian rings.

Given two indecomposable modules  $X$  and  $Y$  in  $\text{mod}(R)$  we view the abelian group

$$\text{Irr}(X, Y) = \text{rad}_R(X, Y) / \text{rad}_R^2(X, Y)$$

as an  $\text{End}(Y)/J \text{End}(Y)$ - $\text{End}(X)/J \text{End}(X)$ -bimodule, and we call it a *bimodule of irreducible morphisms* from  $X$  to  $Y$  (see [27, Section 11.1]).

Some of the results of this paper were presented on the Yamaguchi Conference “The 32nd Symposium on Ring Theory and Representation Theory” in October 1999 (see [34]).

## 2. Bimodules and pure semisimple dimension sequences

We start this section by recalling from [33] some definitions and notation we need throughout this paper.

Assume that  $F$  and  $G$  are division rings and  ${}_F M_G$  is a non-zero  $F$ - $G$ -bimodule. We recall that the matrix ring

$$(2.1) \quad R_M = \begin{pmatrix} F & {}_F M_G \\ 0 & G \end{pmatrix}$$

is hereditary and the modules  $X$  in  $\text{mod}(R_M)$  can be identified with triples  $X = (X'_F, X''_G, t)$ , where  $X'_F, X''_G$  are finite dimensional vector spaces over  $F$  and  $G$ , respectively, and  $t: X'_F \otimes_F M_G \rightarrow X''_G$  is a  $G$ -linear map. We write  $(X'_F, X''_G)$  instead of  $(X'_F, X''_G, t)$ , if the choice of  $t$  is an obvious one. The vector

$$\mathbf{dim} X = (\dim X'_F, \dim X''_G) \in \mathbb{Z}^2$$

is called the *dimension vector* of  $X$ .

Given an  $F$ - $G$ -bimodule  ${}_F N_G$  we set  $\text{l.dim}(N) = \dim_F N$  and  $\text{r.dim}(N) = \dim N_G$  and we define the right dualisation and the left dualisation of  ${}_F N_G$  to be

the  $G$ - $F$ -bimodule

$$(2.2) \quad N^{*r} = \text{Hom}_G({}_F N_G, G) \quad \text{and} \quad N^{*l} = \text{Hom}_F({}_F N_G, F)$$

respectively. To any bimodule  ${}_F M_G$  we associate a sequence of iterated right dualisations of  ${}_F M_G$  by setting  $M^{(0)} = M$  and  $M^{(j)} = (M^{(j+1)})^{*r}$  for  $j \leq -1$ . The sequence of iterated left dualisations of  ${}_F M_G$  is defined by the formula  $M^{(j)} = (M^{(j-1)})^{*l}$  for  $j \geq 1$ . We also set

$$(2.3) \quad d_j^M = \text{r.dim}(M^{(j)}), \quad R_{2j} = \begin{pmatrix} F & M^{(2j)} \\ 0 & G \end{pmatrix}, \quad R_{2j+1} = \begin{pmatrix} G & M^{(2j+1)} \\ 0 & F \end{pmatrix}$$

for any  $j \in \mathbb{Z}$ .

With any  $F$ - $G$ -bimodule  ${}_F M_G$  for which there exists an integer  $m \geq 0$  such that

$$(2.4) \quad d_j^M = \text{r.dim } M^{(j)} \text{ is finite for all } j \leq m \text{ and } d_{m+1}^M = \text{r.dim } M^{(m+1)} = \infty$$

we associate the *infinite dimension-sequence*

$$(2.5) \quad \mathbf{d}_{-\infty}({}_F M_G) = (\dots, d_{-j}(M), \dots, d_{-2}(M), d_{-1}(M), d_0(M), \infty)$$

where  $d_0(M) = d_m^M = \text{r.dim } M^{(m)}$  and  $d_j(M) = d_{m-j}^M = \text{r.dim } M^{(m-j)}$  for all  $j \geq 1$ . The number  $m$  is called the *iterated dimension height* of  ${}_F M_G$ .

Our idea is to study the indecomposable modules over any local right pure semisimple ring  $R$  with radical square zero in terms of the infinite dimension-sequence  $\mathbf{d}_{-\infty}({}_F J_F)$  of the  $F$ - $F$ -bimodule  ${}_F J_F = J(R)$ , where  $F = R/J(R)$ .

For this purpose we recall from [5] that the set

$$(2.6) \quad \mathcal{D} = \mathcal{D}_2 \cup \mathcal{D}_3 \cup \dots \cup \mathcal{D}_m \cup \dots$$

of *dimension-sequences*  $(d_1, \dots, d_m)$ ,  $m \geq 1$ , is defined inductively to be the minimal set satisfying the following two conditions:

- (i)  $\mathcal{D}_2 = \{(0, 0)\}$  and  $\mathcal{D}_3 = \{(1, 1, 1)\}$ .
- (ii) If the set  $\mathcal{D}_m$  is defined we define  $\mathcal{D}_{m+1}$  to be the set of all sequences of the form

$$\xi_{i+1}(d_1, \dots, d_m) = (d_1, \dots, d_{i-1}, d_i + 1, 1, d_{i+1} + 1, d_{i+2}, \dots, d_m),$$

where  $(d_1, \dots, d_m) \in \mathcal{D}_m$  and  $i = 1, \dots, m - 1$ .

We note that for each  $m$  the set  $\mathcal{D}_m$  of dimension-sequences of length  $m$  is closed under the action of cyclic permutations.

We recall from [28] that a sequence  $(d_1, \dots, d_m)$  is said to be a *simple restriction of a dimension-sequence* if it is obtained from a dimension-sequence in  $\mathcal{D}$  by omitting the last coordinate.

Note that the set  $\mathcal{D}^\vee$  of simple restriction of dimension-sequences contains the following sequences and their reversions:  $(0), (1, 1), (1, 2, 1), (2, 1, 2), (1, 2, 2, 1), (2, 2, 1, 3), (2, 1, 3, 1)$ .

It was shown in [29, Proposition 3.1] that in the case the ring  $R_M$  is right pure semisimple and representation-infinite there exists an integer  $m \geq 0$  such that

- (a)  $d_{m+1}^M = \infty$  and  $d_j^M < \infty$  for all  $j \leq m$ , and
- (b) for any pair  $s \leq m$  and  $t \geq 2$  the sequence  $(d_{s-t}^M, d_{s-t+1}^M, \dots, d_{s-1}^M, d_s^M)$  is not a simple restriction of a dimension-sequence.

