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PIECEWISE C^2 PERTURBATION OF LASOTA-YORKE MAPS AND THEIR ERGODIC PROPERTIES

Dedicated to Professor Teturo Kamae on his sixtieth birthday

TAKEHIKO MORITA

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1. Introduction

Between 1970 and 1990 many authors studied the ergodic properties of absolutely continuous invariant measures for a piecewise C^2 uniformly expanding map T of the unit interval ([2], [4], [5], [6], [7], [11], [12] and [14], see also the references of [10]). First, Lasota and Yorke [6] proved the existence of such a measure with density of bounded variation by making use of the so-called Perron-Frobenius operator $L_T\colon L^1(m) \to L^1(m)$ for T defined by

$$L_T g = \frac{d}{dm} \int_{T^{-1}(\cdot)} g\, dm$$

for $g \in L^1(m)$, where $L^1(m)$ denotes the usual L^1-space with respect to the Lebesgue measure m on the unit interval and $(d/dm)\int_{T^{-1}(\cdot)} g\, dm$ denotes the Radon-Nikodym derivative of the complex valued measure $B \mapsto \int_{T^{-1}B} g\, dm$. In this case, it can be shown that L_T also acts on the space BV, where BV is the totality of elements in $L^1(m)$ with versions of bounded variation.

In what follows, an invariant measure means an invariant probability measure, and the terminology absolutely continuous invariant measure is written as a.c.i.m. for short.

After [6], a piecewise C^2 map T of the interval saitisfying ess.inf $|DT^N| > 1$ for some N is called a Lasota-Yorke map (an LY map for short). The ergodic decomposition of an a.c.i.m. is discussed in Li and Yorke [7]. Wagner [14] shows that each ergodic component is decomposed into a finite number of mixing components. More precisely, there exist a finite number of a.c.i.m.'s $\mu_1, \ldots, \mu_P$ such that any a.c.i.m. can be represented as an affine combination of them. The support E_i of each μ_i is decomposed into a finite number of subsets $E_{i,0}, \ldots, E_{i,N_i-1}$ such that $TE_{i,j} = E_{i.j+1}$ a.e. $(\mathrm{mod}\ N)_i$ and the measure-theoretic dynamical systems $(T^{N_i}, \mu_{i.j})$ are mixing,

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where $\mu_{i.j} = N_i\mu|_{E_{i,j}}$ for $j = 1, \dots, N_i$. Bowen [2] shows that each mixing component $(T^{N_i}, \mu_{i.j})$ is Bernoulli and gives sufficient conditions for that T has a unique a.c.i.m. μ and $(T.\mu)$ itself is Bernoulli. We call such a map a Bernoulli Lasota-Yorke map (a BLY map for short) in the sequel.

The central limit problems of LY maps are studied in [4], [5], [9], [11] and so on. In particular, for a BLY map T and for a real valued function f of bounded variation, we can see that the limit variance

$$\sigma_T(f)^2 = \lim_{n\to\infty} \frac{1}{n} \int \left(\sum_{i=0}^{n-1} f \circ T^i - n \int f \, d\mu \right)^2 d\mu$$

exists, where μ is the unique a.c.i.m. In addition, if $\sigma_T(f)^2$ is positive, we can show the central limit theorem

(1.1)
$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| m\left(\left\{ \frac{1}{\sqrt{n}\sigma_T(f)} \left(\sum_{i=0}^{n-1} f \circ T^i - n \int f \, d\mu \right) \le x \right\} \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \, dy \right| = 0.$$

In [5], [9], and [11] the perturbed Perron-Frobenius operators $L_{T,f}(t)\colon BV \to BV$ defined by $g \mapsto L_T(\exp(\sqrt{-1}\,tf)g)$ play important roles in the study of the limit theorems, where t is a comlex parameter. If t is small, the first eigenvalue $\lambda_T(t)$ of $L_{T,f}$ (the eigenvalue of maximal modulus) depends analytically on t. We note that the limit variance and the first eigenvalue are related by the formula

$$\sigma_T(f)^2 = -\frac{d^2\lambda_T}{dt^2}(0) \tag{1.2}$$

In this paper we introduce a natural metric d to the space PC^2 of piecewise C^2 maps of the unit interval so that the curvatures of the graphs of two maps become close to each other if maps themselves are close to each other in PC^2. Temporarily, we call d the piecewise C^2 metric and the topology induced by d, the piecewise C^2 topology. Our main purpose is to show the stability of the ergodic properties mentioned above under a small perturbation in the piecewise C^2 topology. Precisely we shall prove the following:

Theorem 1.1. *Let T_0 be a BLY map satisfying* ess.inf$\,|DT_0| > 2$. *Then there exists $\delta > 0$ such that any piecewise C^2 map T with $d(T, T_0) < \delta$ is a BLY map.*

Theorem 1.2. *Let T_0 be a BLY map satisfying* ess.inf$\,|DT_0| > 2$ *and let f be a real valued function of bounded variation. Then there exist $\delta_0 \in (0, \delta)$ and $t_0 > 0$ such that the mapping $T \mapsto \lambda_T(\cdot)$ from $\{T \in PC^2 : d(T.T_0) < \delta_0\}$ into the Fréchet space of analytic functions on $\{t \in \mathbb{C} : |t| < t_0\}$ is continuous, where $\lambda_T(t)$ is the first eigenvalue of the perturbed Perron-Frobenius operator $L_T(t)$.*

The assumptions of Bernoullicity and ess.inf $|DT| > 2$ in these theorems seems to be very technical at first sight. But we will show by examples that they are essential for the theorems to be valid in the end of Section 4.

Combining the formulas (1.1) and (1.2) with Theorem 1.2, we obtain:

Corollary 1.1. *Under the same assumptions as* Theorem 1.2, $T \mapsto \sigma_T(f)^2$ *is a continuous function on* $\{T \in PC^2 : d(T.T_0) < \delta_0\}$. *In particular, if the central limit theorem* (1.1) *holds for* $\{f \circ T_0^n\}_{n=0}^{\infty}$, *then so does for* $\{f \circ T^n\}_{n=0}^{\infty}$ *for any* T *in a neighborhood of* T_0.

The primitive version of Theorem 1.1 was known to the author in 1983. But in those days it seemed not to be interesting for him. The present form of Theorem 1.1 and the assertions in Theorem 1.2 are inspired by a question asked by Professor Hiroshi Sugita of Kyushu University in 1995. It was concerned with the stability of the invariant density and the limit central theorem under the small deterministic or random perturbation of one dimensional dynamical systems. In general, we do not expect such a sort of stability. For example, the logistic map $T_4x = 4x(1-x)$ has a unique a.c.i.m. with density $1/(\pi\sqrt{x(1-x)})$. But for any $\epsilon > 0$ there exists a subset $F\epsilon$ of $[4-\epsilon, 4]$ with positive Lebesgue measure such that $T_ax = ax(1-x)$ has no a.c.i.m. for any $a \in F_\epsilon$ (see [13]). In contrast with this, any family of LY maps is always guaranteed to have an a.c.i.m. Therefore, it seems meaningful and interesting to study how invariant densities and their limit theorems depend on the parameter change in the case of the family of LY maps. Especially it seems that there is no result that is concerned with the stability of the non degeneracy of the limit variance before this work.

In Section 2, we define the piecewise C^2 metric d and mention some basic properties of the metric space (PC^2, d). The Lasota-Yorke type inequality and the Krylov-Bogolioubov type inequalty are described in Section 3. The forms of these inequalities are more general than we need, but they must be useful for the study of random iterations as in [8]. Section 4 and Section 5 are devoted to the proofs of Theorem 1.1 and Theorem 1.2. which depend heavily on two inequalities in Section 3. We shall give two examples in Section 6, which show that our results can not be obtained by a direct application of the general perturbation theory of linear operators and how the metric d is appropriate to measure the difference between LY maps in the ergodic theoretical point of view. Finally in Appendix, we prove the Lasota-Yorke type inequality and Krylov-Bogolioubov type inequality for reader's convenience.

2. Piecewise C^2 metric

First of all we define the space $PC^2 = PC^2[0,1]$ of piecewise C^2 maps from the unit interval into itself. Our investigations are carried out by using the Lebesgue measure m on $[0,1]$. Therefore, it will be more convenient to modify the usual definition of piecewise C^2 maps the same as we consider L^p-space in the case of integrable functions.

DEFINITION 2.1. An almost everywhere defined map $T : [0,1] \to [0,1]$ is said to be a piecewise C^2 map if there exists a partition $\mathcal{P} = \{[a_{i-1}, a_i]\}_{i=1}^k$ of $[0,1]$ satisfying the following conditions:
(1) $0 = a_0 < \cdots < a_k = 1$.
(2) $T|_{(a_{i-1},a_i)}$ coincides with a C^2 map $T_{[a_{i-1},a_i]}$ almost everywhere on the closed interval $[a_{i-1}, a_i]$ for each i.
(3) $\mathcal{P}$ is minimal in the sense of the refinement among all the partitions satisfying (1) and (2) above. Namely, if a partition $\mathcal{Q} = \{[b_{j-1}, b_j]\}_{j=1}^l$ satisfies (1) and (2), $\mathcal{Q}$ turns out to be a refinement of $\mathcal{P}$. We call the partition $\mathcal{P}$ the defining partition of T.

The number k and the defining partition in the above are uniquely determined by T. So we often write them as $k(T)$ and $\mathcal{P}(T)$.

Let T and S be piecewise C^2 maps with defining partitions $\mathcal{P}(T) = \{[a_{i-1}, a_i]\}_{i=1}^{k(T)}$ and $\mathcal{P}(S) = \{[b_{j-1}, b_j]\}_{j=1}^{k(S)}$. We identify T with S if $k(T) = k(S)$, $a_i = b_i$, and $T|_{(a_{i-1},a_i)} = S|_{(a_{i-1},a_i)}$ hold for $i = 1, \ldots, k(T)$. In the same way as in the case of L^p-space, we define the space PC^2 of piecewise C^2 maps by the totality of the equivalent classes under the identification in the above and we treat each equivalent class as if it is one of its version.

