Kodama, A. and Shimizu, S.
Osaka J. Math.
41 (2004), 85–95

A GROUP-THEORETIC CHARACTERIZATION OF THE SPACE OBTAINED BY OMITTING THE COORDINATE HYPERPLANES FROM THE COMPLEX EUCLIDEAN SPACE

Dedicated to Professor Makoto Namba on his sixtieth birthday

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(Received May 31, 2002)

Introduction

In the study of the holomorphic automorphism group Aut(M) of a complex manifold M, it seems to be natural to direct our attention not only to the abstract group structure of Aut(M) but also to its topological group structure equipped with the compact-open topology. In fact, a well-known theorem of H. Cartan says that the topological group of the holomorphic automorphisms of a bounded domain in $\mathbf{C}^n$ has the structure of a Lie group, and this result enables us to make various kinds of detailed studies of bounded domains in $\mathbf{C}^n$. On the other hand, in contrast to the case of bounded domains, the holomorphic automorphism group Aut$(\mathbf{C}^k \times (\mathbf{C}^*)^l)$ of the unbounded domain $\mathbf{C}^k \times (\mathbf{C}^*)^l$ is terribly big when $k+l \geq 2$, and cannot have the structure of a Lie group. But, by looking at topological subgroups of Aut$(\mathbf{C}^k \times (\mathbf{C}^*)^l)$ with Lie group structures, we can find a lead to apply the Lie group theory to the investigation of the problems related to the structure of Aut$(\mathbf{C}^k \times (\mathbf{C}^*)^l)$. In the present paper, we try to approach from this standpoint to the fundamental problem of what complex manifold has the holomorphic automorphism group isomorphic to Aut$(\mathbf{C}^k \times (\mathbf{C}^*)^l)$ as topological groups. Namely, we prove the following result with the aid of the theory of Reinhardt domains developed in Shimizu [8], [9] (cf. Kruzhilin [6]).

Main Theorem. *Let M be a connected Stein manifold of dimension n. Assume that* Aut(M) *is isomorphic to* Aut$(\mathbf{C}^k \times (\mathbf{C}^*)^{n-k})$ *as topological groups. Then M is biholomorphically equivalent to* $\mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$.

As a consequence of the above theorem, we can obtain the fundamental result on the topological group structure of Aut$(\mathbf{C}^k \times (\mathbf{C}^*)^l)$.

2000 *Mathematics Subject Classification* : Primary 32M05; Secondary 32Q28.
The authors are partially supported by the Grant-in-Aid for Scientific Research (C) No. 14540165 and (C) No. 14540149, the Ministry of Education, Science, Sports and Culture, Japan.

Corollary. *If two pairs* (k, l) *and* (k', l') *of nonnegative integers do not coincide, then the topological groups* $\mathrm{Aut}(\mathbf{C}^k \times (\mathbf{C}^*)^l)$ *and* $\mathrm{Aut}(\mathbf{C}^{k'} \times (\mathbf{C}^*)^{l'})$ *are not isomorphic.*

It should be remarked that, as shown in Ahern and Rudin [1], the groups $\mathrm{Aut}(\mathbf{C}^n)$ and $\mathrm{Aut}(\mathbf{C}^m)$ are isomorphic as abstract groups precisely when $n = m$. Also, as a consequence of the study of $U(n)$-actions on complex manifolds of dimension n, a related result to our Main Theorem has been obtained by Isaev and Kruzhilin [4].

This paper is organized as follows. In Section 1, we collect some preliminary facts. In particular, two main tools for our study are given. One is a tool to obtain the normal form of some compact group action on a Reinhardt domain, and the other is a tool for the standardization of torus actions on complex manifolds. Section 2 is devoted to the proof of the Main Theorem and its corollary. Our method used in Section 2 has interesting applications. As one of such applications, we discuss in Section 3 a new approach to the study of $U(n)$-actions on complex manifolds of dimension n.