The following definition was introduced in [33] in relation with an idea of constructing a large family of potential counter-examples to the pure semisimplicity conjecture.

**DEFINITION 2.7.** The set of *pure semisimple infinite dimension-sequences* is the set  $\mathcal{DS}_{pss} = \mathcal{DS}_{pss}^{(1)} \cup \mathcal{DS}_{pss}^{(2)}$ , where  $\mathcal{DS}_{pss}^{(1)}$  and  $\mathcal{DS}_{pss}^{(2)}$  are defined as follows.

The set  $\mathcal{DS}_{pss}^{(1)}$  is a minimal set of sequences

$$v = (\dots, v_{-m}, v_{-m+1}, \dots, v_{-2}, v_{-1}, v_0, \infty),$$

with  $v_{-j} \in \mathbb{N}$  non-zero for any  $j \in \mathbb{N}$ , satisfying the following two conditions:

- (i)  $\omega = (\dots, 2, 2, \dots, 2, 2, 1, \infty) \in \mathcal{DS}_{pss}^{(1)}$ ;
- (ii) if  $v = (\dots, v_{-m}, \dots, v_{-1}, v_0, \infty)$  is a sequence in  $\mathcal{DS}_{pss}^{(1)}$  then all sequences of the form

$$\xi_{-m}(v) = (\dots, v_{-m-1}, 1 + v_{-m}, 1, 1 + v_{-m+1}, v_{-m+2}, \dots, v_{-2}, v_{-1}, v_0, \infty)$$

belong to  $\mathcal{DS}_{pss}^{(1)}$ , for all  $m \geq 1$ .

Given a dimension-sequence  $u = (\dots, u_{-j}, u_{-j+1}, \dots, u_{-2}, u_{-1}, u_0, \infty)$  in  $\mathcal{DS}_{pss}^{(1)}$  we define the *depth* of  $u$  to be the minimal integer  $l(u) \geq 0$  such that  $u_{-j} = 2$  for all  $j \geq 1 + l(u)$ .

A sequence  $v = (\dots, v_{-m}, v_{-m+1}, \dots, v_{-2}, v_{-1}, v_0, \infty)$  belongs to  $\mathcal{DS}_{pss}^{(2)}$  if there exists a sequence of positive integers  $j(1), j(2), \dots, j(s), \dots$  such that

- (a) for every  $m \geq 0$  the set  $\{s \in \mathbb{N}; j(s) = m\}$  is finite,
- (b)  $\lim_{s \rightarrow \infty} \xi_{-j(s)} \xi_{-j(s-1)} \cdots \xi_{-j(1)}(\omega) = v$ , where  $\lim_{s \rightarrow \infty} w^{(s)} = w$  means that there exists a sequence  $0 < r_1 < r_2 < \dots < r_s < \dots$  of positive integers such that  $w_0^{(s)} = w_0, w_{-1}^{(s)} = w_{-1}, \dots, w_{-r_s}^{(s)} = w_{-r_s}$ ,
- (c) for every integer  $s \geq 0$  there exists an integer  $r_s > s$  such that  $j(r_s) \geq 1 + l(\xi_{-j(r_s-1)} \xi_{-j(r_s-2)} \cdots \xi_{-j(1)}(\omega))$ .

It was shown in [33] that the cardinality of the set  $\mathcal{DS}_{pss}^{(2)}$  is continuum. The set  $\mathcal{DS}_{pss}^{(1)}$  is constructed from the principal sequence

$$\omega = (\dots, 2, 2, \dots, 2, 2, 1, \infty)$$

in a similar way as the set  $\mathcal{D}$  of dimension-sequences was constructed in [5] starting from the trivial dimension-sequence  $(1, 1, 1)$ . In particular, each of the countably many sequences

$$\begin{aligned}
 & (\dots, 2, 2, \dots, 2, 2, 3, 1, 2, \infty), \\
 & (\dots, 2, 2, \dots, 2, 2, 3, 1, 5, 1, 2, 2, \infty), \\
 & (\dots, 2, 2, \dots, 2, 2, 3, 1, 5, 1, 2, 5, 1, 2, 2, \infty), \\
 & (\dots, 2, 2, \dots, 2, 2, 3, 1, 5, 1, 2, 5, 1, 2, 5, 1, 2, 2, \infty), \\
 & (\dots, 2, 2, \dots, 2, 2, 3, 1, 5, 1, 2, 5, 1, 2, 5, 1, 2, 5, 1, 2, 2, \infty), \\
 & \vdots \quad \vdots \quad \vdots \quad \ddots
 \end{aligned}$$

belongs to  $\mathcal{DS}_{pss}^{(1)}$ . The set  $\mathcal{DS}_{pss}^{(2)}$  is constructed from the principal sequence  $\omega$  by applying infinitely many operations  $\xi_{-j(1)}, \dots, \xi_{-j(s)}, \dots$  with the fast growth of the sequence  $j(1), \dots, j(s), \dots$  described by the property (c) in Definition 2.7. Note that the sequence

$$(\dots, 2, 1, 5, \dots, 2, 1, 5, 2, 1, 5, 2, 1, 5, 1, 2, \infty)$$

belongs to the set  $\mathcal{DS}_{pss}^{(2)}$ .

### 3. Small right pure semisimple local rings

Our investigation of potential counter-examples to the pure semisimplicity conjecture of length two or three depends on the following useful observation.

**Lemma 3.1.** *Let  $R$  be a right pure semisimple local ring of infinite representation type. If  $2 \leq l(R_R) \leq 3$ , then  $J(R)^2 = 0$ .*

Proof. If  $l(R_R) = 2$ , then  $J = J(R)$  is a simple right  $R$ -module and therefore  $J^2 = 0$ . Let  $l(R_R) = 3$  and assume to the contrary that  $J^2 \neq 0$ . Let  $x \in J$  be such that its square  $x^2 \in J^2$  is not zero. It follows that  $J^3 = 0$ ,  $J^2$  is a simple right ideal,  $J^2 = x^2R$  and therefore  $x \notin J^2$ . Since  $l(R_R) = 3$  and  $J^2 \neq 0$ , it follows that  $J/J^2$  is a simple module generated by the coset  $\bar{x}$  of  $x$  and therefore  $J = xR + x^2R = xR$ . This shows that  $R$  is right serial. Since  $R$  is of infinite representation type,  $R$  is not left serial, by [8]. On the other hand,  $R$  is right pure semisimple and right serial. It then follows from [26, Theorem 2.2] that  $J^2 = 0$ , and we get a contradiction. This finishes the proof.  $\square$

The above lemma shows that right artinian local rings of right length two or three, that are potential counter-examples to the pure semisimplicity conjecture, are square zero radical rings. Therefore we assume throughout this section that  $R$  is a local right

artinian ring such that  $J(R)^2 = 0$ . It follows that  $F = R/J(R)$  is a division ring and  $J = J(R)$  is an  $F$ - $F$ -bimodule in a natural way.