REMARK 2.1. If $T \in PC^2$ is given, it determines the following stuffs.
(a) The defining partition $\mathcal{P}(T)$.
(b) A family $\{T_J\}_{J\in\mathcal{P}(T)}$ of C^2 maps, where $T_J : J \to [0,1]$ are the C^2 extensions to J of C^2 version of $T|_{\text{int}\,J}$.
(c) The number $k(T)$ of elements in $\mathcal{P}(T)$.
Conversely, if there exist a partition $\mathcal{P} = \{[a_{i-1}, a_i]\}_{i=1}^k$ with $0 = a_0 < \cdots < a_k = 1$ and a family of C^2 maps $T_{[a_{i-1},a_i]} : [a_{i-1}, a_i] \to [0,1]$, $i = 1, \ldots, k$ such that if $k \geq 2$, there is no C^2 map U on $[a_{i-1}, a_{i+1}]$ such that $U|_{[a_{i-1},a_i]} = T_{[a_{i-1},a_i]}$ and $U|_{[a_i,a_{i+1}]} = T_{[a_i,a_{i+1}]}$, then we have an element $T \in PC^2$ so that $\mathcal{P}(T) = \mathcal{P}$.

We would like to consider only elements in PC^2 whose n-fold iterations can be defined as elements in PC^2.

DEFINITION 2.2. An element T in PC^2 is said to be nondegenerate if $DT_J(x) \neq 0$ holds for any $x \in J$ and for any $J \in \mathcal{P}(T)$, where T_J's are as in Remark 2.1 and DF denotes the derivative of a function F.

If $T \in PC^2$ is nondegenerate, it is obvious that $T_J : J \to T_J J$ is a homeomorphism for each $J \in \mathcal{P}(T)$. Therefore, we can define the n-fold iteration of T^n as follows:

For a while we write $\mathcal{P}(T) = \{J(i)\}_{i=1}^{k}$. For each $(J(i_0), J(i_1), \ldots, J(i_{n-1})) \in \mathcal{P}(T)^n$, we consider the set

$$J(i_0, \ldots, i_{n-1}) = J(i_0) \cap T_{J(i_0)}^{-1} J(i_1) \cap \cdots \cap T_{J(i_0)}^{-1} \cdots T_{J(i_{n-2})}^{-1} J(i_{n-1}).$$

If $\int J(i_0, \ldots, i_{n-1}) \neq \emptyset$, we put

$$S_{J(i_0,\ldots,i_{n-1})} = T_{J(i_{n-1})} \cdots T_{J(i_0)}.$$

Then $\mathcal{Q} = \{J(i_0, \ldots, i_{n-1})\}_{J(i_0,\ldots,i_{n-1}):\mathrm{int}\, J(i_0,\ldots,i_{n-1}) \neq \emptyset}$ turns out to be a partition of $[0, 1]$ into closed intervals and determines an element S in PC^2 so that $S_{J(i_0,\ldots,i_{n-1})} = S_{J(i_0,\ldots,i_{n-1})}$. It is natural to denote S by T^n and call it the n-fold iteration of T. Note that $\mathcal{Q}$ is not necessarily the defining partition $\mathcal{P}(S)$ of S since it does not always satisfy the minimality condition (3) in Definition 2.1. For example consider the map

$$Tx = \begin{cases} 2x + \dfrac{1}{2}, & \left(x \in \left[0, \dfrac{1}{4}\right)\right) \\ 2\left(x - \dfrac{1}{4}\right), & \left(x \in \left[\dfrac{1}{4}, \dfrac{3}{4}\right)\right) \\ 2\left(x - \dfrac{3}{4}\right), & \left(x \in \left[\dfrac{3}{4}, 1\right]\right). \end{cases}$$

The defining partition of T is $\{J(1) = [0, 1/4], J(2) = [1/4, 3/4], J(3) = [3/4, 1]\}$. From the construction above, $J(1, 3) = [1/8, 1/4]$ and $J(2, 1) = [1/4, 3/8]$. But we see that $S_{J(1,3)}x = 4(x - 1/8)$ and $S_{J(2,1)}x = 4(x - 1/8)$. This shows that $\mathcal{Q}$ does not always satisfy the minimality condition.

For the sake of later convenience, we denote S_J by T_J^n for each $J \in \mathcal{P}(T^n)$. The inverse $(T_J^n)^{-1} \colon T_J^n J \to J$ is denoted by T_J^{-n}.

DEFINITION 2.3. A nondegenerate element T is called a Lasota-Yorke map (an LY map for short) if there exists a positive integer N such that $\mathrm{ess.inf}\,|DT^N| > 1$.

REMARK 2.2. If T is a nondegenerate element in PC^2, so is T^n. Conversely, if $T \in PC^2$ has a version $\tilde{T}$ for which we can define n-fold iteration $\tilde{T}^n$ almost everywhere and $\tilde{T}^n$ becomes a version of a nondegenerate element in PC^2, then it is not hard to see that T itself is nondegenerate. Thus we can define an LY map as an element in PC^2 having a version $\tilde{T}$ such that the N-fold iteration $\tilde{T}^N$ is defined almost everywhere, $\tilde{T}^N$ becomes a version of a nondegenerate element in PC^2, and $\mathrm{ess.inf}\,|D\tilde{T}^N| > 1$ holds for some N.

Next we introduce a metric to PC^2. Let T be an element in PC^2 and $\mathcal{P}(T) = \{[a_{i-1}, a_i]\}_{i=1}^{k}$ the defining partition of T as in Definition 2.1. For any positive integer

i, define a map $T_i\colon [0,1] \to [0,1]$ and nonnegative number $l_i(T)$ by

$$T_i = \begin{cases} T_{[a_{i-1},a_i]} \circ \alpha_{T,i} & \text{if } 1 \le i \le k, \\ 0 & \text{if } i > k, \end{cases}$$

and

$$l_i(T) = \begin{cases} a_i - a_{i-1} & \text{if } 1 \le i \le k, \\ 0 & \text{if } i > k, \end{cases}$$

where $\alpha_{T,i}\colon [0,1] \to [a_{i-1}, a_i]$ is a linear map given by $\alpha_{T,i}x = (a_i - a_{i-1})x + a_{i-1}$.

For T and S in PC^2, we define $d\colon PC^2 \times PC^2 \to \mathbb{R}$ by

$$d(T,S) = |k(T) - k(S)| + \sum_{i=1}^{\infty} \|T_i - S_i\|_{C^2} + \sum_{i=1}^{\infty} |l_i(T) - l_i(S)|,$$

where $\|F\|_{C^2}$ denotes the usual C^2 norm given by

$$\|F\|_{C^2} = \max_{x\in[0,1]} |F(x)| + \max_{x\in[0,1]} |DF(x)| + \max_{x\in[0,1]} |D^2F(x)|.$$

It is easy to see that d is a metric on the space PC^2. We call it the piecewise C^2 metric. We summarize the basic properties of the metric d as the following proposition.

Proposition 2.1. (1) *The metric space (PC^2, d) is separable but not complete.*
(2) *For $a > 0$, let $PC^2(a)$ denote the totality of elements T in PC^2 with $k(T) \ge 2$ such that*

$$|T(a_i+) - T(a_i-)| + |DT(a_i+) - DT(a_i-)| + |D^2T(a_i+) - D^2T(a_i-)| \ge a$$

holds for each i with $1 \le i \le k(T)-1$, where $0 = a_0 < \cdots < a_{k(T)} = 1$ are the division points as in Definition 2.1. *Then $(PC^2(a), d)$ becomes a complete metric space.*
(3) *If a sequence $T_n \in PC^2$ converges to T in PC^2 as n goes to ∞, then T_n converges to T almost everywhere. Moreover, for any point $x \ne a_i(T)$ $(i = 0, 1, \dots, k(T))$, there exists a neighborhood of x where T_n converges to T in C^2 topology.*
(4) *For any $a > 0$, the set $\{T \in PC^2 : \text{ess.inf}\,|DT| > a\}$ is an open set in PC^2. In particular, the totality of nondegenerate elements is an open set in PC^2.*
(5) *On the subspace consisting of nodegenerate elements, we can define the map $T \mapsto T^n$ for any positive integer n. But it is not continuous at every nondegenerate T.*

Proof. (1) The separability is an easy consequence of the separability of the Banach space $(C^2[0,1], \|\cdot\|_{C^2})$. In fact, as a countable dense subset we can take the totality of elements T in PC^2 such that the division points a_i's are rational and $T|_{[a_{i-1},a_i]}$'s are polynomials with rational coefficients. The following example shows that (PC^2, d) is not complete. For $n \geq 2$, consider the sequence T_n given by

$$T_n x = \begin{cases} x & \text{if } x \in \left[0, \frac{1}{2}\right) \\ x - \frac{1}{n} & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Then T_n is a Cauchy sequence in PC^2. If the limit T exists, the unique candidate is the identity. But we have $k(T_n) = 2$ and $k(T) = 1$. Therefore $d(T, T_n) \geq 1$. This implies that the limit does not exist.

(2) First we note that if T_n is a Cauchy sequence in PC^2, the number of division points $k(T_n)$ are independent of n (say k) for sufficiently large n and $a_i(T_n)$ converges to a point a_i for each $i = 0, \dots, k$. Therefore we see that there exists $T \in PC^2$ such that T_n, DT_n, D^2T_n converge to T, DT, D^2T uniformly on any compact set in $[0, 1] \setminus \{a_0, a_1, \dots, a_k\}$, respectively. The example above, however, shows that the points a_i possibly become regular points of T i.e. $k(T) < k(T_n)$. The condition on $PC^2(a)$ removes such a possibility. Therefore we can show the validity of the assertion (2).

