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CORRECTED ENERGY OF DISTRIBUTIONS ON RIEMANNIAN MANIFOLDS

PABLO M. CHACÓN and A.M. NAVEIRA

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1. Introduction

In the mathematical literature there are several functionals which let us measure how the vector fields defined over any Riemannian manifold M^n are *ordered*. We can ask ourselves which are the *optimal* vector fields. In fact, we try to measure how far from being parallel our vector field is. We can also extend this question to distributions.

Gluck, Ziller [5] and Johnson [6], among others, studied the *volume* of unit vector fields. They define the volume of a unit vector field X to be the volume of the submanifold in the unit tangent bundle defined by $X(M)$. For this, we regard the vector field as a map $X\colon M \to T^1M$ and in T^1M we consider the Sasaki metric. We know [5] that in the ambient manifold S^3 the Hopf vector fields, and no others, minimize this functional. For higher dimensional spheres, we know [6] that the Hopf vector fields are unstable critical points; that is, they are not even local minima.

Wiegmink [8] defined the *total bending* of a unit vector field X. This functional is related to the *energy* of the map $X\colon M \to T^1M$, as we shall see in Section 3. Brito [1] proved that the Hopf vector fields in S^3 are the only minima of the total bending. Furthermore, he proved a more general result giving an absolute minimum in any dimension of the total bending corrected by the second fundamental form of the orthogonal distribution to the field X. The coefficient of this correction vanishes in dimension 3 and then the corrected total bending agrees with the total bending.

Similarly to the situation for vector fields, the energy of a q-distribution $\mathcal{V}$ in a compact oriented Riemannian manifold M is the energy of the section of the Grassmann manifold of q-planes in M induced by $\mathcal{V}$.

In this paper, we add to the energy the norms of the mean curvatures of $\mathcal{V}$ and its orthogonal distribution (with different weights) introducing in this way the *corrected energy*.

In Theorem 1, we find a lower bound for the corrected energy of a foliation, and

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in Theorem 3 we prove the Hopf fibrations are minima of this corrected energy.

2. Notations

Let (M^n, g) be a compact oriented Riemannian manifold of dimension $n = p + q$, over which an almost product structure $(\mathcal{H}^p, \mathcal{V}^q)$ is defined. We shall call $\mathcal{H}$ the *horizontal* distribution and $\mathcal{V}$ the *vertical* distribution. We only consider bases of the tangent space adapted to the almost product structure, that is, if $x \in M$, for an orthonormal local frame $\{e_1, \dots, e_n\}_x \subset T_x M$, we demand

$$\{e_1, \dots, e_p\}_x \subset \mathcal{H}_x \qquad \text{and} \qquad \{e_{p+1}, \dots, e_{p+q}\}_x \subset \mathcal{V}_x \ .$$

We shall use the index convention $1 \leq a, b, c \leq n$, $1 \leq i, j, k \leq p$ and $p + 1 \leq \alpha, \beta, \gamma \leq n = p + q$.

We denote the dual basis and the connection forms respectively by

$$\{\theta_1, \dots, \theta_n\} \qquad ; \qquad \omega_{ab}(e_c) = g(\nabla_{e_c} e_a, e_b), \tag{1}$$

where ∇ is the Levi-Civita connection. The curvature 2-forms will be denoted by

$$\Omega_{ab}(X, Y) = g\big(R(X, Y)e_a, e_b\big), \tag{2}$$

where R is the curvature tensor. The sectional curvature of the plane spanned by the vectors $\{e_a, e_b\}$ will be expressed by $c_{ab} = -\Omega_{ab}(e_a, e_b)$.

The second fundamental form of the distribution $\mathcal{H}$ in the direction e_α is determined by the matrix $(h^\alpha_{ij})_{i,j}$ where $h^\alpha_{ij} = -g(\nabla_{e_i} e_\alpha, e_j)$. Analogously the second fundamental form of $\mathcal{V}$ in the direction e_i is $h^i_{\alpha\beta} = -g(\nabla_{e_\alpha} e_i, e_\beta)$.