Following Gabriel [10], we associate with  $R$  the hereditary right artinian ring

$$R_J = \begin{pmatrix} R/J & (R/J)J(R/J) \\ 0 & R/J \end{pmatrix} = \begin{pmatrix} F & FJ_F \\ 0 & F \end{pmatrix}$$

and the reduction functor

$$(3.2) \quad \mathbb{F} : \text{mod}(R) \longrightarrow \text{mod}(R_J)$$

defined by attaching to any module  $Y$  in  $\text{mod}(R)$  the triple  $\mathbb{F}(Y) = (Y', Y'', t)$ , where  $Y' = Y/YJ$  and  $Y'' = YJ$  are viewed as right  $R/J$ -modules and  $t: Y' \otimes_{R/J} J_{R/J} \rightarrow Y''_{R/J}$  is a  $R/J$ -homomorphism defined by formula  $t(\bar{y} \otimes r) = y \cdot r$  for  $\bar{y} = y + J$  and  $r \in J$ . The triple  $\mathbb{F}(Y)$  is viewed as a right  $R_J$ -module in a natural way. If  $f: Y \rightarrow Z$  is an  $R$ -homomorphism we set  $\mathbb{F}(f) = (f', f'')$ , where  $f'': Y'' \rightarrow Z''$  is the restriction of  $f$  to  $Y'' = YJ$  and  $f': Y' \rightarrow Z'$  is the  $R/J$ -homomorphism induced by  $f$ .

Now we collect the main properties of the functor  $\mathbb{F}$  we need later.

**Lemma 3.3.** *Let  $R$  be a local right artinian ring such that  $J(R)^2 = 0$ . Let us view  $J = J(R)$  as an  $F$ - $F$ -bimodule, where  $F = R/J(R)$  is a division ring. Under the notation introduced above the functor  $\mathbb{F}$  (3.2) has the following properties.*

- (i)  $\mathbb{F}$  is full and establishes a representation equivalence between  $\text{mod}(R)$  and the category  $\text{Im } \mathbb{F}$ , that is, a homomorphism  $f: X \rightarrow Y$  is an isomorphism if and only if  $\mathbb{F}(f)$  is an isomorphism.
- (ii) A right  $R_J$ -module  $X$  belongs to  $\text{Im } \mathbb{F}$  if and only if  $X$  has no non-zero summand isomorphic to a simple projective right  $R_J$ -module.
- (iii) The functor  $\mathbb{F}$  preserves the indecomposability, projectivity and the length. Moreover,  $\mathbb{F}$  defines a bijection between the isomorphism classes of indecomposable modules in  $\text{mod}(R)$  and the isomorphism classes of indecomposable modules in  $\text{mod}(R_J)$ , which are not simple and projective.
- (iv) The functor  $\mathbb{F}$  carries a homomorphism  $f: Y \rightarrow Z$  in  $\text{mod}(R)$  to zero if and only if  $\text{Im } f \subseteq ZJ$ . For any pair  $Y, Z$  of indecomposable modules in  $\text{mod}(R_J)$  the functor  $\mathbb{F}$  induces ring isomorphisms

$$\text{End}(Y)/J \text{End}(Y) \cong \text{End}(\mathbb{F}(Y))/J \text{End}(\mathbb{F}(Y))$$

and

$$\text{End}(Z)/J \text{End}(Z) \cong \text{End}(\mathbb{F}(Z))/J \text{End}(\mathbb{F}(Z)).$$

If, in addition,  $Y$  is not isomorphic to a direct summand of  $ZJ$  then  $\mathbb{F}$  induces an  $\text{End}(Y)/J \text{End}(Y)$ - $\text{End}(Z)/J \text{End}(Z)$ -bimodule isomorphism

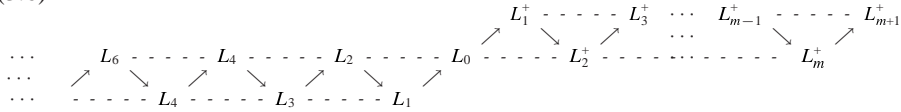
$$\text{Irr}(Y, Z) \cong \text{Irr}(\mathbb{F}(Y), \mathbb{F}(Z)).$$





if  $m \geq 1$  is odd, and the form

(3.6)



if  $m \geq 0$  is even, where  $L_1^+ = R$ ,  $L_0$  is a unique simple right  $R$ -module and  $L_1 \cong E_R(L_0)$  is an injective envelope of  $L_0$ . Here we draw a dashed edge between indecomposable modules  $X$  and  $Z$  if they are connected by an almost split sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ .

(c) There exists an integer  $m \geq 0$  such that  $d_{m+1}^J = \text{r.dim } J^{(m+1)} = \infty$ ,  $d_j^J = \text{r.dim } J^{(j)} < \infty$  for all  $j \leq m$  and the infinite dimension-sequence  $\mathbf{d}_{-\infty}(FJ_F)$  of the  $F$ - $F$ -bimodule  $FJ_F$  belongs to the set  $\mathcal{DS}_{pss} = \mathcal{DS}_{pss}^{(1)} \cup \mathcal{DS}_{pss}^{(2)}$ .

(d) The infinite radical  $\text{rad}_R^\infty = \text{rad}^\infty(\text{mod } R)$  of the category  $\text{mod}(R)$  is non-zero, whereas its square  $(\text{rad}_R^\infty)^2$  is zero.

If any of the conditions (a)–(d) is satisfied, then the infinite Jacobson radical  $\text{rad}_R^\infty$  of  $\text{mod}(R)$  is generated by all  $R$ -module homomorphisms  $L_0 \rightarrow L_{j+1}$  and all  $R$ -module homomorphisms  $L_i^+ \rightarrow L_j$  for  $j = 0, 1, 2, \dots$  and  $i \geq 1$ .