The assertions (3), and (4) are easy exercises. For (5), we just give an example showing that $T \mapsto T^2$ is not continuous. Consider the maps T and T_n with $n \geq 2$ defined by

$$Tx = \begin{cases} x + \frac{1}{2} & \text{if } x \in \left[0, \frac{1}{2}\right) \\ -2(x-1) & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}, \quad T_n x = \begin{cases} \left(1 + \frac{2}{n}\right) x + \frac{1}{2} - \frac{1}{n} & \text{if } x \in \left[0, \frac{1}{2}\right) \\ -2(x-1) & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Clearly $d(T_n, T) \to 0$ as $n \to \infty$, But $d(T_n^2, T^2) \geq 1$ since $k(T_n^2) = 4$ and $k(T^2) = 3$. □

3. Perturbed Perron-Frobenius operators for LY maps

In the present section, we briefly describe some basic results on the perturbed Perron-Frobenius operators for LY maps ([10], see also [5] and [11]). To begin with, we recall a general definition of the Perron-Frobenius operator. Let $(X, \mathcal{B}, \mu)$ be a probability space. For $1 \leq p \leq \infty$, $L^p(\mu)$ denotes the usual L^p-space with L^p-norm $\|\cdot\|_{p,\mu}$. Let $T\colon X \to X$ be a nonsingular transformation, i.e. T is $\mathcal{B}/\mathcal{B}$-measurable and $\mu(T^{-1}B) = 0$ for any $B \in \mathcal{B}$ with $\mu(B) = 0$. The Perron-Frobenius operator $L_{T,\mu}\colon L^1(\mu) \to L^1(\mu)$ for T with respect to μ is defined by

$$L_{T.\mu} g = \frac{d}{d\mu} \int_{T^{-1}(\cdot)} g \, d\mu$$

for each $g \in L^1(\mu)$, where the right hand side denotes the Radon-Nikodym derivative of the complex valued measure $B \mapsto \int_{T^{-1}B} g\,d\mu$ with respect to μ. Then it is easy to show that $L_{T,\mu}$ is characterized by the identity

$$\int (g_1 \circ T)\cdot g_2\,d\mu = \int g_1 \cdot L_{T,\mu} g_2\,d\mu \tag{3.1}$$

for any $g_1 \in L^\infty(\mu)$ and for any $g_2 \in L^1(\mu)$. The following assertions are the consequences of the identity (3.1) (see [8], [10] for the proof).

Proposition 3.1. (1) *$L_{T^n,\mu} = L^n_{T,\mu}$ for any positive interger n.*
(2) *$L^n_{T,\mu}((g_1 \circ T^n)g_2) = g_1 L^n_{T,\mu} g_2$ for any $g_1 \in L^\infty(\mu)$ and for any $g_2 \in L^1(\mu)$.*
(3) *If μ is T-invariant, $E_\mu(g \mid T^{-n}\mathcal{B}) = (L^n_{T,\mu} g)\circ T^n$ for any $g \in L^1(\mu)$, where $E_\mu(g \mid T^{-n}\mathcal{B})$ is the conditional expectation of g with respect to the sub σ-algebra $T^{-n}\mathcal{B}$ of $\mathcal{B}$.*
(4) *For $g \in L^1(\mu)$, $L_{T,\mu} g = g$ if and only if g is a density function of a μ-absolutely continuous invariant complex valued measure for T.*

Proposition 3.2. *Let h be an element in $L^1(\mu)$ with $h \geq 0$, $L_{T,\mu} h = h$, and $\int h\,d\mu = 1$. Put $\nu = h\,\mu$. For $g \in L^1(\nu)$ and for a measurable function φ with modulus 1, the following are equivalent to one another.*
(i) $g \circ T = \varphi g$ *in* $L^1(\nu)$.
(ii) $L_{T,\nu}(\varphi g) = g$ *in* $L^1(\nu)$.
(iii) $L_{T,\mu}(\varphi g h) = gh$ *in* $L^1(\mu)$.

For a real valued bounded measurable function f and $t \in \mathbb{C}$, the perturbed Perron-Frobenius operator $L_{T,\mu,f}(t)\colon L^1(\mu) \to L^1(\mu)$ is defined by

$$L_{T,\mu,f}(t)g = L_{T,\mu}\big(\exp(\sqrt{-1}\,tf)g\big)$$

for each $g \in L^1(\mu)$. By the assertion (2) in Proposition 2.1, we see that

$$L_{T,\mu,f}(t)^n g = L^n_{T,\mu}\left(\exp\left(\sqrt{-1}\,t\sum_{i=0}^{n-1} f\circ T^i\right) g\right) \tag{3.2}$$

holds. Since $L_{T,\mu}$ preserves the value of the integral by the equation (3.1), the formula (3.2) yields

$$\int \exp\left(\sqrt{-1}\,t\sum_{i=0}^{n-1} f\circ T^i\right) g\,d\mu = \int L_{T,\mu,f}(t)^n g\,d\mu.$$

Therefore one can imagine that the asymptotic behavior of the characteristic function of the partial sum $\sum_{i=0}^{n-1} f\circ T^i$ as n goes to ∞ with respect to the measure $g\,\mu$ and the spectral properties of the perturbed Perron-Frobenius operators are closely related.

Since we work on the unit interval with the Lebesgue measure in the sequel, we write L_T instead of $L_{T,m}$. Since any LY map is nonsingular with respect to the Lebesgue measure m, we can define the Perron-Frobenius operator for it with respect to m. Moreover if T^n is the n-fold iteration of an LY map T with defining partition $\mathcal{P}(T^n) = \{J\}$ and T_J^n denotes the C^2 version of $T^n|_J$, we have

$$(3.3) \qquad L_{T,m}^n g(x) = \sum_{J\in\mathcal{P}(T^n)} \chi_{T_J^n J}(x)|DT_J^n(T_J^{-n}x)|^{-1} g(T_J^{-n}x) \qquad m\text{-a.e.},$$

where the notation T_J^{-n} is the same as we introduced just before Definition 2.3 and χ_A denotes the indicator function of the set A. As mentioned in Introduction, if T is an LY map, L_T acts on the space BV of measurable functions with version of bounded variation as well as $L^1(m)$. For $g \in BV$ we put

$$\bigvee g = \inf\left\{\tilde{\bigvee}\tilde{g} : \tilde{g} \text{ is a function of bounded variation which is a version of } g\right\},$$

where $\tilde{\bigvee}\tilde{g}$ is the total variation of $\tilde{g}$. Then

$$\|g\|_{BV,p} = \|g\|_{p,m} + \bigvee g$$

becomes a Banach norm of BV for each $1 \le p \le \infty$. Since we have $\|g\|_{\infty,m} \le \bigvee g + \|g\|_{1,m}$ for $g \in BV$, we can show $\|g\|_{BV,1} \le \|g\|_{BV,\infty} \le 2\|g\|_{BV,1}$. Therefore we employ $\|g\|_{BV,1}$ as the norm of BV and write it by $\|g\|_{BV}$.

We need some notations in order to describe the Lasota-Yorke type inequality, which plays an important role in the study of the spectral properties of the perturbed Perron-Frobenius operators. Let T be an LY map. For simplicity, we assume that $\rho = \rho(T) < 1$, where ρ is given by $1/\rho = \text{ess.inf}\,|DT|$. Set

$$R = R(T) = \text{ess.sup}\left|\frac{D^2T}{(DT)^2}\right| = \sup_{J\in\mathcal{P}(T)}\sup_{x\in J}\left|\frac{D^2T_J(x)}{(DT_J(x))^2}\right|.$$

We put

$$\Delta_n = \Delta(T^n) = \min_{J\in\mathcal{P}(T^n)} m(T^n J).$$

Then we have the following:

Lemma 3.1 (Lasota-Yorke type inequality). *Let T be an LY map with $\rho < 1$. Let n be a positive integer. Assume that $f_0, \ldots, f_{n-1}$ are functions of bounded variation* *with modulus not greater than* $H \geq 1$. *Then we have*

$$\bigvee L_T^n \left(\left(\prod_{i=0}^{n-1} f_i \circ T^i \right) g \right)$$
$$\leq 2 \left(1 + \sum_{i=0}^{n-1} \bigvee f_i \right) H^n \rho^n \bigvee g$$
$$+ 2 \left(\left(1 + \sum_{i=0}^{n-1} \bigvee f_i \right) \frac{1}{\Delta_n} + \left(2 + \sum_{i=0}^{n-1} \bigvee f_i \right) nR \right) H^n \|g\|_{1,m}$$

for any $g \in BV$.

We need another inequality of Krylov-Bogolioubov type. To describe it we introduce the following notations. For each positive integer n and $J \in \mathcal{P}(T^n)$, choose a point $x_J \in J$. We define a bounded linear operator $\Pi_n \colon BV \to BV$ by

$$\Pi_n g(x) = \sum_{J \in \mathcal{P}(T)} \overline{g}(x_J) \chi_J(x),$$

where $\overline{g}$ is a version of g such that it is right continuous on $[0, 1)$ and satisfies $\overline{g}(1) = \overline{g}(1-)$. Note that $\bigvee g = \tilde{\bigvee} \overline{g}$ holds. Then we have:

Lemma 3.2 (Krylov-Bogolioubov type inequality). *Let T be an LY map with $\rho < 1$. Let n be a positive integer. Assume that $f_0, \ldots, f_{n-1}$ are functions of bounded variation with modulus not greater than $H \geq 1$. Then we have*

$$\left\| L_T^n \left(\left(\prod_{i=0}^{n-1} f_i \circ T^i \right) \Pi_n g \right) - L_T^n \left(\left(\prod_{i=0}^{n-1} f_i \circ T^i \right) g \right) \right\|_{BV}$$
$$\leq \left(5 + 2nR + 2 \sum_{i=0}^{n-1} \bigvee f_i \right) H^n \rho^n \bigvee g$$

for any $g \in BV$.

We shall prove Lemma 3.1 and Lemma 3.2 later in Appendix for reader's convenience.