The mean curvature vector of the horizontal and vertical distributions are respectively

$$\vec{H}_{\mathcal{H}} = \sum_{\alpha=p+1}^{n} \Big(\frac{1}{p} \sum_{i=1}^{p} h^\alpha_{ii}\Big) e_\alpha \quad , \quad \vec{H}_{\mathcal{V}} = \sum_{i=1}^{p} \Big(\frac{1}{q} \sum_{\alpha=p+1}^{n} h^i_{\alpha\alpha}\Big) e_i \ . \tag{3}$$

For the sake of simplicity in notation, we shall write $\sum_i$, $\sum_\alpha$, $\sum_a$ instead of $\sum_{i=1}^p$, $\sum_{\alpha=p+1}^n$, $\sum_{a=1}^n$, respectively, through the following.

3. Corrected energy for higher dimensions

For maps between Riemannian manifolds $f \colon (M, g) \to (N, h)$, the *energy* is defined to be (see for example [3])

$$\mathcal{E}(f) = \frac{1}{2} \int_M \sum_a h\big(df(e_a), df(e_a)\big) \ \nu, \tag{4}$$

where ν is the canonical volume form in M. Now, we get an expression for the energy of a q-distribution.

We can regard a distribution like a section of $\pi\colon G(q,M) \to M$ where

$$G(q,M) = \bigcup_{x\in M} G(q,T_xM)$$

and $G(q,T_xM)$ is the Grassmann manifold of oriented q-planes in the n-dimensional space T_xM. We can define a metric g_S in $G(q,M)$ called, as in the one dimensional case, the Sasaki metric. In fact, the splitting of the space $G(q,M)$ by the connection ∇ and the so-called connection map $\mathcal{K}\colon TG(q,M) \to G(q,M)$ are the natural generalizations of the same objects in T^1M. For a clear definition in T^1M and a brief comment for the higher dimensional case see [7].

We write the vertical distribution $\mathcal{V}$ as the map $\xi\colon M \to G(q,M)$ where $\xi(x)$ is the q-vector in T_xM determined by $\mathcal{V}_x$; that is,

$$\xi(x) = e_{p+1}(x) \wedge \cdots \wedge e_n(x).$$

Now, we calculate $\|d\xi\|$ from the definition of g_S,

$$g_S\big(d\xi(e_a), d\xi(e_a)\big) = g\big(\pi_*(d\xi(e_a)), \pi_*(d\xi(e_a))\big) + g\big(\mathcal{K}(d\xi(e_a)), \mathcal{K}(d\xi(e_a))\big).$$

Note that we denote by the same letter g the metric in T^1M and in $G(q,T_xM)$. Since ξ is a section we have $\pi_* \circ d\xi = d(\pi\circ\xi) = d(\mathrm{id}_M) = \mathrm{id}_{T^1M}$. We also know [7] that $\mathcal{K}(d\xi(e_a)) = \nabla_{e_a}\xi$, then

$$\sum_a g_S\big(d\xi(e_a), d\xi(e_a)\big) = \sum_a g(e_a,e_a) + g(\nabla_{e_a}\xi, \nabla_{e_a}\xi)$$

and the energy (4) of the distribution $\mathcal{V}$ is

$$\mathcal{E}(\mathcal{V}) = \frac{1}{2}\int_M \sum_a \|\nabla_{e_a}\xi\|^2\nu + \frac{n}{2}\,\mathrm{vol}(M).$$

Wiegmink in [8] defined the *total bending* for a unit vector field X as

$$\mathcal{B}(X) = \frac{1}{(n-1)\,\mathrm{vol}(\mathsf{S}^n)}\int_M \sum_a \|\nabla_{e_a}X\|^2\nu \qquad \text{for} \quad n \geq 2\ .$$

The relation between total bending and energy of vector fields is

$$\mathcal{E}(X) = \frac{(n-1)\,\mathrm{vol}(\mathsf{S}^n)}{2}\mathcal{B}(X) + \frac{n}{2}\,\mathrm{vol}(M).$$

With this, the study of the possible minima of the total bending $\mathcal{B}$ is the same as the study of the possible minima of the energy $\mathcal{E}$.