1. Lie group actions, Reinhardt domains and torus actions

We begin with a basic fact on Lie group actions on complex manifolds. Let M be a complex manifold. An *automorphism of* M means a biholomorphic mapping of M onto itself. We denote by $\mathrm{Aut}(M)$ the topological group of all automorphisms of M equipped with the compact-open topology. Let G be a Lie group and consider a continuous group homomorphism $\rho\colon G \to \mathrm{Aut}(M)$. Then the mapping

$$G \times M \ni (g, p) \longmapsto (\rho(g))(p) \in M$$

is continuous. It follows from Akhiezer [2] that this mapping is actually of class C^ω, and therefore G acts on M as a Lie transformation group. In view of this, when a continuous group homomorphism $\rho\colon G \to \mathrm{Aut}(M)$ is given, we say that *G acts on M as a Lie transformation group through* ρ. Also, the action of G on M is called *effective* if ρ is injective.

We now recall basic concepts and results on Reinhardt domains (cf. [8], [9]). We denote by $U(k)$ the *unitary group of degree* k. Write $T^n = (U(1))^n$. The n-dimensional torus T^n acts as a group of automorphisms on $\mathbf{C}^n$ by the standard rule

$$\alpha \cdot z = (\alpha_1 z_1, \dots, \alpha_n z_n) \quad \text{for } \alpha = (\alpha_1, \dots, \alpha_n) \in T^n \text{ and } z = (z_1, \dots, z_n) \in \mathbf{C}^n.$$

By definition, a *Reinhardt domain* D in $\mathbf{C}^n$ is a domain in $\mathbf{C}^n$ which is stable under this action of T^n. Each element α of T^n then induces an automorphism π_α of D given by $\pi_\alpha(z) = \alpha \cdot z$, and the mapping ρ_D sending α to π_α is an injective continuous group homomorphism of T^n into $\mathrm{Aut}(D)$. The subgroup $\rho_D(T^n)$ of $\mathrm{Aut}(D)$ is denoted by $T(D)$.

Let f be a holomorphic function on a Reinhardt domain D in $\mathbf{C}^n$. Then f can be expanded uniquely into a "Laurent series"

$$f(z) = \sum_{\nu \in \mathbf{Z}^n} a_\nu z^\nu,$$

which converges absolutely and uniformly on any compact set in D, where $z = (z_1, \ldots, z_n)$, $\nu = (\nu_1, \ldots, \nu_n)$, and $z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$. The following lemma is a consequence of the uniqueness of the Laurent series expansion.

Lemma 1.1. *If f satisfies the condition that, for some $\nu_0 \in \mathbf{Z}^n$,*

$$f(\alpha \cdot z) = \alpha^{\nu_0} f(z) \quad \text{for all } \alpha \in T^n \text{ and all } z \in D,$$

then f has the form $f(z) = a_{\nu_0} z^{\nu_0}$.

Proof. Since we have

$$f(\alpha \cdot z) = \sum_{\nu \in \mathbf{Z}^n} \alpha^\nu a_\nu z^\nu \quad \text{and} \quad \alpha^{\nu_0} f(z) = \sum_{\nu \in \mathbf{Z}^n} \alpha^{\nu_0} a_\nu z^\nu,$$

it follows from the assumption that, for every $\nu \in \mathbf{Z}^n$, we have

$$\alpha^\nu a_\nu = \alpha^{\nu_0} a_\nu \quad \text{for all } \alpha \in T^n.$$

This implies that if $a_\nu \neq 0$, then $\nu = \nu_0$, and our lemma is proved. □

We denote by $\Pi(\mathbf{C}^n)$ the group of all automorphisms of $\mathbf{C}^n$ of the form

$$\mathbf{C}^n \ni (z_1, \ldots, z_n) \longmapsto (\alpha_1 z_1, \ldots, \alpha_n z_n) \in \mathbf{C}^n,$$

where $(\alpha_1, \ldots, \alpha_n) \in (\mathbf{C}^*)^n$. For a Reinhardt domain D in $\mathbf{C}^n$, we denote by $\Pi(D)$ the subgroup of $\Pi(\mathbf{C}^n)$ consisting of all elements of $\Pi(\mathbf{C}^n)$ leaving D invariant. Identifying $\Pi(\mathbf{C}^n)$ with the multiplicative group $(\mathbf{C}^*)^n$, we see that, when $\Pi(D)$ is regarded as a topological subgroup of $\mathrm{Aut}(D)$, it is isomorphic to a closed Lie subgroup of $(\mathbf{C}^*)^n$. Using Lemma 1.1, we obtain a characterization of $\Pi(D)$ as a subgroup of $\mathrm{Aut}(D)$.