REMARK 3.1. The versions of Lasota-Yorke type inequality and Krylov-Bogolioubov type inequality are more general than we need in the present work. But if one considers random iteration of nondegenerate maps as in [8], he or she can recognize that the present forms of these inequalities are more useful. In this paper, we just apply these inequalities to the case when $f_0 = f_1 = \cdots = f_{n-1} = \exp(\sqrt{-1}\, tf)$

and f being a real valued element in BV. In such a case, we have

$$\begin{aligned}&\bigvee L^n_{T,f}(t)g\\ &\leq 2\left(1+n|t|e^{|t|\|f\|_{\infty,m}}\bigvee f\right)(e^{|t|\|f\|_{\infty,m}}\rho)^n\bigvee g\\ &\quad+2\left(\left(1+n|t|e^{|t|\|f\|_{\infty,m}}\bigvee f\right)\frac{1}{\Delta_n}+\left(2+n|t|e^{|t|\|f\|_{\infty,m}}\bigvee f\right)nR\right)e^{n|t|\|f\|_{\infty,m}}\|g\|_{1,m}\end{aligned}$$

and

$$\|L_{T.f}(t)g-L^n_{T,f}(t)\Pi_n g\|_{BV}\leq\left(5+2nR+2n|t|e^{|t|\|f\|_{\infty,m}}\bigvee f\right)(e^{|t|\|f\|_{\infty,m}}\rho)^n\bigvee g.$$

We notice that

$$L^n_{T,f}(t)\Pi_n g=\sum_{J\in\mathcal{P}(T^n)}\bar{g}(x_J)L^n_{T,f}(t)\chi_J$$

holds. Thus $L^n_{T,f}(t)\Pi_n$ is an operator of finite rank, therefore a compact operator. This implies that $L_{T,f}(t)$ is a quasicompact operator on BV with the essential spectral radius not greater than $e^{|t|\|f\|_{\infty,m}}\rho$. Hence the spectrum in the domain $\{\lambda : |\lambda|>e^{|t|\|f\|_{\infty,m}}\rho\}$ consists of isolated eigenvalues with finite multiplicity. (see [1]).

Our main concern is the following class of LY maps.

DEFINITION 3.1. A Lasota-Yorke map T is called a Bernoulli Lasota-Yorke map (a BLY map for short) if it has a unique a.c.i.m. $\mu=\mu_T$ and the measure-theoretic dynamical system (T,μ) is Bernoulli.

In the following proposition, we enumerate the properties of BLY maps and their perturbed Perron-Frobenius operators which we need in the later investigations. Proofs can be found in [4], [5], [8], and [12]. We have to note that the Lasota-Yorke type inequality and the Krylov-Bogolioubove type inequality play importnat roles in these references too.

In the sequel, $f\in BV$ is real valued. $\mathcal{L}(V)$ denotes the Banach space of bounded linear operators on a Banach space V with operator norm. Since f is fixed, we write $L_T(t)$ instead of $L_{T,f}(t)$.

Proposition 3.3. (1) *Let T be an LY map and L_T the Perron-Frobenius operator for T. Then T is Bernoulli if and only if* 1 *is a unique eigenvalue with modulus* 1 *of $L_T\colon L^1(m)\to L^1(m)$ and it is simple.*
(2) *The mapping $\mathbb{C}\ni t\mapsto L_T(t)\in\mathcal{L}(BV)$ is analytic.*
From now on, we assume that T is a BLY map.
(3) *For $t\in\mathbb{R}\setminus\{0\}$, the spectral radius of $L_T(t)$ as an element in $\mathcal{L}(BV)$ is less than* 1.

(4) *There exists* $\tau = \tau(T, f) > 0$ *such that the spectral decomposition*

$$L_T(t)^n = \lambda_T(t)^n P_T(t) + R_T(t)^n \tag{3.4}$$

satisfying the following properties is valid for any $t \in \mathbb{C}$ *with* $|t| < \tau$.

(4-i) $P_T(t)^2 = P_T(t)$ *and* $P_T(t)R_T(t) = R_T(t)P_T(t) = O$.

(4-ii) $\lambda_T(\cdot)$, $P_T(\cdot)$, *and* $R_T(\cdot)$ *are anlytic in* $\{t : |t| < \tau\}$. $\lambda_T(t)$ *is the first eigenvalue, i.e. the eigenvalue with maximal modulus, of* $L_T(t) \in \mathcal{L}(BV)$ *and* $P_T(t)$ *is the projection onto the one dimensional eigenspace corresponding to* $\lambda_T(t)$

(4-iii) *There exist* $r > 0$ *and* $r' > 0$ *with* $r + r' < 1$ *independent of* t *such that* $|\lambda_T(t) - 1| < r'$ *and the spectrum of* $L_T(t) \in \mathcal{L}(BV)$ *except for the eigenvalue* $\lambda_T(t)$ *is contained in the* disc$\{t : |t| < r\}$. *Moreover,* $P_T(t)$ *and* $R_T(t)^n$ *is written in terms of the Dunford integral by*

$$P_T(t) = \frac{1}{2\pi\sqrt{-1}} \int_{(|\lambda-1|=r')} (\lambda I - L_T(t))^{-1}\, d\lambda$$
$$R_T(t)^n = \frac{1}{2\pi\sqrt{-1}} \int_{(|\lambda|=r)} \lambda^n (\lambda I - L_T(t))^{-1}\, d\lambda.$$

In particular, the spectral radius of $R_T(t) \in \mathcal{L}(BV)$ *is not greater than* r.

(4-iv) $h_T = P_T(0)1$ *is the density of the unique a.c.i.m.* μ_T *of* T *and the following formulas hold.*

$$\frac{d\lambda_T}{dt}(0) = \sqrt{-1}\int f\, d\mu_T \quad \text{and} \quad \frac{d^2\lambda_T}{dt^2}(0) = -\sigma_T(f)^2,$$

where $\sigma_T(f)^2$ *is the limit variance mentioned in Introduction.*

(4-v) *If* $t = 0$ *the spectral decomposition* (3.4) *and the assertion* (4-i) *are valid for* $L_T(0) = L_T$ *as an element in* $\mathcal{L}(L^1(m))$.

4. Auxiliary lemmas

We shall prove some lemmas in this section. As a consequence we can easily prove Theorem 1.1. But in order to prove Theorem 1.2 we need more investigations.

Lemma 4.1. *Let* T_0 *be an LY map satisfying* ess.inf$|DT_0| > 2$ *and* f *a real valued function of bounded variation. Then there exist* $t_1 > 0$, $\delta_1 > 0$, $K > 0$, *and* $0 < \alpha < 1$ *such that the following assertions are valid whenever* $T \in PC^2$ *and* $t \in \mathbb{C}$ *satisfy* $d(T, T_0) < \delta_1$ *and* $|t| < t_1$, *respectively.*

(1) *The spectrum of* $L_T(t)$ *in* $\{\lambda \in \mathbb{C} : |\lambda| > \alpha/2\}$ *consists of isolated eigenvalues with finite multiplicity.*

(2)
$$\bigvee L_T(t)g \le \alpha \bigvee g + K\|g\|_{1,m}$$

holds for any $g \in BV$.

Proof. By the definition of the metric d, if $d(T, T_0)$ is small enough, T is also an LY map with ess.inf $|DT_0| > 2$. Moreover, $\rho(T)$, $R(T)$, and $\Delta(T)$ which appeared in the inequalities in Remark 3.1 become close to $\rho(T_0)$, $R(T_0)$, and $\Delta(T_0)$, respectively. Hence we can choose $t_1 > 0$, $\delta_1 > 0$, $K > 0$, and $0 < \alpha < 1$ with the desired properties. □

Next we show a sort of uniform estimate on the total variation of eigenfunctions of the perturbed Perron-Frobenius operators $L_{T,f}$ for T being close to T_0.

Lemma 4.2. *Let T_0 be an LY map satisfying* ess.inf $|DT_0| > 2$ *and f a real valued function of bounded variation. The numbers $t_1 > 0$, $\delta_1 > 0$, $K > 0$, and $0 < \alpha < 1$ are the same as in* Lemma 4.1. *If $L_T(t)g = \lambda g$ holds for some $g \in BV$, T with $d(T, T_0) < \delta_1$, $t \in \mathbb{C}$ with $|t| < t_1$, and $\lambda \in \mathbb{C}$ with $|\lambda| > \alpha$, then we have $\bigvee g \le C\|g\|_{1,m}$ for some positive constant C depending on $t_1 > 0$, $\delta_1 > 0$, $K > 0$, and $0 < \alpha < 1$ but not on g, T, t, and λ.*

Proof. For the sake of simplicity, we put $M = \lambda^{-1}L_T(t)$. Applying Lemma 4.1 repeatedly we have

$$\begin{aligned}&\bigvee M^n g \\ &\le |\lambda^{-1}\alpha|^n \bigvee g + K|\lambda^{-1}|\big(\|M^{n-1}g\|_{1,m} + |\lambda^{-1}\alpha|\|M^{n-2}g\|_{1,m} + \cdots + |\lambda^{-1}\alpha|^{n-1}\|g\|_{1,m}\big).\end{aligned}$$

Therefore $Mg = g$ implies

$$\bigvee g \le |\lambda^{-1}\alpha|^n \bigvee g + \frac{K|\lambda^{-1}|}{1 - |\lambda^{-1}\alpha|}\|g\|_{1,m}.$$

Hence the desired inequality holds with $C = K|\lambda^{-1}|/(1 - |\lambda^{-1}\alpha|)$. □

Now we can prove:

Lemma 4.3. *Let T_0 be a BLY map satisfying* ess.inf $|DT_0| > 2$. *Then there exist $\delta_2 \in (0, \delta_1)$ and $r_1 \in (\alpha, 1)$ such that $d(T, T_0) < \delta_2$ implies that 1 is the unique eigenvalue of L_T contained in the region $\{\lambda \in \mathbb{C} : |\lambda| \ge r_1\}$.*

Proof. Note that the modulus of any eigenvalue of L_T is not greater than 1. If the lemma is not valid, we can choose LY maps $T_n \in PC^2$, functions $g_n \in BV$, and complex numbers $\lambda_n \neq 1$ such that

$$L_n g_n = \lambda_n g_n, \quad \|g_n\|_{BV} = 1, \quad \text{and} \quad |\lambda_n| \to 1 \quad (n \to \infty),$$

where $L_n = L_{T_n}$.