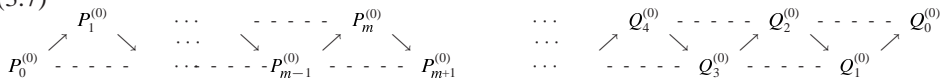
Proof. Consider the reduction functor  $\mathbb{F} : \text{mod}(R) \rightarrow \text{mod}(R_J)$  of (3.2) associated with  $R$ , where

$$R_J = \begin{pmatrix} F & FJ_F \\ 0 & F \end{pmatrix}$$

is hereditary and right artinian. We claim that every indecomposable non-projective module  $L$  in  $\text{mod}(R_J)$  admits an almost split sequence  $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ . For, since  $L$  is not projective,  $L$  is in the image of  $\mathbb{F}$  and according to Lemma 3.3 there exists a non-projective indecomposable module  $Z$  in  $\text{mod}(R)$  such that  $L \cong \mathbb{F}(Z)$ . By our assumption, there exists an almost split sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  in  $\text{mod}(R)$  and applying Lemma 3.3 (v) one shows that the derived sequence  $0 \rightarrow \mathbb{F}(X) \rightarrow \mathbb{F}(Y) \rightarrow \mathbb{F}(Z) \rightarrow 0$  in  $\text{mod}(R_J)$  is almost split. In view of the isomorphism  $L \cong \mathbb{F}(Z)$  our claim follows. It follows from [25, Corollary 1.9] that the number  $d_j^J = \text{r.dim } J^{(j)}$  is finite for any  $j \leq 0$ .

(c)  $\Rightarrow$  (b) Assume (c) is satisfied. By Theorem 4.16, Proposition 4.17 and Corollary 4.18 of [33] the hereditary ring  $R_J$  is of infinite representation type, the Auslander-Reiten translation quiver  $\Gamma(\text{mod } R_J)$  of  $\text{mod}(R_J)$  has two connected components and is of the form

(3.7)



if  $m \geq 1$  is odd, and of the form

$$(3.8) \quad \begin{array}{ccccccc} & P_1^{(0)} & \cdots & \cdots & P_{m+1}^{(0)} & \cdots & \cdots \\ & \nearrow & \cdots & \searrow & \nearrow & \cdots & \searrow \\ P_0^{(0)} & \cdots & \cdots & \cdots & P_m^{(0)} & \cdots & \cdots \\ & \searrow & \cdots & \nearrow & \searrow & \cdots & \nearrow \\ & & & & & & & Q_4^{(0)} & \cdots & Q_2^{(0)} & \cdots & Q_0^{(0)} \\ & & & & & & & \nearrow & \cdots & \searrow & \nearrow & \searrow \\ & & & & & & & \cdots & & \cdots & \cdots & \cdots \\ & & & & & & & & & & & & & Q_3^{(0)} & \cdots & Q_1^{(0)} & \cdots & Q_0^{(0)} \\ & & & & & & & & & & & & & \nearrow & \cdots & \searrow & \nearrow & \searrow \end{array}$$

if  $m \geq 0$  is even, where the left hand component is preprojective and finite, whereas the other one is preinjective and infinite. The infinite radical  $\text{rad}_{R_J}^\infty = \text{rad}^\infty(\text{mod } R_J)$  of the category  $\text{mod}(R_J)$  is non-zero, whereas its square  $(\text{rad}_{R_J}^\infty)^2$  is zero.

Since  $R_J$  is of infinite representation type, in view of Lemma 3.3 (ii), the module  $Q_m^{(0)}$  is in the image of the functor  $\mathbb{F}$  for any  $m \geq 0$ , because it follows from [6] that none of the modules  $Q_m^{(0)}$  is simple projective. For any  $j \in \mathbb{N}$  and  $1 \leq i \leq m + 1$ , we denote by

$$(3.9) \quad L_j = \mathbb{F}^{-1}(Q_j^{(0)}) \text{ and } L_i^+ = \mathbb{F}^{-1}(P_i^{(0)})$$

an indecomposable module in  $\text{mod}(R)$  corresponding, via the functor  $\mathbb{F}$ , to  $Q_j^{(0)}$  and to  $P_i^{(0)}$  in  $\Gamma(\text{mod } R_J)$ , respectively, that is,  $L_j$  and  $L_i^+$  are indecomposable modules in  $\text{mod}(R)$  such that  $\mathbb{F}(L_j) \cong Q_j^{(0)}$  and  $\mathbb{F}(L_i^+) \cong P_i^{(0)}$  (apply Lemma 3.3 (iii)).

By Lemma 3.3 (i)–(v), the preinjective component of  $\Gamma(\text{mod}(R_M))$  corresponds to the part of the Auslander-Reiten translation quiver of  $\text{mod}(R)$  formed by the modules  $L_0, L_1, \dots, L_s, \dots$  shown in (3.5) and (3.6). It follows from Lemma 3.3 (iii) that the module  $L_0$  is simple, and therefore  $J(R) \cong L_0 \oplus \cdots \oplus L_0$  (a direct sum of  $\dim J_F$  copies of  $L_0$ ). Since the inclusion  $\text{soc}(R) = J(R) \hookrightarrow R$  is an irreducible morphism and  $L_0$  is a direct summand of  $J(R)$ , there is an irreducible morphism  $u: L_0 \rightarrow R$  such that  $\mathbb{F}(u) = 0$ . The preprojective component of  $\Gamma(\text{mod}(R_J))$  starts with two projective modules

$$(0, F) = P_0^{(0)} \hookrightarrow P_1^{(0)} = (F, J_F).$$

It follows from Lemma 3.3 (i)–(iii) that  $\mathbb{F}(R) \cong P_1^{(0)}$  and  $P_0^{(0)}$  is not in the image of  $\mathbb{F}$ . We recall from Lemma 3.3 (iv) that  $\mathbb{F}$  carries irreducible morphisms to irreducible ones or to zero. Consequently, the Auslander-Reiten translation quiver of  $\text{mod}(R)$  is obtained from  $\Gamma(\text{mod } R_J)$  via  $\mathbb{F}$  as a gluing of the preprojective component of  $\text{mod}(R_J)$  with its preinjective component by the identification of  $P_0^{(0)}$  with  $Q_0^{(0)}$ . It follows that  $\Gamma(\text{mod } R)$  is connected and has the required shape shown in (3.5) and (3.6). This finishes the proof of the implication (c) $\Rightarrow$ (b).

(c) $\Rightarrow$ (d) Apply Lemma 3.3 (v) and the facts used above in the proof of the implication (c) $\Rightarrow$ (b).