DEFINITION 1. For a q-distribution $\mathcal{V}$ we define the corrected energy to be

$$\mathcal{D}(\mathcal{V}) = 2\mathcal{E}(\mathcal{V}) - n\,\mathrm{vol}(M) + \int_M \left(p(p-2)\|\vec{H}_{\mathcal{H}}\|^2 + q^2\|\vec{H}_{\mathcal{V}}\|^2\right)\nu,$$

or more explicitly,

$$\mathcal{D}(\mathcal{V}) = \int_M \left(\sum_a \|\nabla_{e_a}\xi\|^2 + p(p-2)\|\vec{H}_{\mathcal{H}}\|^2 + q^2\|\vec{H}_{\mathcal{V}}\|^2\right)\nu, \tag{5}$$

where ξ is the q-vector defined by $\mathcal{V}$ as before.

REMARK. This corrected energy is not an extension of the corrected total bending of [1]. However, for vector fields the two functionals have the same lower bound and the same minimality conditions.

We can calculate $\nabla\xi$ and obtain its norm in terms of the second fundamental form of $\mathcal{H}$ and $\mathcal{V}$. Recall that the connection acts as a derivation in the multivector algebra. We have,

$$\sum_a \|\nabla_{e_a}\xi\|^2 = \sum_{i,j,\alpha}\left(h_{ij}^{\alpha}\right)^2 + \sum_{i,\alpha,\beta}\left(h_{\alpha\beta}^{i}\right)^2 \ . \tag{6}$$

It is clear that this expression is independent of the adapted local basis. Note that the trivial minima of the energy are the totally geodesic distributions with horizontal also totally geodesic.

Theorem 1. *If $\mathcal{V}$ is integrable, then*

$$\mathcal{D}(\mathcal{V}) \geq \int_M \sum_{i,\alpha} c_{i\alpha}\ \nu,$$

where $c_{i\alpha}$ is the sectional curvature of the plane spanned by $e_i \in \mathcal{H}$ and $e_\alpha \in \mathcal{V}$.

Proof. From the definition of mean curvature vector (3) and from (6), we have

$$\begin{aligned}
&\sum_a \|\nabla_{e_a}\xi\|^2 + p(p-2)\|\vec{H}_{\mathcal{H}}\|^2 \\
&= \sum_\alpha \left[\sum_{i,j}\left(h_{ij}^{\alpha}\right)^2 + \sum_{i,\beta}\left(h_{\alpha\beta}^{i}\right)^2 + \frac{p-2}{p}\left(\sum_i h_{ii}^{\alpha}\right)^2\right] \\
&= \sum_\alpha \left[\sum_i \frac{2p-2}{p}\left(h_{ii}^{\alpha}\right)^2 + \sum_{i\neq j}\left(h_{ij}^{\alpha}\right)^2 + \frac{2(p-2)}{p}\sum_{i<j} h_{ii}^{\alpha}h_{jj}^{\alpha} + \sum_{i,\beta}\left(h_{\alpha\beta}^{i}\right)^2\right].
\end{aligned} \tag{7}$$

But the sums $\sum\left(h_{ii}^{\alpha}\right)^2$ and $\sum\left(h_{ij}^{\alpha}\right)^2$ may be written, for each α, in the following way

$$\begin{aligned}(p-1)\sum_i\left(h_{ii}^{\alpha}\right)^2 &= \sum_{i<j}\left(h_{ii}^{\alpha}-h_{jj}^{\alpha}\right)^2+2h_{ii}^{\alpha}h_{jj}^{\alpha}\ ,\\ \sum_{i\neq j}\left(h_{ij}^{\alpha}\right)^2 &= \sum_{i<j}\left(h_{ij}^{\alpha}+h_{ji}^{\alpha}\right)^2-2h_{ij}^{\alpha}h_{ji}^{\alpha}\ .\end{aligned}\tag{8}$$