In virtue of Helly's selection theorem, we may assume that there exist $g \in BV$ and λ with $|\lambda| = 1$ such that $g_n \to g$ a.e. and in $L^1(m)$ and $\lambda_n \to \lambda$ as $n \to \infty$ by taking subsequences if necessary. Clearly we get $L_0 g = \lambda g$, where $L_0 = L_{T_0}$.

We claim that $g = 0$. First, by Proposition 3.3 (1), $\lambda = 1$, i.e. $L_0 g = g$ since T_0 is Bernoulli. Since L_n preserves the value of the integration, we see

$$\int g_n \, dm = \int L_n g_n \, dm = \lambda_n \int g_n \, dm$$

for each $n \geq 1$. This yields $\int g_n \, dm = 0$. Consequently we have $\int g \, dm = 0$. Since the eigenspace of L_0 corresponding to 1 is one dimensional, we can see that $g = 0$.

On the other hand, from Lemma 4.2, we have

$$1 = \|g_n\|_{BV} = \bigvee g_n + \|g_n\|_{1,m} \leq (C + 1)\|g_n\|_{1,m}.$$

But the claim implies that the right hand side goes to 0 as $n \to \infty$. Now we arrive at a contradiction. □

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Assume that T_0 is a BLY map. In virtue of the assertion (1) in Proposition 3.3 and Lemma 4.3, it suffices to show the following.

CLAIM. There exists $\delta_3 \in (0, \delta_2)$ such that if $d(T, T_0) < \delta_3$, 1 is a simple eigenvalue of $L_T : BV \to BV$.

Note that if 1 is not simple, we can choose at least two ergodic a.c.i.m.'s. Thus if Claim is not true, we can find sequences $T_n \in PC^2$, $g_n \in BV$, and $h_n \in BV$ such that

$$d(T_n, T_0) \to 0 \quad (n \to \infty), \quad g_n h_n = 0,$$
$$g_n \geq 0, \quad \int g_n \, dm = 1, \quad L_n g_n = g_n, \quad \text{and} \quad h_n \geq 0, \quad \int h_n \, dm = 1, \quad L_n h_n = h_n,$$

where $L_n = L_{T_n}$. Since we see $\bigvee g_n \leq C$ and $\bigvee h_n \leq C$ from Lemma 4.2, we can apply Helly's theorem to g_n and h_n. Therefore choosing a subsequence if necessary, we may assume that there exist g, $h \in BV$ such that $g_n \to g$ and $h_n \to h$ a.e. and in $L^1(m)$ as $n \to \infty$. Then it is easy to show that

$$L_0 g = g, \quad L_0 h = h, \quad \text{and} \quad \int g \, dm = \int h \, dm = 1,$$

where $L_0 = L_{T_0}$.

Since T_0 is Bernoulli, we conclude that $g = h = h_0$, where h_0 is the density of the unique a.c.i.m. of T_0. Hence we must have

$$2 = \|g_n - h_n\|_{1,m} \to \|g - h\|_{1,m} = 0, \quad (n \to \infty).$$

This is a contradiction. □

Corollary 4.1. *Let T_0 be a BLY map with* ess.inf $|DT_0| > 2$ *having a unique a.c.i.m. μ_0 with density h_0. If $T_n \in PC^2$ converges to T_0 in PC^2 as $n \to \infty$, then for any n large enough, T_n turns out to be a BLY map having a unique a.c.i.m. μ_n with density $h_n \in BV$ and h_n converges to h_0 a.e. and in $L^1(m)$.*

Proof. There exists n_0 such that $d(T_n, T_0) < \delta_3$ for any $n \geq n_0$. In addition, $\bigvee h_n \leq C$ holds for any $n \geq n_0$ from Lemma 4.2. Therefore by Helly's theorem, any subsequence h_n contains a subsequence g_n converges to some $g \in BV$ a.e. and in $L^1(m)$. Clearly $\int g\,dm = 1$ and $L_{T_0} g = g$ are true. This implies $g = h_0$. Thus we reach the desired result. □

REMARK 4.1. We need the condition ess.inf $|DT_0| > 2$ to prove Lemma 4.1. From the proof one can easily recognize that the assertions in Lemma 4.1 are still valid if T_0 is an LY map satisfying

(A) There exists a positive integer N such that ess.inf $|DT_0^N| > 2$ and the mapping $T \mapsto T^N$ is continuous at T_0.

Hence we can see that any result in the present section can be obtained if we replace the condition ess.inf $|DT_0| > 2$ by (A).

The rest of the present section is devoted to giving examples which show that the conditions Bernoullicity and ess.inf $|DT| > 2$ are essential. The following example shows that without ess.inf $|DT_0| > 2$, the Bernoullicity is not necessarily stable by PC^2 perturbation.

EXAMPLE 4.1. For any sufficiently small $\epsilon \geq 0$, we set

$$T_\epsilon x = \begin{cases} \dfrac{4}{1-4\epsilon}x & \text{if } x \in \left[0, \dfrac{1-4\epsilon}{4}\right) \\ -2x + \dfrac{3-4\epsilon}{2} & \text{if } x \in \left[\dfrac{1-4\epsilon}{4}, \dfrac{1-2\epsilon}{2}\right) \\ 2x - \dfrac{1}{2} & \text{if } x \in \left[\dfrac{1-2\epsilon}{2}, \dfrac{3}{4}\right) \\ -2x + \dfrac{5}{2} & \text{if } x \in \left[\dfrac{3}{4}, 1\right]. \end{cases}$$

Note that T_ϵ restricted to $[1/2, 1]$ is the so-called tent map for any ϵ. It easy to see that $d(T_\epsilon, T_0) \to 0$ as $\epsilon \to 0$. If $\epsilon > 0$ is small enough, we have $T_\epsilon[1-2\epsilon/2, 1/2] = [1-2\epsilon/2, 1/2]$ and $T_\epsilon[1/2, 1] = [1/2, 1]$. Therefore T_ϵ can not be a BLY.

On the other hand, T_0 is shown to be a BLY as follows. T_0 restricted to $[1/2, 1]$ is the tent map. Thus it is mixing. Since it is obvious that m almost every point in $[0, 1/2]$ is attracted by the interval $[1/2, 1]$, T_0 restricted to $[1/2, 1]$ is the unique mixing component of T_0.

If $\text{ess.inf}\,|DT_0| > 2$ were satisfied, such a phenomenon could not happen by Theorem 1.1. But we have $\text{ess.inf}\,|DT_0| = 2$ in the present case.

The assumption of Bernoullicity in Theorem 1.1 means that T_0 has a unique mixing component. Therefore we may regard the assertion in Theorem 1.1 as the stability on the number of mixing components under PC^2 perturbation. If T_0 is an LY map with more than one mixing components, it is natural to ask whether such a stability result holds or not. More precisely, is there a neghborhood of T_0 in the PC^2 topology whose members have the same number of mixing components as T_0? Next we show that if we do not impose the Bernoullicity condition on T_0, we can not obtain the stability of number of mixing components even if the slope condition $\text{ess.inf}\,|DT_0| > 2$. is satisfied. We recall one of Bowen's criterion in [2, Theorem 2] for convenience.

Proposition 4.1 (Bowen [2]). *Let T be an LY map with defining partition $\mathcal{P}$. If the conditions that* $\text{ess.inf}\,|DT| > 2$ *and* $m(T^n J) \to 1$ $(n \to \infty)$ *holds for each* $J \in \mathcal{P}$ *are satisfied, then T turns out to be a BLY map.*

EXAMPLE 4.2. For sufficiently small $\epsilon \geq 0$ we define T_ϵ by

$$T_\epsilon x = \begin{cases} 3(1+2\epsilon)x & \text{if } x \in \left[0, \dfrac{1}{6}\right) \\ -3(1+2\epsilon)x + 1 + 2\epsilon & \text{if } x \in \left[\dfrac{1}{6}, \dfrac{1}{3}\right) \\ 3x - 1 & \text{if } x \in \left[\dfrac{1}{3}, \dfrac{2}{3}\right) \\ -3(1+2\epsilon)x + 3 - 4\epsilon & \text{if } x \in \left[\dfrac{2}{3}, \dfrac{5}{6}\right) \\ 3(1+2\epsilon)x - 2 - 6\epsilon & \text{if } x \in \left[\dfrac{5}{6}, 1\right). \end{cases}$$

Put $J(1) = I[0, 1/6]$, $J(2) = [1/6, 1/3]$, $J(3) = [1/3, 2/3]$, $J(4) = [2/3, 5/6]$, and $J(5) = [5/6, 1)$. Clearly these intervals form the defining partition $\mathcal{P}(T_\epsilon)$ for any $\epsilon \geq 0$ and we have $d(T_\epsilon, T_0) \to 0$ as $\epsilon \to 0$.

We can easily see that $T_0[0, 1/2] = [0, 1/2]$, $T_0[1/2, 1] = [1/2, 1]$ and T_0 preserves the Lebesgue measure m. Since $\text{ess.inf}\,|DT_0| = 3$, Bowen's criterion implies that $(T_0, 2m|_{[0,1/2]})$ and $(T_0, 2m|_{[1/2,1]})$ are mixing componets. Thus T_0 has exactly two mixing components. On the other hand, we can show that T_ϵ is a BLY map for any

$\epsilon > 0$ i.e. it has only one mixing component. This means that for an LY map T with more than one mixing components, we do not always find an $\epsilon_0 > 0$ such that any element in the ϵ_0-neighborhood of T has the same number of mixing components as T even if T satisfies ess.inf $|DT| > 2$.