Lemma 1.2. *Let D be a Reinhardt domain in $\mathbf{C}^n$. Then $\Pi(D)$ is the centralizer $C_{\mathrm{Aut}(D)}(T(D))$ of $T(D)$ in* $\mathrm{Aut}(D)$.

Proof. It is immediate that $\Pi(D) \subset C_{\mathrm{Aut}(D)}(T(D))$. To prove the reverse inclusion, let φ be any element of $C_{\mathrm{Aut}(D)}(T(D))$ and write $\varphi = (\varphi_1, \ldots, \varphi_n)$, where $\varphi_1, \ldots, \varphi_n$ are holomorphic functions on D. Then, for every $i = 1, \ldots, n$, we have

$$\varphi_i(\alpha \cdot z) = \alpha_i \varphi_i(z) = \alpha^{e_i} \varphi_i(z) \quad \text{for all } \alpha \in T^n \text{ and all } z \in D,$$

where each e_i denotes the element of $\mathbf{Z}^n$ whose i-th component is equal to 1 and whose components except the i-th are all equal to 0. By Lemma 1.1, it follows from this property that every function $\varphi_i(z)$ has the form

$$\varphi_i(z) = a_{e_i} z^{e_i} = a_{e_i} z_i .$$

This implies that $\varphi \in \Pi(D)$, and the reverse inclusion $C_{\mathrm{Aut}(D)}(T(D)) \subset \Pi(D)$ is shown, as desired. $\square$

The argument used in Shimizu [9] for determining the automorphisms of bounded Reinhardt domains has the following consequence, which plays a crucial role in our study.

Proposition 1.1. *Let D be a bounded Reinhardt domain in $\mathbf{C}^n$ and suppose that*

$$\begin{aligned} D \cap \{z_i = 0\} &\neq \emptyset, \quad 1 \le i \le m, \\ D \cap \{z_i = 0\} &= \emptyset, \quad m+1 \le i \le n. \end{aligned}$$

If G is a connected compact subgroup of $\mathrm{Aut}(D)$ *containing $T(D)$, then there exists a transformation*

$$\varphi\colon \mathbf{C}^m \times (\mathbf{C}^*)^{n-m} \ni (z_1, \dots, z_n) \longmapsto (w_1, \dots, w_n) \in \mathbf{C}^m \times (\mathbf{C}^*)^{n-m},$$
$$\begin{cases} w_i = r_i z_{\sigma'(i)} (z'')^{\nu_i''}, & 1 \le i \le m, \\ w_i = r_i z_{\sigma''(i)}, & m+1 \le i \le n, \end{cases}$$

such that, for $\tilde{D} = \varphi(D)$ and $\tilde{G} = \varphi G \varphi^{-1} \subset \mathrm{Aut}(\tilde{D})$, one has

$$\begin{aligned} &\tilde{G} = U(k_1) \times \cdots \times U(k_s) \times U(k_{s+1}) \times \cdots \times U(k_t), \\ &k_1 + \cdots + k_s + k_{s+1} + \cdots + k_t = n, \\ &k_1 + \cdots + k_s = m, \\ &k_{s+1} = \cdots = k_t = 1, \end{aligned}$$

where $r_1, \dots, r_n$ are positive constants, σ' and σ'' are permutations of $\{1, \dots, m\}$ and $\{m+1, \dots, n\}$, respectively, z'' denotes the coordinates $(z_{m+1}, \dots, z_n)$, and $\nu_1'', \dots, \nu_m''$ are elements of $\mathbf{Z}^{n-m}$.

We give a useful form of this proposition as a corollary.