Then from (7) and (8)

$$\begin{aligned}\sum_a\|\nabla_{e_a}\xi\|^2+p(p-2)\|\vec{H}_{\mathcal{H}}\|^2 &= \frac{2}{p}\sum_{i<j,\alpha}(h_{ii}^{\alpha}-h_{jj}^{\alpha})^2\\ &+\sum_{i<j,\alpha}(h_{ij}^{\alpha}+h_{ji}^{\alpha})^2+2\sum_{i<j,\alpha}\left(h_{ii}^{\alpha}h_{jj}^{\alpha}-h_{ij}^{\alpha}h_{ji}^{\alpha}\right)+\sum_{i,\alpha,\beta}(h_{\alpha\beta}^{i})^2\\ &\geq 2\sum_{i<j,\alpha}\left(h_{ii}^{\alpha}h_{jj}^{\alpha}-h_{ij}^{\alpha}h_{ji}^{\alpha}\right)+\sum_{i,\alpha,\beta}(h_{\alpha\beta}^{i})^2=2\sum_{\alpha}\sigma_2^{\alpha}+\sum_{i,\alpha,\beta}(h_{\alpha\beta}^{i})^2,\end{aligned}\tag{9}$$

with σ_2^{α} the second elementary symmetric function of the second fundamental form of $\mathcal{H}$ in the direction e_{α}.

Under the integrability assumption of $\mathcal{V}$, i.e. $h_{\alpha\beta}^{i}=h_{\beta\alpha}^{i}$, we can relate the mean curvature of $\mathcal{V}$ with the second symmetric function σ_2^{i} in the following way:

$$\begin{aligned}\left(\sum_{\alpha}h_{\alpha\alpha}^{i}\right)^2 &= \sum_{\alpha,\beta}h_{\alpha\alpha}^{i}h_{\beta\beta}^{i}\\ &= \sum_{\alpha}(h_{\alpha\alpha}^{i})^2+\sum_{\alpha\neq\beta}\left(h_{\alpha\alpha}^{i}h_{\beta\beta}^{i}-h_{\alpha\beta}^{i}h_{\beta\alpha}^{i}+h_{\alpha\beta}^{i}h_{\beta\alpha}^{i}\right)\\ &= \sum_{\alpha}(h_{\alpha\alpha}^{i})^2+2\sum_{\alpha<\beta}\left(h_{\alpha\alpha}^{i}h_{\beta\beta}^{i}-h_{\alpha\beta}^{i}h_{\beta\alpha}^{i}\right)+\sum_{\alpha\neq\beta}(h_{\alpha\beta}^{i})^2\\ &= 2\sigma_2^{i}+\sum_{\alpha,\beta}(h_{\alpha\beta}^{i})^2.\end{aligned}\tag{10}$$

Now, considering that the mean curvature (3) is normalized , we can write, using equation (10),

$$q^2\|\vec{H}_{\mathcal{V}}\|^2=\sum_i\left(\sum_{\alpha}h_{\alpha\alpha}^{i}\right)^2=\sum_i 2\sigma_2^{i}+\sum_{i,\alpha,\beta}(h_{\alpha\beta}^{i})^2.\tag{11}$$

With (9) and (11) we get

$$\begin{aligned}&\sum_a\|\nabla_{e_a}\xi\|^2+p(p-2)\|\vec{H}_{\mathcal{H}}\|^2+q^2\|\vec{H}_{\mathcal{V}}\|^2\\ &\quad\geq 2\sum_{\alpha}\sigma_2^{\alpha}+2\sum_i\sigma_2^{i}+2\sum_{i,\alpha,\beta}(h_{\alpha\beta}^{i})^2.\end{aligned}\tag{12}$$

To evaluate the integral of σ_2^α and σ_2^i, we need a lemma proved in [4]. There, with the definitions (1) and (2), we can find the definition of the following differential forms:

$$\varphi = \sum_{\sigma\in\mathfrak{S}_p}\sum_{\tau\in\mathfrak{S}^q}\epsilon(\sigma)\epsilon(\tau)\omega_{\sigma(1)\tau(p+1)}\wedge\theta_{\sigma(2)}\wedge\cdots\wedge\theta_{\sigma(p)}\wedge\theta_{\tau(p+2)}\wedge\cdots\wedge\theta_{\tau(n)}$$

$$\phi_1 = \sum_{\sigma\in\mathfrak{S}_p}\sum_{\tau\in\mathfrak{S}^q}\epsilon(\sigma)\epsilon(\tau)\left(\sum_\alpha \omega_{\sigma(1)\alpha}\wedge\omega_{\alpha\sigma(2)}\right)\wedge\theta_{\sigma(3)}\wedge\cdots \wedge\theta_{\sigma(p)}\wedge\theta_{\tau(p+1)}\wedge\cdots\wedge\theta_{\tau(n)}$$