It remains to show that T_ϵ ($\epsilon > 0$) is BLY. We verify that $m(T_\epsilon^n J(j)) \to 1$ ($n \to \infty$) for each $j = 1, 2, \ldots, 5$. For $J(3)$ this is trivial. We consider $J(1)$. $T_\epsilon J(1) = [0, 1/2] \cup [1/2, 1/2 + \epsilon)$. It is easy to see that $m(T_\epsilon^n[1/2, 1/2 + \epsilon)) \supset [1/2, 1]$ for some n. Therefore we have $T_\epsilon^{n+1} J(1) = [0, 1]$. For $J(2)$, $J(4)$, and $J(5)$ we can obtain the desired result in the same way. Since ess.inf $|DT_\epsilon| = 3(1+2\epsilon) > 2$, T_ϵ is BLY in virtue of Bowen's criterion.

5. Continuity of the first eigenvalue of $L_T(t)$ and Proof of Theorem 1.2

Throughout the section, T_0 is a BLY map satisfying the condition ess.inf $|DT_0| > 2$ unless otherwise stated. The numbers r_1 and δ_3 are the same as in Lemma 4.3 and Claim in Proof of Theorem 1.1, respectively. Recall that if $d(T, T_0) < \delta_3$, we are in the following situation. In the region $\{\lambda \in \mathbb{C} : |\lambda| \geq r_1\}$, there is no spectrum of $L_T \in \mathcal{L}(BV)$ except for the simple eigenvalue 1. Thus we can choose $r_2 > 0$ with $r_1 + r_2 < 1$ independent of T so that the punctured disc$\{\lambda : |\lambda - 1| \leq r_2\} \setminus \{1\}$ is contained in the resolvent set of L_T.

Put $\Lambda = \{\lambda \in \mathbb{C} : |\lambda| \geq r_1,\ |\lambda - 1| \geq r_2\}$. We need the following technical lemma.

Lemma 5.1. *There exists $\delta_4 \in (0, \delta_3)$ such that*

$$\beta = \sup_{T : d(T,T_0) < \delta_4} \sup_{\lambda \in \Lambda} \|(\lambda I - L_T)^{-1}\|_{BV} < \infty.$$

Proof. We know that the set Λ is contained in the resolvent set of $L_T \in \mathcal{L}(BV)$ and

$$\sup_{\lambda \in \Lambda} \|(\lambda I - L_T)^{-1}\|_{BV} < \infty$$

is valid for each fixed T with $d(T, T_0) < \delta_3$. Note that we can replace $\sup_{\lambda \in \Lambda}$ by $\sup_\Omega$, where $\Omega = \Lambda \cap \{\lambda ; |\lambda| \leq 2\}$ because of the following reason.

Combining Lemma 4.1 and the fact $\|L_T\|_{1,m} = 1$, we have

$$\bigvee L_T^n g \leq \alpha^n \bigvee g + \frac{K}{1-\alpha} \|g\|_{1,m}$$

for any $T \in PC^2$ with $d(T, T_0) < \delta_3 < \delta_1$. It follows that

$$\|L_T^n\|_{BV} \leq \alpha^n + 1 + \frac{K}{1-\alpha} \leq \frac{K + 2 - \alpha}{1 - \alpha}.$$

This yields

$$\|(\lambda I - L_T)^{-1}\|_{BV} \le \sum_{n=0}^{\infty} \frac{\|L_T^n\|_{BV}}{|\lambda|^{n+1}} \le \frac{K+2-\alpha}{1-\alpha}$$

if $|\lambda| \ge 2$.

To obtain the desired result, we have to show that we can choose $\delta_4 \in (0, \delta_3)$ such that

$$\inf_{T:d(T.T_0)<\delta_4} \inf_{\lambda\in\Omega} \inf_{g:\|g\|_{BV}=1} \|(\lambda I - L_T)g\|_{BV} > 0.$$

If we can not find such a δ_4, there exist $T_n \in PC^2$, $g_n \in BV$ with $\|g_n\|_{BV} = 1$, $\lambda_n \in \Omega$ such that

$$d(T_n, T_0) \to 0, \quad \text{and} \quad h_n = (\lambda_n I - L_n)g_n \to 0 \text{ in } BV \quad (n \to \infty),$$

where $L_n = L_{T_n}$. Apply Lemma 4.1 to $g_n = \lambda_n^{-1}h_n + \lambda_n^{-1}L_n g_n$, we have

$$\bigvee g_n \le |\lambda_n^{-1}| \bigvee h_n + |\lambda_n^{-1}|\alpha \bigvee g_n + K|\lambda_n^{-1}|\|g_n\|_{1,m}.$$

This yields

$$\bigvee g_n \le \frac{|\lambda_n^{-1}|}{1-|\lambda_n^{-1}|\alpha}\left(\bigvee h_n + K\|g_n\|_{1,m}\right). \tag{5.1}$$

Next, since $\|g_n\|_{BV} = 1$, we can apply Helly's theorem to g_n. Thus we may assume that there exist $g \in BV$ and $\lambda \in \Omega$ such that $g_n \to g$ a.e. and in $L^1(m)$ and $\lambda_n \to \lambda$ as $n \to \infty$. Clearly,

$$(\lambda_n I - L_n)g_n \to (\lambda I - L_0)g$$

a.e. and in $L^1(m)$ as $n \to \infty$, where $L_0 = L_{T_0}$.

On the other hand the left hand side in the above converges to 0 in BV by our assumption. Hence we see $L_0 g = \lambda g$. Since $L_0 \in \mathcal{L}(BV)$ can not have an eigenvalue with $|\lambda| \ge r_1$, the choice of r_2 implies $g = 0$. Combining this fact with the inequality (5.1), we conclude that $\bigvee g_n \to 0$ as $n \to \infty$. Consequently we can get

$$1 = \|g_n\|_{BV} = \bigvee g_n + \|g_n\|_{1,m} \to 0 \quad (n \to \infty).$$

This is a contradiction. □

Next we give a uniform bound for the resolvent operators of the perturbed Perron-Frobenius operators $L_T(t)$ with $d(T, T_0) < \delta_4$ for any t with sufficiently small absolute value.

Lemma 5.2. *There exists $t_2 \in (0, t_1)$ such that if $d(T, T_0) < \delta_4$, the set Λ is contained in the resolvent set of $L_T(t) \in \mathcal{L}(BV)$ for any $t \in \mathbb{C}$ with $|t| < t_2$. In particular, we can choose t_2 so that*

$$\sup_{T:d(T,T_0)<\delta_4} \sup_{t:|t|<t_2} \sup_{\lambda\in\Lambda} \|(\lambda I - L_T(t))^{-1}\|_{BV} \le 2\beta$$

holds.

Proof. Recall the following elementary fact (see [3, VII-6]).

Let A and B be bounded linear operators on a Banach space $\mathcal{X}$. Assume that A is invertible and $\|(A - B)A^{-1}\| < 1$. Then B is invertible and B^{-1} can be expressed by $B^{-1} = \sum_{n=0}^{\infty} A^{-1}((A - B)A^{-1})^n$.

We show that we can choose t_2 so that if $|t| < t_2$, we can apply this fact to $A = \lambda I - L_T$ and $B = \lambda I - L_T(t)$ for any $\lambda \in \Lambda$. To this end we estimate $\|L_T(t) - L_T\|_{BV}$. We can easily show that for any $t \in \mathbb{C}$ and for any $g \in BV$

$$\begin{aligned} \|\big(e^{\sqrt{-1}tf} - 1\big)g\|_{1,m} &\le \big(e^{|t|\|f\|_{BV}} - 1\big)\|g\|_{1,m}, \\ \bigvee\big((e^{\sqrt{-1}tf} - 1)g\big) &\le \big(e^{|t|\|f\|_{BV}} - 1\big)\bigvee g + |t|\big(e^{|t|\|f\|_{BV}} - 1\big)\bigvee f\, \|g\|_{1,m}. \end{aligned}$$

Thus we have

$$\|\big(e^{\sqrt{-1}tf} - 1\big)g\|_{BV} \le \big(e^{|t|\|f\|_{BV}} - 1\big)\Big(1 + |t|\bigvee f\Big)\|g\|_{BV} = \gamma(t)\|g\|_{BV}.$$

Therefore it follows that

$$\|(L_T(t) - L_T)g\|_{BV} \le \|L_T\|_{BV}\|(e^{\sqrt{-1}tf} - 1)g\|_{BV} \le \|L_T\|_{BV}\gamma(t)\|g\|_{BV}.$$

On the other hand, by Lemma 4.1, if $d(T, T_0) < \delta_4 < \delta_1$, we have $\|L_T\|_{BV} \le K + 1$ in the same way as in the proof of Lemma 5.1. Hence if $d(T, T_0) < \delta_4$ and $\lambda \in \Lambda$, we obtain

$$\|(L_T(t) - L_T)(\lambda I - L_T)^{-1}\|_{BV} \le \|L_T\|_{BV}\gamma(t)\beta \le (K + 1)\gamma(t)\beta$$

in virtue of Lemma 5.1.

Notice that $\gamma(t) \downarrow 0$ as $t \downarrow 0$. Therefore we can choose $t_2 \in (0, t_1)$ so that $|t| < t_2$ implies $(K + 1)\gamma(t)\beta < 1/2$. Then we have the desired results □

In virtue of Lemma 5.2, we see that if $d(T, T_0) < \delta_4$ and $|t| < t_2$, we can define the projections

$$\begin{aligned} P_T(t) &= \frac{1}{2\pi\sqrt{-1}} \int_{(|\lambda-1|=r_2)} (\lambda I - L_T(t))^{-1}\, d\lambda, \\ Q_T(t) &= \frac{1}{2\pi\sqrt{-1}} \int_{(|\lambda|=r_1)} (\lambda I - L_T(t))^{-1}\, d\lambda \end{aligned} \tag{5.2}$$

by using the Dunford integral. In addition we have

(5.3)
$$\|P_T(t)\|_{BV} \le 2\beta r_2, \|Q_T(t)\|_{BV} \le 2\beta r_2, \text{ and } \|R_T(t)^n\|_{BV} = \|L_T(t)^n Q_T(t)\|_{BV} \le 2\beta r_1^{n+1}.$$

In particular,

$$\begin{aligned}\|(\lambda I - L_T(t))^{-1} - (\lambda I - L_T))^{-1}\|_{BV} =& \|(\lambda I - L_T(t))^{-1}(L_T - L_T(t))(\lambda I - L_T))^{-1}\|_{BV} \\ \le& 2\beta^2(K+1)\gamma(t)\end{aligned}$$

and

$$\|P_T(t) - P_T\|_{BV} \le 2r_2\beta^2(K+1)\gamma(t) \tag{5.4}$$

hold. Therefore we can show the following:

Lemma 5.3. *There exists* $t_3 \in (0, t_2)$ *such that if* $T \in PC^2$ *satisfies* $d(T, T_0) < \delta_4$, *the mapping* $\{t \in \mathbb{C} : |t| < t_3\} \ni t \mapsto P_T(t) \in \mathcal{L}(BV)$ *is analytic and* dim $P_T(t)BV = 1$.