Corollary. *In the above proposition, if G is isomorphic to $U(k) \times (U(1))^{n-k}$ as topological groups and if $k \ge 2$, then $m \ge k$.*

Proof. Since G is necessarily isomorphic to $U(k) \times (U(1))^{n-k}$ as Lie groups, we have $\dim G = k^2 + (n-k)$. On the other hand, Proposition 1.1 implies that $\dim G = \dim \tilde{G} = k_1^2 + \cdots + k_s^2 + (n-m)$. Therefore, if $m < k$, then it follows that

$$k^2 = k_1^2 + \cdots + k_s^2 + (k-m) \quad \text{and} \quad k = k_1 + \cdots + k_s + (k-m).$$

By noting that $k \geq 2$ and $k - m > 0$, this is a contradiction. Thus we obtain $m \geq k$. □

We recall the fundamental result on torus actions on complex manifolds, which is a part of Barrett, Bedford and Dadok [3, Theorem 1].

Standardization Theorem. *Let M be a connected Stein manifold of dimension n. Assume that T^n acts effectively on M as a Lie transformation group through ρ. Then there exist a biholomorphic mapping F of M into $\mathbf{C}^n$ and a continuous group automorphism θ of T^n such that*

$$F\big((\rho(\alpha))(p)\big) = \theta(\alpha) \cdot F(p) \quad \textit{for all } \alpha \in T^n \textit{ and all } p \in M.$$

Consequently, $D := F(M)$ is a Reinhardt domain in $\mathbf{C}^n$, and one has $F\rho(T^n)F^{-1} = T(D)$.

To apply the Standardization Theorem to our study, we need a lemma.

Lemma 1.3. *In the Standardization Theorem, if $M = \mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$, then we have $D = F(M) = \mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$ after a suitable permutation of coordinates, if necessary.*

Proof. We first show that $D \cap (\mathbf{C}^*)^n = D - \{z_1 \cdots z_n = 0\} = (\mathbf{C}^*)^n$. Suppose contrarily that $D \cap (\mathbf{C}^*)^n \neq (\mathbf{C}^*)^n$. Since $D \cap (\mathbf{C}^*)^n$ is a Stein manifold, the logarithmic image of the Reinhardt domain $D \cap (\mathbf{C}^*)^n$ is a convex domain contained in a half space of $\mathbf{R}^n$. Hence, there exists a nonconstant bounded plurisubharmonic function u on $D \cap (\mathbf{C}^*)^n$. Since u extends to the whole of D, we have a nonconstant bounded plurisubharmonic function on D. This contradicts the fact that D is biholomorphically equivalent to $M = \mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$. Thus we obtain $D \cap (\mathbf{C}^*)^n = (\mathbf{C}^*)^n$.

Since D is a Stein manifold, it follows from what we have shown above that, after a suitable permutation of coordinates, D has the form $D = \mathbf{C}^h \times (\mathbf{C}^*)^{n-h}$ (cf. [7, p. 46, Theorem 1.5]). Note that $\mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$ and $\mathbf{C}^h \times (\mathbf{C}^*)^{n-h}$ are homeomorphic precisely when $k = h$. Therefore we have $D = \mathbf{C}^k \times (\mathbf{C}^*)^{n-k}$, because D and M are biholomorphically equivalent. □

2. The characterization of $\mathbf{C}^k \times (\mathbf{C}^*)^l$: Proof of the Main Theorem and its corollary

For brevity, we write $X_{k,l} = \mathbf{C}^k \times (\mathbf{C}^*)^l$ and $\Omega_k = X_{k,n-k}$.

Now, as in the Main Theorem stated in the introduction, let M be a connected Stein manifold of dimension n and assume that there exists an isomorphism $\Phi\colon \mathrm{Aut}(\Omega_k) \to \mathrm{Aut}(M)$. Since Ω_k is a Reinhardt domain in $\mathbf{C}^n$, we have the injective continuous group homomorphism $\rho_{\Omega_k}\colon T^n \to \mathrm{Aut}(\Omega_k)$. Thus we obtain an injective continuous group homomorphism $\Phi \circ \rho_{\Omega_k}\colon T^n \to \mathrm{Aut}(M)$. Hence, by the Standardization Theorem, there exists a biholomorphic mapping F of M into $\mathbf{C}^n$ such that $D := F(M)$ is a Reinhardt domain in $\mathbf{C}^n$ and we have $F(\Phi \circ \rho_{\Omega_k})(T^n)F^{-1} = T(D)$. Therefore we may assume that M is a Reinhardt domain D in $\mathbf{C}^n$ and we have an isomorphism $\Phi\colon \mathrm{Aut}(\Omega_k) \to \mathrm{Aut}(D)$ such that $\Phi(T(\Omega_k)) = T(D)$.