$$\phi_2 = \sum_{\sigma\in\mathfrak{S}_p}\sum_{\tau\in\mathfrak{S}^q}\epsilon(\sigma)\epsilon(\tau)\theta_{\sigma(1)}\wedge\cdots\wedge\theta_{\sigma(p)}\wedge\left(\sum_i \omega_{\tau(p+1)i}\wedge\omega_{i\tau(p+2)}\right)\wedge\cdots \wedge\theta_{\tau(p+3)}\wedge\cdots\wedge\theta_{\tau(n)}$$

$$\Omega = \sum_{\sigma\in\mathfrak{S}_p}\sum_{\tau\in\mathfrak{S}^q}\epsilon(\sigma)\epsilon(\tau)\Omega_{\sigma(1)\tau(p+1)}\wedge\theta_{\sigma(2)}\wedge\cdots\wedge\theta_{\sigma(p)}\wedge\theta_{\tau(p+2)}\wedge\cdots\wedge\theta_{\tau(n)}\ ,$$

where $\mathfrak{S}_p$ denotes the group of permutations of $\{1,\dots,p\}$, $\mathfrak{S}^q$ the permutations of $\{p+1,\dots,p+q\}$ and $\epsilon(\tau)$ denotes the signature of the permutation τ. The forms φ, ϕ_1, ϕ_2 and Ω are invariant under adapted orthonormal frame changes. These forms satisfy the following lemma.

Lemma 2 ([4]). *For φ, ϕ_1, ϕ_2 and Ω defined as above,*

$$d\varphi = (-1)^p\left[\frac{p-1}{q}\phi_1 + \frac{q-1}{p}\phi_2\right] + \Omega.$$

For the proof of the lemma, we only use the structure equations of M and the properties of the group of permutations (in [4], the authors work under the assumption of integrability of both distributions, but it is not necessary for the proof of this lemma). Evaluating the n-forms ϕ_1, ϕ_2 and Ω on the basis $\{e_1,\dots,e_n\}$ we get

$$\begin{aligned}\phi_1(e_1,\dots,e_n) &= -q!(p-2)!\sum_\alpha 2\sigma_2^\alpha,\\ \phi_2(e_1,\dots,e_n) &= -p!(q-2)!\sum_i 2\sigma_2^i\ ,\\ \Omega(e_1,\dots,e_n) &= (-1)^p(p-1)!(q-1)!\sum_{i,\alpha} c_{i\alpha}.\end{aligned}\tag{13}$$

Now, applying Stokes' Theorem to Lemma 2, with the help of (13) we deduce that

$$\int_M\left(\sum_\alpha 2\sigma_2^\alpha + \sum_i 2\sigma_2^i\right)\nu = \int_M\sum_{i,\alpha} c_{i\alpha}\ \nu.\tag{14}$$

To obtain the corrected energy (5), we integrate equation (12) and use (14):

$$\mathcal{D}(\mathcal{V}) \geq \int_M \sum_{i,\alpha} c_{i\alpha} + 2 \sum_{i,\alpha,\beta} (h^i_{\alpha\beta})^2 \nu \geq \int_M \sum_{i,\alpha} c_{i\alpha} \nu \ , \tag{15}$$

as we claimed. □

In the inequalities (9) and (15) of the proof, we have lost several terms. These terms will give us the conditions for a foliation $\mathcal{V}$ to be a minimum of $\mathcal{D}$. The conditions are

$$\sum_{\alpha,\beta,i} (h^i_{\alpha\beta})^2 = 0 \quad ; \sum_{i<j,\alpha} (h^\alpha_{ii} - h^\alpha_{jj})^2 = 0 \quad ; \sum_{i<j,\alpha} (h^\alpha_{ij} + h^\alpha_{ji})^2 = 0. \tag{16}$$

The first condition means that $\mathcal{V}$ is totally geodesic. The second and third conditions mean that the vertical vectors $\{e_{p+1}, \ldots, e_n\}$ are conformal vector fields for the horizontal ones. That is, we have

$$\mathcal{L}_{e_\alpha} g(X, Y) = \lambda g(X, Y) \text{ for any } \alpha \text{ and } X,\ Y \in \mathcal{H},$$

where $\mathcal{L}_Z$ is the Lie derivative in the direction Z and λ is a function on M. Killing vector fields are conformal vector fields with $\lambda \equiv 0$.