Proof. The assertion in analyticity is obvious. Thus we have only to prove the second assertion. Recall the fact that if E_1 and E_2 are projections on a Banach space $\mathcal{X}$ satisfying $\|E_1 - E_2\| < \min(\|E_1\|^{-1}, \|E_2\|^{-1})$, then $\dim E_1\mathcal{X} = \dim E_2\mathcal{X}$ holds.

Therefore if we choose $t_3 < t_2$ so small that $\gamma(t_3)$ satisifies $\beta(K+1)\gamma(t_3) < 1$, we get the desired result by the inequalities (5.3) and (5.4). □

We have seen that the spectral decomposition (3.4) in Proposition 3.3 is valid simultaneously in T with $d(T, T_0) < \delta_4$ and t with $|t| < t_3$ in the following sense.

Proposition 5.1. *Let* T_0 *be a BLY map satisfying* ess.inf $|DT_0| > 2$. *Then there exist* $r_1 > 0$, $r_2 > 0$, $t_3 > 0$ *and* $\delta_4 > 0$ *with* $r_1 + r_2 < 1$ *such that whenever* $T \in PC^2$ *and* $t \in \mathbb{C}$ *satisfy* $d(T, T_0) < \delta_4$ *and* $|t| < t_3$, *the perturbed Perron-Frobenius operator* $L_T(t) \in \mathcal{L}(BV)$ *has the spectral decomposition*

$$L_T(t)^n = \lambda_T(t)^n P_T(t) + R_T(t)^n,$$

where $P_T(t)$ *defined by* (5.2) *is the projection onto the one dimensional eigenspace corresponding to the first eigenvalue* $\lambda_T(t)$ *of* $L_T(t)$ *and* $R_T(t)$ *satisfies the estimate in* (5.3). *Moreover,* $\{\lambda_T(\cdot)\}_{T:d(T,T_0)<\delta_4}$ *is a normal family of analytic functions in* $\{t \in \mathbb{C} : |t| < t_3\}$ *satisfying*

$$\sup_{T:d(T,T_0)<\delta_4} \sup_{t:|t|<t_3} |\lambda_T(t)| \le 1 + r_2.$$

Proof. The proposition is an easy consequence of Propositon 3.3 and the observation in the above. Note that the last assertion follows from Montel's theorem. □

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. Let $\mathcal{H}$ be the Fréchet space of analytic functions in $\{t : |t| < t_3\}$. We show that $\{T : d(T, T_0) < \delta_4\} \ni T \mapsto \lambda_T(\cdot) \in \mathcal{H}$ is continuous. We just prove the continuity at the center T_0 since we can prove the continuity at the other points in the same way.

In virtue of Proposition 5.1, $\{\lambda_T(\cdot) : d(T, T_0) < \delta_4\}$ is relatively compact in $\mathcal{H}$. It remains to show that if $\lambda_n(\cdot) = \lambda_{T_n}(\cdot)$ converges to $\lambda(\cdot)$ uniformly on any compact set in $\{t : |t| < t_3\}$ as $d(T_n, T_0) \to 0$, we have $\lambda(t) = \lambda_0(t)$ for any t with $|t| < t_3$, where $\lambda_0(\cdot) = \lambda_{T_0}(\cdot)$.

Let $g_n \in BV$ satisfy

$$L_n(t)g_n = \lambda_n(t)g_n, \quad \text{and} \quad \|g_n\|_{BV} = 1,$$

where $L_n(t) = L_{T_n}(t)$. By Helly's theorem, we may assume that there exists $g \in BV$ such that $g_n \to g$ a.e. and in $L^1(m)$. Thus we have $L_0(t)g = \lambda(t)g$, where $L_0(t) = L_{T_0}(t)$. If we can show that $g \neq 0$, $|\lambda(t)| \leq 1 + r_2$ implies this must coincides with $\lambda_0(t)$ by Lemma 5.2.

On the other hand $\bigvee g_n \leq C\|g_n\|_{1,m}$ is valid for each n by Lemma 4.2. It follows that

$$1 = \|g_n\|_{BV} = \bigvee g_n + \|g_n\|_{1,m} \leq (C + 1)\|g_n\|_{1,m}.$$

Thus we see $\|g_n\|_{1,m} \geq 1/(C+1)$. Consequently, $\|g\|_{1,m} \geq 1/(C+1)$. Hence we arrive at the desired result. □

REMARK 5.1. It is not hard to see that all the results in the present section are true for T_0 satisfying the assumption (A) in Remark 5.1

6. Examples

In this section we cite two examples which illustrate how the metric d is appropriate to measure the difference between an LY map and the other element in PC^2 from a viewpoint of the ergodic theory of a.c.i.m.'s.

By Corollary 4.1 if T_n converges to a BLY map T_0 with ess.inf $|DT_0| > 2$ in the metric d as $n \to \infty$, then T_n is a BLY map for sufficiently large n and the invariant density h_n of T_n converges to the invariant density h_0 of T_0 a.e. and in $L^1(m)$. The first example shows that even if T_n is assumed to be a continuous BLY map for any $n \geq 0$, $\|T_n - T_0\|_{\infty,m}$ $(n \to \infty)$ does not always imply the convergence of the invariant density.

EXAMPLE 6.1. Consider the following maps with $n \geq 4$.

$$T_n x = \begin{cases} 3x & \text{if } x \in \left[0, \frac{1}{3n}\right), \\ -3\left(x - \frac{2}{3n}\right) & \text{if } x \in \left[\frac{1}{3n}, \frac{2}{3n}\right), \\ 3\left(x - \frac{2}{3n}\right) & \text{if } x \in \left[\frac{2}{3n}, \frac{1}{n}\right), \\ \frac{3(n-1)}{n-3}\left(x - \frac{1}{n}\right) + \frac{1}{n} & \text{if } x \in \left[\frac{1}{n}, \frac{1}{3}\right), \\ -3\left(x - \frac{2}{3}\right) & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right), \\ 3\left(x - \frac{2}{3}\right) & \text{if } x \in \left[\frac{2}{3}, 1\right], \end{cases}$$

$$T_0 x = \begin{cases} 3x & \text{if } x \in \left[0, \frac{1}{3}\right), \\ -3\left(x - \frac{2}{3}\right) & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right), \\ 3\left(x - \frac{2}{3}\right) & \text{if } x \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

Then $\|T_n - T_0\|_{\infty,m} \leq 3/n$. T_n is a BLY map with unique a.c.i.m. $\mu_n = n\, m|_{[0,1/n]}$ and T_0 is a BLY with unique a.c.i.m. $\mu_0 = m$. Clearly, $\mu_n \to \delta_0$ weakly as $(n \to \infty)$. On the other hand we can see $d(T_n, T_0) \geq 3$ for $n \geq 4$ since $k(T_n) = 6$ and $k(T_0) = 3$.

Next, when one gives a glance at our result, one may consider the possibility of the phenomenon as $\lim_{d(T_n,T_0)\to 0} \|L_{T_n} - L_{T_0}\|_{BV} = 0$. If such a phenomenon could occur in general, our results would be no interest and would be easy consequences of the general perturbation theory for linear operators. The second example shows that $\lim_{d(T_n,T_0)\to 0} \|L_{T_n} - L_{T_0}\|_{BV} \neq 0$ in general. This means that the topology induced by the metric d is not so large that it can make the map $T \mapsto L_T \in \mathcal{L}(BV)$ continuous while it is large enough to distinguish the ergodic properties of a.c.i.m. of T_0 from those of T_n.

EXAMPLE 6.2. Let $\epsilon \geq 0$ be small enough. We consider the maps:

$$T_\epsilon x = \begin{cases} 3(1-\epsilon)x + \epsilon & \text{if } x \in \left[0, \frac{1}{3}\right), \\ 3\left(x - \frac{1}{3}\right) & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right), \\ 3\left(x - \frac{2}{3}\right) & \text{if } x \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

Clearly we have $d(T_\epsilon, T_0) \leq 2\epsilon$ and we can apply Theorem 1.1 and Theorem 1.2 to this family. Choose $f_\epsilon = \chi_{[0,\epsilon]} \in BV$. Then

$$(L_{T_\epsilon} - L_{T_0})f_\epsilon(x) = \chi_{[\epsilon,1]}(x)\frac{1}{3(1-\epsilon)}\chi_{[0,\epsilon]}\left(\frac{x}{3(1-\epsilon)} - \frac{\epsilon}{3(1-\epsilon)}\right) - \frac{1}{3}\chi_{[0,\epsilon]}\left(\frac{x}{3}\right)$$
$$= \frac{1}{3(1-\epsilon)}\chi_{[\epsilon,4\epsilon-3\epsilon^2]}(x) - \frac{1}{3}\chi_{[0,3\epsilon]}(x)$$

yields

$$\bigvee\left((L_{T_\epsilon} - L_{T_0})f_\epsilon\right) = \frac{3-\epsilon}{3(1-\epsilon)}, \quad \text{and} \quad \|(L_{T_\epsilon} - L_{T_0})f_\epsilon\|_{1,m} = \frac{2\epsilon}{3}.$$

Thus we have

$$\|L_{T_\epsilon} - L_{T_0}\|_{BV} \geq \left(\frac{3-\epsilon}{3(1-\epsilon)} + \frac{2\epsilon}{3}\right)\frac{1}{1+\epsilon} = \frac{3+\epsilon-2\epsilon^2}{3(1-\epsilon^2)} \to 1 \quad (\epsilon \downarrow 0)$$

since $\|f_\epsilon\|_{BV} = 1 + \epsilon$ holds for any small $\epsilon > 0$. Hence one can not say that our theorems are easy consequences of the general perturbation theory of linear operators in [3, VII-6].