We show that $(\mathbf{C}^*)^n \subset D$. Since $\Phi\colon \mathrm{Aut}(\Omega_k) \to \mathrm{Aut}(D)$ is a group isomorphism and since $\Phi(T(\Omega_k)) = T(D)$, we see that Φ gives rise to a topological group isomorphism $\Phi\colon C_{\mathrm{Aut}(\Omega_k)}(T(\Omega_k)) \to C_{\mathrm{Aut}(D)}(T(D))$ between the centralizers. Moreover, by Lemma 1.2 we have $C_{\mathrm{Aut}(\Omega_k)}(T(\Omega_k)) = \Pi(\Omega_k)$, and it is immediate that $\Pi(\Omega_k) = \Pi(\mathbf{C}^n)$. On the other hand, again by Lemma 1.2 we have $C_{\mathrm{Aut}(D)}(T(D)) = \Pi(D)$. Therefore we obtain

$$2n = \dim \Pi(\mathbf{C}^n) = \dim C_{\mathrm{Aut}(\Omega_k)}(T(\Omega_k)) = \dim C_{\mathrm{Aut}(D)}(T(D)) = \dim \Pi(D).$$

Since $\Pi(D)$ is a closed Lie subgroup of $\Pi(\mathbf{C}^n)$, it follows that $\Pi(D) = \Pi(\mathbf{C}^n)$. By taking a point z_0 in $D \cap (\mathbf{C}^*)^n$, this shows that

$$(\mathbf{C}^*)^n = \Pi(\mathbf{C}^n) \cdot z_0 = \Pi(D) \cdot z_0 \subset D,$$

as required.

Since D is a Stein manifold by assumption, we see from the result of the preceding paragraph that D has the form $D = \Omega_h$ after a suitable permutation of coordinates.

When $n = 1$, we have $D = \Omega_0 = \mathbf{C}^*$ or $D = \Omega_1 = \mathbf{C}$. Moreover, since $\mathrm{Aut}(\mathbf{C}^*)$ and $\mathrm{Aut}(\mathbf{C})$ are not isomorphic, the condition that $\mathrm{Aut}(\Omega_k)$ and $\mathrm{Aut}(D)$ are isomorphic implies that, according to the cases of $k = 0$ and $k = 1$, we must have $D = \Omega_0$ and $D = \Omega_1$. This proves the Main Theorem when $n = 1$. Therefore, in what follows, we assume that $n \geq 2$.

We show that $h \geq k$. When $k = 0$, there is nothing to prove. To prove our assertion when $k \neq 0$, we divide into the two cases of $k = 1$ and $k \geq 2$.

First consider the case of $k \geq 2$. Noting that $\mathrm{Aut}(\Omega_k)$ contains the subgroup $U(k) \times (U(1))^{n-k}$, we set $G = \Phi(U(k) \times (U(1))^{n-k})$, which is a connected compact subgroup of $\mathrm{Aut}(D)$ containing $T(D)$, because $U(k) \times (U(1))^{n-k} \supset T(\Omega_k)$ and $\Phi(T(\Omega_k)) = T(D)$. Take a relatively compact subdomain U of D and put

$$D_0 = \{g(z) \in D \mid g \in G,\ z \in U\} = \bigcup_{g \in G} g(U) = \bigcup_{z \in U} G \cdot z.$$

Then D_0 is a bounded Reinhardt domain contained in D and G can be regarded as a connected compact subgroup of the Lie group $\mathrm{Aut}(D_0)$ containing $T(D_0)$. Recalling that G is isomorphic to $U(k) \times (U(1))^{n-k}$ and $k \geq 2$, we can apply the corollary to Proposition 1.1 to D_0 and $G \subset \mathrm{Aut}(D_0)$. Therefore, after a suitable permutation of coordinates, we have for some $m \geq k$,

$$\emptyset \neq D_0 \cap \{z_i = 0\} \subset D \cap \{z_i = 0\}, \quad 1 \leq i \leq m.$$

This implies that $\Omega_m \subset D$, and, when we write $D = \Omega_h$, we must have $h \geq m \geq k$, as required.