(d) $\Rightarrow$ (c) By Lemma 3.3 (v), the infinite radical  $\text{rad}_{R_J}^\infty = \text{rad}^\infty(\text{mod } R_J)$  of the category  $\text{mod}(R_J)$  is non-zero, whereas its square  $(\text{rad}_{R_J}^\infty)^2$  is zero. It follows from [32, Theorem 4.4] and [36] that there exists an integer  $m \geq 0$  such that  $d_{m+1}^J = \text{r.dim } J^{(m+1)} = \infty$ ,  $d_j^J = \text{r.dim } J^{(j)} < \infty$  for all  $j \leq m$  and the infinite dimension-  
 sequence  $\mathbf{d}_{-\infty}(F, J_F)$  of the  $F$ - $F$ -bimodule  ${}_F J_F$  belongs to the set  $\mathcal{DS}_{pss} = \mathcal{DS}_{pss}^{(1)} \cup$



low, the theorem is proved. □

**Proposition 3.10.** *Assume that  $R$  is a local right artinian ring such that  $J(R)^2 = 0$  and view  $J = J(R)$  as a bimodule over the division ring  $F = R/J(R)$ . Assume also that there exists an integer  $m \geq 0$  such that  $d_{m+1}^J = \text{r.dim } J^{(m+1)} = \infty$ ,  $d_j^J = \text{r.dim } J^{(j)} < \infty$  for all  $j \leq m$  and the dimension-sequence  $\mathbf{d}_{-\infty}(FJF) = (\dots, d_{-j}(J), \dots, d_{-1}(J), d_0(J), \infty)$  belongs to  $\mathcal{DS}_{pss} = \mathcal{DS}_{pss}^{(1)} \cup \mathcal{DS}_{pss}^{(2)}$ . Then the following statements hold.*

- (a) *The ring  $R$  is right pure semisimple of infinite representation type, that is,  $R$  is a counter-example to the pure semisimplicity conjecture.*
- (b) *The ring  $R$  is not self-injective and the global dimension of  $R$  is infinite. The length  $l(R_R)$  of the right  $R$ -module  $R_R$  is  $1 + \dim J_F$ .*
- (c) *The Auslander-Reiten translation quiver  $\Gamma(\text{mod } R)$  of the category  $\text{mod}(R)$  consists of the modules  $L_j$  and  $L_i^+$  (3.9) with  $j \geq 0$  and  $0 \leq i \leq m + 1$ . It has the form (3.5) if  $m$  is odd, and the form (3.6) if  $m$  is even, where  $L_1^+ = R$ ,  $L_0$  is a unique simple right  $R$ -module and  $L_1 \cong E_R(L_0)$  is an injective envelope of  $L_0$ .*
- (d) *For any  $s \geq 2$  and  $0 \leq n \leq m - 1$  there exist almost split sequences in  $\text{mod}(R)$*

$$0 \longrightarrow L_s \longrightarrow (L_{s-1})^{d_s^J} \longrightarrow L_{s-2} \longrightarrow 0$$

and

$$0 \longrightarrow L_n^+ \longrightarrow (L_{n+1}^+)^{d_n^J} \longrightarrow L_{n+2}^+ \longrightarrow 0,$$

where  $d_j^J = \text{r.dim } J^{(j)}$ ,  $L_s$  and  $L_n^+$  are the modules (3.9), and we set  $L_0^+ = L_0$  and  $L_1^+ = R$ .

- (e) *There is no almost split sequence in  $\text{mod}(R)$  starting from an indecomposable module  $L$  if and only if  $L$  is isomorphic to  $L_1$ ,  $L_m^+$  or  $L_{m+1}^+$ .*
- (f) *The infinite Jacobson radical  $\text{rad}_R^\infty$  of  $\text{mod}(R)$  is generated by all  $R$ -module homomorphisms from  $L_0$  to  $L_{j+1}$  and all  $R$ -module homomorphisms from  $L_i^+$  to  $L_j$  for  $j = 0, 1, 2, \dots$  and arbitrary  $i \geq 1$ .*
- (g) *If  $\mathbf{d}_{-\infty}(FJF) = \boldsymbol{\omega} = (\dots, 2, 2, \dots, 2, 2, 2, 1, \infty)$ , then  $J(R) \cong L_0^{d_0^J}$ ,  $l(L_j) = 2j + 1$  for  $j \geq 0$ ,  $l(R) = l(L_1^+) = 1 + d_0^J$ ,  $l(L_j^+) = 1 + jd_0^J$  for  $j = 1, \dots, m + 1$ , all irreducible morphisms  $L_m \rightarrow L_{m-1}$  are surjective, all irreducible morphisms  $L_n^+ \rightarrow L_{n+1}^+$  are injective, and the number of indecomposable modules in  $\text{mod}(R)$  of length  $s$  is 0, 1 or 2, for every  $s \geq 1$ .*

**Proof.** Consider the reduction functor  $\mathbb{F}: \text{mod}(R) \rightarrow \text{mod}(R_J)$  (3.2) with the properties collected in Lemma 3.3, where  $R_J = \begin{pmatrix} F & FJF \\ 0 & F \end{pmatrix}$ .

- (a) Since  $\mathbf{d}_{-\infty}(FJF) = (\dots, d_{-j}(J), \dots, d_{-1}(J), d_0(J), \infty)$  belongs to the set  $\mathcal{DS}_{pss} = \mathcal{DS}_{pss}^{(1)} \cup \mathcal{DS}_{pss}^{(2)}$ , Theorem 4.16, Proposition 4.17 and Corollary 4.18 of [33] apply to the hereditary ring  $R_J$ . In particular, it follows that  $R_J$  is right pure semisimple of infinite representation type and therefore the ring  $R$  is also right pure semisimple of infinite representation type.

finite representation type, by Lemma 3.3.

(b) Let  $L_0$  denote a unique simple right  $R$ -module. Since  $R$  is a local non-simple ring,  $L_0 \cong R/J$  is not projective. It follows that the semisimple right  $R$ -module  $J \cong L_0 \oplus \dots \oplus L_0$ , a direct sum of  $l(J_R)$  copies of  $L_0$ , is not projective and the global dimension of  $R$  is infinite. In view of (a), we conclude that  $R$  is not self-injective, because self-injective right pure semisimple rings are finite representation type, by [13, Corollary 5.3]. The remaining statement of (b) is obvious, because  $J(R)^2 = 0$ .

The statement (c) is a consequence of Theorem 3.4.