Note that the lower bound in Theorem 1 depends on the distribution for an arbitrary manifold. In any case, the lower bound is interesting because it is the integral of the cross sectional curvature of the almost product structure. This cross sectional curvature is an invariant of order 2 (called linear invariants) of the Riemannian almost product structure [2]. In the case $M = \mathsf{S}^n$, the lower bound depends only on n and q.

Theorem 3. *Among the integral distributions of dimension* 1 (*resp.* 3) *of* S^{2n+1} (*resp.* S^{4n+3}), *Hopf fibrations* $\mathsf{S}^1 \hookrightarrow \mathsf{S}^{2n+1} \to \mathbb{C}\mathsf{P}^n \quad \forall n \geq 1$ (*resp.* $\mathsf{S}^3 \hookrightarrow \mathsf{S}^{4n+3} \to \mathbb{H}\mathsf{P}^n$) *minimize* $\mathcal{D}$.

Proof. To be more explicit, here we show the case of the $(4n + 3)$-sphere with $q = 3$. In the other case, the proof is very similar.

By definition, the fibers of $\mathsf{S}^3 \hookrightarrow \mathsf{S}^{4n+3}$ are the intersection of the sphere $\mathsf{S}^{4n+3} \subset \mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1}$ with quaternionic lines in $\mathbb{H}^{n+1}$ (a 4-plane in $\mathbb{R}^{4n+4}$). Then, the fibers are great 3-spheres inside S^{4n+3} and the distribution tangent to the fibers is integrable and totally geodesic.

In order to prove the other minimality conditions (16), we use the natural almost complex structures $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$ defined on $\mathbb{H}^{n+1}$. For each point $x \in \mathsf{S}^{4n+3} \subset \mathbb{H}^{n+1}$, the vector tangents to the fiber will be $\mathbf{I}(\vec{x}), \mathbf{J}(\vec{x})$ and $\mathbf{K}(\vec{x})$. We consider a real basis $\{\vec{x}, \mathbf{I}\vec{x}, \mathbf{J}\vec{x}, \mathbf{K}\vec{x}, v_1, \mathbf{I}v_1, \mathbf{J}v_1, \mathbf{K}v_1, v_2, \ldots, \mathbf{K}v_n\}$ in $\mathbb{H}^{n+1}$ which is adapted to the sphere and to the fibration. In this nice basis we can calculate the second fundamental form of $\mathcal{H}$. For that, we need use that the almost complex structures are isometries and parallels (in $\mathbb{R}^{4n+4}$). The calculations give us the conditions of (16) we need. $\square$

REMARK. The Hopf fibration $\mathsf{S}^7 \hookrightarrow \mathsf{S}^{15} \to \mathsf{CaP}$ is not considered in this paper. The fibers are not the intersection of S^{15} with Cayley lines in $\mathbb{R}^{16} \cong \mathsf{Ca}^2$ (note that this does not make a fibration of S^{15}). Then the fibers are not determined by the almost complex structures induced by the Cayley product with imaginary units. Therefore the argument in Theorem 3 cannot be applied. Thus we cannot decide whether this Hopf fibration is a minimum of the corrected energy.

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Pablo M. Chacón
Dpto. de Matemática
Instituto de Matemática e Estatística
Universidade de São Paulo
R. do Matão 1010
São Paulo-SP
05508-900
BRAZIL
e-mail: pablom@ime.usp.br

Current address:
Dpto. Matemática Aplicada
Facultad de Informática
Campus de Espinardo
Universidad de Murcia
30100 Murcia
SPAIN
e-mail: pmchacon@um.es

A.M. Naveira
Dpto. de Geometría y Topología
Facultad de Matemáticas
Universidad de Valencia
Avda. Andrés Estellés nº 1
46100 Burjassot (Valencia)
SPAIN
e-mail: naveira@uv.es