Appendix

We prove Lemma 3.1 and Lemma 3.2 (cf. [1], [6], and [9]). Let T be an LY map with ess.inf$|DT| > 1$. The notations ρ, R, and Δ_n are the same as in Section 3. Note that it is not hard to show that

$$\rho(T^n) \leq \rho^n \quad \text{and} \quad R(T^n) \leq nR.$$

Let $\mathcal{P}_n = \mathcal{P}(T^n)$ be the defining partition of T^n. For each $J \in \mathcal{P}_n$, g_J is an element in BV specified later. For $i = 0, \ldots, n-1$, let f_i be an element in BV satisfying $\|f_i\|_{\infty,m} \leq H$ for some $H \geq 1$. In what follows if we consider an element in BV, we choose a version which is right continuous on $[0,1)$ and left continuous at 1. For simplicity, we put $F = \prod_{i=0}^{n-1} f_i(T^i)$.

We note that in the sequel we use the notations $\sup_J$ and $\inf_J$ to denote the supremum and the infimum taken over all $x \in J$, respectively. First of all we show that

$$\sup_J \frac{1}{|DT_J^n|} \leq \left(nR + \frac{1}{m(T_J^n J)}\right) m(J) \tag{A.1}$$

holds for any $J \in \mathcal{P}_n$. In fact, by Mean Value Theorem, we have

$$\left|\frac{1}{|DT_J^n(x)|} - \frac{1}{|DT_J^n(y)|}\right| \leq \sup_J \left|\frac{D^2T_J^n}{(DT_J^n)^2}\right| |x-y| \leq R(T^n)m(J) \leq nRm(J)$$

for any $x,\ y \in J$ and for any $J \in \mathcal{P}_n$. Thus we have

$$\sup_J \frac{1}{|DT_J^n|} \leq nRm(J) + \inf_J \frac{1}{|DT_J^n|} \leq \left(nR + \frac{1}{m(T_J^n J)}\right) m(J)$$

since

$$m(T_J^n J) = \int_J |DT_J^n|\, dm \leq \sup_J |DT_J^n| m(J)$$

Now we carry out the estimation as follows.

$$\begin{aligned}
&\bigvee \sum_{J \in \mathcal{P}_n} \left(\chi_{T_J^n J} F(T_J^{-n}) |DT_J^n(T_J^{-n})|^{-1} g_J(T_J^{-n})\right) \\
&\quad \leq \sum_{J \in \mathcal{P}_n} \bigvee_{TJ} \left(F(T_J^{-n}) |DT_J^n(T_J^{-n})|^{-1} g_J(T_J^{-n})\right) + \\
&\qquad\qquad \sum_{J \in \mathcal{P}_n} \left(|F(a_J)| DT_J^n|^{-1}(a_J) g_J(a_J)| + |F(b_J)| DT_J^n|^{-1}(b_J) g_J(b_J)|\right) \\
\text{(A.2)}\quad &\quad \leq \sum_{J \in \mathcal{P}_n} \bigvee_J \left(F |DT_J^n|^{-1} g_J\right) + \\
&\qquad\qquad + \sum_{J \in \mathcal{P}_n} \left(|F(a_J)| DT_J^n|^{-1}(a_J) g_J(a_J)| + |F(b_J)| DT_J^n|^{-1}(b_J) g_J(b_J)|\right) \\
&\quad \leq 2 \sum_{J \in \mathcal{P}_n} \bigvee_J \left(F |DT_J^n|^{-1} g_J\right) + 2 \sum_{J \in \mathcal{P}_n} \left(\frac{1}{m(J)} \int_J |F||DT_J^n|^{-1} g_J|\, dm\right) \\
&\quad = 2 \left(\sum_{J \in \mathcal{P}_n} I_J + \sum_{J \in \mathcal{P}_n} II_J\right),
\end{aligned}$$

where $J = [a_J, b_J]$ and $\bigvee_J$ denots the total variation on J. Note that in the third inequality in the above, we have used the fact that

$$|h(a)| + |h(b)| \leq |h(a) - h(x)| + |h(x) - h(b)| + 2|h(x)| \leq \bigvee_{[a,b]} h + 2|h(x)|$$

holds for any $x \in [a, b]$.

Next, using $\bigvee_J F \leq H^{n-1} \sum_{i=0}^{n-1} \bigvee f_i$, we have

$$\begin{aligned}
&I_J \leq \int_J |F||g_J| \left|\frac{D^2 T_J^n}{(DT_J^n)^2}\right| dm + \bigvee_J (F g_J) \sup_J(|DT_J^n|^{-1}) \\
\text{(A.3)}\quad &\quad \leq nRH^n \int_J |g_J|\, dm + \left(H^{n-1} \sum_{i=0}^{n-1} \bigvee f_i \cdot \sup_J |g_J| + H^n \bigvee_J g_J\right) \sup_J(|DT_J^n|^{-1}) \\
&\quad \leq nRH^n \int_J |g_J|\, dm + H^n \sum_{i=0}^{n-1} \bigvee f_i \cdot \sup_J |g_J| \sup_J(|DT_J^n|^{-1}) + H^n \rho^n \bigvee_J g_J
\end{aligned}$$

and

$$\text{(A.4)} \qquad II_J \leq H^n \sup_J(|DT_J^n|^{-1}) \frac{1}{m(J)} \int_J |g_J| \, dm.$$

If we put $g_J = g$ for any J, then the inequalities (A.3) and (A.4) become

(A.5)

$$\begin{aligned} I_J &\leq nRH^n \int_J |g| \, dm + H^n \sum_{i=0}^{n-1} \bigvee f_i \left(\bigvee_J g + \frac{1}{m(J)} \int_J |g| \, dm \right) \sup_J(|DT_J^n|^{-1}) \\ &\qquad + H^n \rho^n \bigvee_J g \\ &\leq nRH^n \int_J |g| \, dm + H^n \rho^n \left(\sum_{i=0}^{n-1} \bigvee f_i \right) \bigvee_J g \\ &\qquad + H^n \left(\sum_{i=0}^{n-1} \bigvee f_i \right) \left(nR + \frac{1}{\Delta_n} \right) \int_J |g| \, dm + H^n \rho^n \bigvee_J g \\ &\leq H^n \rho^n \left(1 + \sum_{i=0}^{n-1} \bigvee f_i \right) \bigvee_J g + \left(\left(1 + \sum_{i=0}^{n-1} \bigvee f_i \right) nR + \frac{1}{\Delta_n} \sum_{i=0}^{n-1} \bigvee f_i \right) H^n \int_J |g| \, dm \end{aligned}$$

and

$$\text{(A.6)} \qquad II_J \leq H^n \sup_J(|DT_J^n|^{-1}) \frac{1}{m(J)} \int_J |g| \, dm \leq H^n \left(nR + \frac{1}{\Delta_n} \right) \int_J |g| \, dm.$$

In the above we have used the inequalities $\sup_J g \leq \bigvee_J g + (1/m(J)) \int_J |g| \, dm$ and (A.1) to obtain the first inequality and the second inequality, respectively. (A.1) is also used in the last inequality. Since the identity

$$L_T^n \left(\prod_{i=0}^{n-1} f_i(T^i) \cdot g \right) = \sum_{J \in \mathcal{P}_n} \chi_{T_J^n J} F(T_J^{-n}) |DT_J^n(T_J^{-n})|^{-1} g(T_J^{-n})$$

holds, we obtain the Lasota-Yorke type inequality by combining (A.2), (A.5), and (A.6).

Finally, we put $g_J = g(x_J) - g$ in (A.3) and (A.4), where x_J is a point in J chosen beforehand. Then we have

$$\text{(A.7)} \qquad I_J \leq nRH^n \rho^n \bigvee_J g + H^n \rho^n \sum_{i=0}^{n-1} \bigvee f_i \cdot \bigvee_J g + H^n \rho^n \bigvee_J g$$

and

$$\text{(A.8)} \qquad II_J \leq H^n \rho^n \bigvee_J g.$$

In addition, we have

$$\int \left| L_T^n\left(\left(\prod_{i=0}^{n-1} f_i(T^i)\right)\Pi_n g\right) - L_T^n\left(\left(\prod_{i=0}^{n-1} f_i(T^i)\right)g\right)\right| dm$$

(A.9)
$$\le \int \prod_{i=0}^{n-1} |f_i(T^i)(\Pi_n g - g)|\, dm$$

$$\le H^n \sum_{J\in\mathcal{P}_n} \int_J |g_J - g|\, dm \le H^n\rho^n \sum_{J\in\mathcal{P}_n} \bigvee_J g \le H^n\rho^n \bigvee g.$$

Since

$$L_T^n\left(\left(\prod_{i=0}^{n-1} f_i(T^i)\right)\Pi_n g\right) - L_T^n\left(\left(\prod_{i=0}^{n-1} f_i(T^i)\right)g\right)$$
$$= \sum_{J\in\mathcal{P}_n} \chi_{T_J^n J} F(T_J^{-n})|DT_J^n(T_J^{-n})|^{-1}(g(T_J^{-n}) - g(x_J))$$

holds, we can reach the Krylov-Bogolioubov type inequality in virtue of (A.2), (A.7), (A.8), and (A.9).

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Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima, 739-8526
Japan