(d) Fix  $s \geq 2$ . By Theorem 3.4, the Auslander-Reiten translation quiver of  $\text{mod}(R)$  has one of the forms (3.5) and (3.6) and is obtained via the reduction functor  $\mathbb{F}: \text{mod}(R) \rightarrow \text{mod}(R_J)$  of (3.2) from the Auslander-Reiten translation quiver of  $\text{mod}(R_J)$  shown in (3.7) and (3.8). The ring  $R_J$  is of infinite representation type. It follows from [33, Corollary 2.11] applied to  $F = G$ ,  ${}_F M_G = {}_F J_F$  and  $R_M = R_J$  that there exist ring isomorphisms  $\text{End}(Q_{2j}^{(0)}) \cong F$ ,  $\text{End}(Q_{2j+1}^{(0)}) \cong F$  for all  $j \geq 0$ , an  $F$ - $F$ -bimodule isomorphism  $\text{Irr}(Q_s^{(0)}, Q_{s-1}^{(0)}) \cong \text{Hom}_{R_J}(Q_s^{(0)}, Q_{s-1}^{(0)}) \cong J^{(-s-1)}$  and an almost split sequence

$$(3.11) \quad 0 \longrightarrow Q_s^{(0)} \xrightarrow{\varphi_s} (Q_{s-1}^{(0)})^{d_{-s}^J} \xrightarrow{\psi_s} Q_{s-2}^{(0)} \longrightarrow 0$$

in  $\text{mod}(R_J)$ , where  $d_{-s}^J = \text{r.dim } J^{(-s)} = 1.\text{dim } J^{(-s-1)} = 1.\text{dim Irr}(Q_s^{(0)}, Q_{s-1}^{(0)})$ . Since  $Q_j^{(0)} \cong \mathbb{F}(L_j)$  for  $j \geq 0$  and the functor  $\mathbb{F}$  is full, there exist  $R$ -module homomorphisms

$$L_s \xrightarrow{f_s} (L_{s-1})^{d_{-s}^J} \xrightarrow{g_s} L_{s-2}$$

such that  $g_s f_s = 0$ ,  $\varphi_s = \mathbb{F}(f_s)$  and  $\psi_s = \mathbb{F}(g_s)$ , that is,  $\mathbb{F}$  carries the above sequence to the exact sequence (3.11), up to isomorphism. Hence, by applying the definition of the functor  $\mathbb{F}$ , we easily conclude that the sequence

$$(3.12) \quad 0 \longrightarrow L_s \xrightarrow{f_s} (L_{s-1})^{d_{-s}^J} \xrightarrow{g_s} L_{s-2} \longrightarrow 0$$

is exact in  $\text{mod}(R)$ . By Lemma 3.3 (v) and the observation made above, there is a ring isomorphism  $\text{End}(L_s)/J \text{End}(L_s) \cong \text{End}(\mathbb{F}(L_s))/J \text{End}(\mathbb{F}(L_s)) \cong \text{End}(Q_s^{(0)}) \cong F$ , and an  $F$ - $F$ -bimodule isomorphisms

$$\begin{aligned} \text{Irr}(L_s, L_{s-1}) &\cong \text{Irr}(\mathbb{F}(L_s), \mathbb{F}(L_{s-1})) \cong \text{Irr}(Q_s^{(0)}, Q_{s-1}^{(0)}) \cong J^{(-s-1)}, \\ \text{Irr}(L_{s-1}, L_{s-2}) &\cong \text{Irr}(\mathbb{F}(L_{s-1}), \mathbb{F}(L_{s-2})) \cong \text{Irr}(Q_{s-1}^{(0)}, Q_{s-2}^{(0)}) \cong J^{(-s)}, \end{aligned}$$

and  $J^{(-s-1)} \cong \text{Hom}_F(J^{(-s)}, F)$ . It follows that

$$\begin{aligned} 1.\text{dim Irr}(L_s, L_{s-1}) &= 1.\text{dim } J^{(-s-1)} \\ &= \text{r.dim } J^{(s)} \\ &= d_{-s}^J \\ &= \text{r.dim Irr}(L_{s-1}, L_{s-2}). \end{aligned}$$

Hence, in view of [27, Proposition 11.13] applied to the category  $\mathcal{A} = \text{mod}(R)$ , we conclude that (3.12) is an almost split sequence in  $\text{mod}(R)$ .

The existence of the second almost split sequence in (d) can be proved in a similar way by applying the functor  $\mathbb{F}$  and using an almost split sequence

$$0 \longrightarrow P_n^{(0)} \xrightarrow{\varphi'_n} (P_{n+1}^{(0)})^{d'_n} \xrightarrow{\psi'_n} P_{n+2}^{(0)} \longrightarrow 0$$

in  $\text{mod}(R_j)$  for  $0 \leq n \leq m - 2$  (see [33, Corollary 2.11]).

(e) Apply (d) and the shape of the Auslander-Reiten translation quiver of  $\text{mod}(R)$  described in (3.5) and (3.6).

(f) First we show that  $\text{Hom}_R(L_i^+, L_s) = \text{rad}_R^\infty(L_i^+, L_s)$  for all  $s \geq 0$  and  $i \geq 1$ . Assume that  $s \geq 2$  and let  $h: L_i^+ \rightarrow L_{s-2}$  be a non-zero  $R$ -homomorphism. Note that  $L_j$  is not isomorphic to  $L_i^+$ , because  $\mathbb{F}(L_i^+)$  is preprojective, while  $\mathbb{F}(L_j)$  is not preprojective for all  $j \geq 0$ . Since (3.12) is an almost split sequence, there is an  $R$ -module homomorphism  $h^{(s-1)} = (h_j^{(s-1)}): L_i^+ \rightarrow (L_{s-1})^{d_{L_s}^+}$  of  $h$  such that  $h = g_s h^{(s-1)}$  and  $h_j^{(s-1)}: L_i^+ \rightarrow L_{s-1}$  belongs to  $\text{rad}(\text{mod } R)$  for all  $j$ . It follows that  $h^{(s-1)}$  also belongs to  $\text{rad}(\text{mod } R)$ . Since (3.12) is an almost split sequence,  $g_s$  is an irreducible morphism and therefore  $g_s$  belongs to  $\text{rad}(\text{mod } R)$ . Consequently,  $h = g_s h^{(s-1)}$  belongs to the square of  $\text{rad}(\text{mod } R)$ . Applying the above arguments to each of the homomorphisms  $h_j^{(s-1)}: L_i^+ \rightarrow L_{s-1}$ , we show that  $h_j^{(s-1)}$  belongs to the square of  $\text{rad}(\text{mod } R)$ . It follows that  $h^{(s-1)}$  belongs to the square of  $\text{rad}(\text{mod } R)$  and consequently  $h = g_s h^{(s-1)}$  belongs to the cube of  $\text{rad}(\text{mod } R)$ . Continuing this way we show that  $h$  belongs to  $\text{rad}^j(\text{mod } R)$  for any  $j \geq 0$ , and therefore  $h \in \text{rad}^\infty(\text{mod } R)$  (compare with [40]).

The above arguments also yield  $\text{Hom}_R(L_0, L_{s+1}) = \text{rad}_R^\infty(L_0, L_{s+1})$  for all  $s \geq 0$ . Consequently,  $\text{rad}_R^\infty$  contains the set

$$\mathcal{X} = \bigcup_{i \geq 1} \bigcup_{s \geq 0} \text{Hom}_R(L_i^+, L_s) \cup \text{Hom}_R(L_0, L_{s+1}).$$

Now we show that  $\mathcal{X}$  generates the infinite radical  $\text{rad}_R^\infty$  of  $\text{mod}(R)$ . For this purpose we note first that any  $R$ -module homomorphism  $h \in \text{rad}^\infty(L_n, L_j)$  has a factorisation through a direct sum of monomorphisms  $\text{soc } L_t \hookrightarrow L_t$  for some  $t \geq 1$ . Assume for simplicity that  $n < j$ . Then  $\mathbb{F}(h) \in \text{Hom}_{R_j}(\mathbb{F}(L_n), \mathbb{F}(L_j)) = 0$  and according to Lemma 3.3,  $h$  factorises through  $L_j J \subseteq \text{soc } L_j \subseteq L_j$  as we required. The remaining cases follow in a similar way. Since the monomorphism  $\text{soc } L_j \hookrightarrow L_j$  is a sum of homomorphisms  $L_0 \hookrightarrow L_j$ , it follows that  $\text{rad}^\infty(L_n, L_j)$  is contained in the two-sided ideal of  $\text{mod}(R)$  generated by the set  $\mathcal{X}$ .

Further we note that any  $R$ -module homomorphism  $h \in \text{rad}^\infty(L_n^+, L_s^+)$  has a factorisation through a direct sum of monomorphisms  $\text{soc } L_t^+ \hookrightarrow L_t^+$  for some  $t \geq 1$ , and therefore  $h$  has a factorisation through a homomorphism  $L_n^+ \rightarrow \text{soc } L_t^+$ , which is a sum of homomorphisms  $L_n^+ \rightarrow L_0$ . It follows that  $\text{rad}^\infty(L_n^+, L_s^+)$  is contained in the ideal of  $\text{mod}(R)$  generated by the set  $\mathcal{X}$ .

Finally, take any homomorphism  $h \in \text{rad}^\infty(L_j, L_n^+)$ . Since  $\mathbb{F}(h) = 0$ , according to Lemma 3.3,  $h$  factorises through  $L_n^+J \subseteq \text{soc } L_n^+ \subseteq L_n^+$ . It follows that there is a factorisation  $h = h''h'$ , where  $h' \in \text{rad}^\infty(L_j, \text{soc } L_n^+)$ . Consequently,  $h'$  is a sum of homomorphisms in  $\text{rad}^\infty(L_j, L_0)$ . It follows that  $\text{rad}^\infty(L_j, L_n^+)$  is contained in the ideal of  $\text{mod}(R)$  generated by the set  $\mathcal{X}$ . This finishes the proof of (f).

(g) Since we assume that  $\mathbf{d}_{-\infty}(F F_F) = \boldsymbol{\omega}$ ,  $d_j^J = 2$  for all  $j \leq m-1$ ,  $d_m^J = 1$  and  $d_{m+1}^J = \infty$ , where  $m \geq 0$ . Recall that the Auslander-Reiten translation quiver of  $\text{mod}(R_J)$  has one of the forms (3.5) or (3.6), the module  $Q_0^{(0)}$  is simple injective and  $Q_1^{(0)}$  is the injective envelope of  $P_0^{(0)} \cong (0, F)$ . It follows that  $Q_0^{(0)} \cong (F, 0)$ ,  $Q_1^{(0)} \cong (J_F^{(-2)}, F)$  (see [25]) and therefore  $\mathbf{dim} Q_0^{(0)} = (1, 0)$ ,  $\mathbf{dim} Q_1^{(0)} = (d_{-2}^J, 1) = (2, 1)$ . Furthermore, the almost split sequence (3.11) in  $\text{mod}(R_J)$  yields

$$\mathbf{dim} Q_s^{(0)} = d_{-s}^J \mathbf{dim} Q_{s-1}^{(0)} - \mathbf{dim} Q_{s-2}^{(0)} = 2 \mathbf{dim} Q_{s-1}^{(0)} - \mathbf{dim} Q_{s-2}^{(0)}$$

for all  $s \geq 2$ . Hence, for  $s = 2$ , we get  $\mathbf{dim} Q_2^{(0)} = 2 \mathbf{dim} Q_1^{(0)} - \mathbf{dim} Q_0^{(0)} = (3, 2)$ , and applying inductively the above equality yields  $\mathbf{dim} Q_s^{(0)} = (s + 1, s)$  and  $l(Q_s^{(0)}) = 2s + 1$  for any  $s \geq 0$ . Hence, in view of Lemma 3.3 (iii), we conclude that  $l(L_s) = l(\mathbb{F}(L_s)) = l(Q_s^{(0)}) = 2s + 1$ . We recall that every irreducible morphism between indecomposable modules is either injective or surjective (see [3, Lemma 5.1] and [27, Section 11.1]). It follows that any irreducible morphism  $L_s \rightarrow L_{s-1}$  is surjective for  $s \geq 1$ , because it is not injective.

Now we note that the second almost split sequence in (d) yields

$$l(L_{n+2}^+) = d_n^J l(L_{n+1}^+) - l(L_n^+) = 2l(L_{n+1}^+) - l(L_n^+)$$

for  $n = 0, 1, \dots, m-1$ . Since  $L_0^+$  is simple and  $L_1^+ \cong R$ ,  $l(L_0^+) = 1$  and  $l(L_1^+) = 1 + d_0^J \leq 3$ . Hence we get  $l(L_2^+) = l(L_1^+) - l(L_0^+) = 2(1 + d_0^J) - 1 = 1 + 2d_0^J$ , and inductively we show that  $l(L_j^+) = 1 + jd_0^J$  for  $j = 1, \dots, m+1$ . Consequently the statement (g) follows. □

The following corollary shows how potential local counter-examples  $R$  to the pure semisimplicity conjecture of length two or three should look like, and gives the structure of their Auslander-Reiten translation quiver  $\Gamma(\text{mod } R)$ .

**Corollary 3.13.** *Assume that  $R$  is a local right pure semisimple ring of infinite representation type such that  $2 \leq l(R_R) \leq 3$ . Then  $J(R)^2 = 0$ ,  $J = J(R)$  is a bimodule over the division ring  $F = R/J(R)$ , there exists an integer  $m \geq 0$  such that  $d_{m+1}^J = \text{r.dim } J^{(m+1)} = \infty$ ,  $d_j^J = \text{r.dim } J^{(j)} < \infty$  for all  $j \leq m$ , the infinite dimension-sequence  $\mathbf{d}_{-\infty}(F J_F) = (\dots, d_{-j}(J), \dots, d_{-1}(J), d_0(J), \infty)$ , with  $d_{-j}(J) = d_{m-j}^J$ , (2.5) is defined and the following conditions are equivalent:*

(a) *The Auslander-Reiten quiver  $\Gamma(\text{mod } R)$  of  $\text{mod}(R)$  is infinite and connected of the form  $\dots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \dots \rightarrow \bullet \rightarrow \bullet$ .*

(b) *The infinite dimension-sequence  $\mathbf{d}_{-\infty}(FJ_F)$  of  ${}_FJ_F$  belongs to the set  $\mathcal{DS}_{pss} = \mathcal{DS}_{pss}^{(1)} \cup \mathcal{DS}_{pss}^{(2)}$ .*

(c) *The infinite radical  $\text{rad}_R^\infty = \text{rad}^\infty(\text{mod } R)$  of the category  $\text{mod}(R)$  is non-zero, whereas its square  $(\text{rad}_R^\infty)^2$  is zero.*

*If any of the conditions (a)–(c) is satisfied, then  $R$  is a counter-example to the pure semisimplicity conjecture, the Auslander-Reiten translation quiver  $\Gamma(\text{mod } R)$  has one of the forms (3.5) or (3.6), and  $R$  has the properties presented in Proposition 3.10.*

*Proof.* We know from Lemma 3.1 that  $J(R)^2 = 0$ . Since  $R$  is right pure semisimple, according to [25, Proposition 2.4] every indecomposable non-projective module  $X$  in  $\text{mod}(R)$  admits an almost split sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  and Theorem 3.4 and Proposition 3.10 apply. □

In connection with [28, Remark 2.4] the following observation is useful.

**Corollary 3.14.** *Assume  $F \subset G$  are division rings such that  $F \cong G$ ,  $\dim_F G = \infty$  and that the associated infinite dimension-sequence  $\mathbf{d}_{-\infty}(F_G G)$  (3.2) of the  $F$ - $G$ -bimodule  ${}_F G_G$  belongs to  $\mathcal{DS}_{pss} = \mathcal{DS}_{pss}^{(1)} \cup \mathcal{DS}_{pss}^{(2)}$ . Then*

(a) *the trivial extension  $T_G = F \rtimes {}_F G_G$  of  $F$  by  ${}_F G_G$  is a local ring and it is a counter-example to the pure semisimplicity conjecture of length two (that is,  $l(T_G) = 2$ , when  $T_G$  is viewed as a right  $T_G$ -module),*

(b) *the ring  $T_G$  is not self-injective,*

(c) *the global dimension of  $T_G$  is infinite, and*

(d) *the Auslander-Reiten quiver  $\Gamma(\text{mod } T_G)$  of  $\text{mod}(T_G)$  is connected of the form*  
 $\cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \bullet.$

*Proof.* Apply Theorem 3.4. □

**REMARK 3.15.** Since for any  $v = (\dots, v_{-m}, \dots, v_{-1}, v_0, \infty) \in \mathcal{DS}_{pss}$  there exists  $j \geq 1$  such that  $v_{-j} = 1$ , according to [28, Remark 4.5] the existence of an  $F$ - $G$ -bimodule  ${}_F M_G$  such that  $\mathbf{d}_{-\infty}({}_F M_G) = v$  is an infinite version of the Artin problem for division ring extensions studied in [4], [20], [28] and [29] (see [28, Section 4]). In the situation we study in Corollary 3.14 we assume in addition that  $F \cong G$ .

We hope that, by applying a modification of the bimodule amalgam rings construction of Schofield [21, Chapter 13], one can construct a division ring embedding  $F \subseteq G \cong F$  such that  $\mathbf{d}_{-\infty}({}_F G_G) = v$  for some of the dimension-sequences  $v \in \mathcal{DS}_{pss}$ .

A solution of this problem is strongly related with the main problems studied in [15], [38] and [39] of finding special classes of artinian rings without self-



extensions (compare with [1], [41]).

We finish the paper by raising the following problems related with the one stated in [33, Problem 4.21] for hereditary rings of the form  $R_M$  (2.1).

**Problem 3.16.** Assume that  $R$  is a right artinian local ring with the Jacobson radical  $J = J(R)$ , such that  $J^2 = 0$ ,  $F = R/J$  and the associated infinite dimension-sequence  $\mathbf{d}_{-\infty}(FJ_F)$  of (2.5) associated to the  $F$ - $F$ -bimodule  ${}_FJ_F$  belongs to the set  $\mathcal{DS}_{pss} = \mathcal{DS}_{pss}^{(1)} \cup \mathcal{DS}_{pss}^{(2)}$ . Let  $L_0, L_1, L_2, \dots, L_s, \dots$  be pairwise non-isomorphic indecomposable  $R$ -modules shown in (3.5) and defined by (3.9) (see Theorem 3.4).

(a) Find a decomposition of the right  $R$ -module

$$(3.17) \quad \mathcal{L}(R) = \prod_{m=0}^{\infty} L_m / \bigoplus_{m=0}^{\infty} L_m$$

in a direct sum of indecomposable modules.

(b) Give a characterization of local rings  $R$  for which the  $R$ -module  $\mathcal{L}(R)$  is projective.

In [16] a partial solution of the problem [33, Problem 4.21] is presented for hereditary rings of the form  $R_M$  (2.1).

The following interesting problem stated in [31, Problem 3.2] remains unsolved.

**Problem 3.18.** Give a characterisation of semiperfect rings  $R$  for which every indecomposable right  $R$ -module is pure-projective or pure-injective. Is every such a ring  $R$  right artinian or right pure semisimple?

Let us finish the paper by the following open question related with Theorem 3.4.

**Problem 3.19.** Prove that under the assumption in Theorem 3.4 the statement (a) is equivalent to the following one:

(a') The Auslander-Reiten quiver  $\Gamma(\text{mod } R)$  is infinite and connected.

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