Now consider the case of $k = 1$. It suffices to show that $\mathrm{Aut}(\Omega_1)$ and $\mathrm{Aut}(\Omega_0)$ are not isomorphic. Suppose contrarily that we have an isomorphism $\Phi\colon \mathrm{Aut}(\Omega_1) \to \mathrm{Aut}(\Omega_0)$. Then, by the Standardization Theorem and Lemma 1.3, we may assume that we have an isomorphism $\Phi\colon \mathrm{Aut}(\Omega_1) \to \mathrm{Aut}(\Omega_0)$ such that $\Phi(T(\Omega_1)) = T(\Omega_0)$. For $s = 0, 1$, let us set

$$T'(\Omega_s) = \{(1, \alpha_2, \ldots, \alpha_n) \in T(\Omega_s) \mid \alpha_2, \ldots, \alpha_n \in U(1)\}.$$

Then $\Phi(T'(\Omega_1))$ is an $(n-1)$-dimensional subtorus of $T(\Omega_0)$, and, after a suitable change of coordinates by a transformation of the form

$$\begin{aligned} \Omega_0 &= (\mathbf{C}^*)^n \ni (z_1, \ldots, z_n) \longmapsto (w_1, \ldots, w_n) \in (\mathbf{C}^*)^n = \Omega_0, \\ w_i &= z^{\nu_i}, \quad 1 \leq i \leq n, \end{aligned}$$

where $\nu_1, \ldots, \nu_n$ are elements of $\mathbf{Z}^n$, we have $\Phi(T'(\Omega_1)) = T'(\Omega_0)$. Since $\Phi\colon \mathrm{Aut}(\Omega_1) \to \mathrm{Aut}(\Omega_0)$ is a group isomorphism, we see that Φ maps the centralizer Z_1 of $T'(\Omega_1)$ in $\mathrm{Aut}(\Omega_1)$ onto the centralizer Z_0 of $T'(\Omega_0)$ in $\mathrm{Aut}(\Omega_0)$. Therefore, for the groups Z_0 and Z_1, their commutator groups $[Z_0, Z_0]$ and $[Z_1, Z_1]$ must be isomorphic. To derive a contradiction, it is sufficient to see that $[Z_0, Z_0]$ is an abelian group, while $[Z_1, Z_1]$ is not an abelian group. We verify this only in the case of $n = 2$, because the verification in the case of $n > 2$ is almost identical. Using a method similar to that in the proof of Lemma 1.2, we can show that Z_1 and Z_0 are the groups of all elements

$$g_1 \in \mathrm{Aut}(\Omega_1) = \mathrm{Aut}(\mathbf{C} \times \mathbf{C}^*) \quad \text{and} \quad g_0 \in \mathrm{Aut}(\Omega_0) = \mathrm{Aut}((\mathbf{C}^*)^2)$$

having the forms

$$g_1(z) = \bigl(\alpha z_1 + \beta,\, f(z_1) z_2\bigr) \tag{$*$}$$

and

$$g_0(z) = \bigl(\alpha z_1,\, f(z_1) z_2\bigr),$$

respectively, where $\alpha \in \mathbf{C}^*$, $\beta \in \mathbf{C}$, and $f(z_1)$ is a nowhere vanishing holomorphic function that is defined on $\mathbf{C}$ for g_1 and on $\mathbf{C}^*$ for g_0. Take any two transformations $K_{\alpha,\beta,f}$ and $K_{\alpha',\beta',f'}$ of the form $(*)$ given by

$$K_{\alpha,\beta,f}(z) = \bigl(\alpha z_1 + \beta, f(z_1)z_2\bigr) \quad \text{and} \quad K_{\alpha',\beta',f'}(z) = \bigl(\alpha' z_1 + \beta', f'(z_1)z_2\bigr)$$

and write $[K_{\alpha,\beta,f}, K_{\alpha',\beta',f'}](z) = (K_1(z), K_2(z))$ in terms of the coordinates in $\mathbf{C}^2$, where $[\varphi, \psi] := \varphi^{-1} \circ \psi^{-1} \circ \varphi \circ \psi$ denotes the commutator of transformations φ and ψ. Then we have

$$\begin{aligned} K_1(z) &= \frac{\alpha\alpha' z_1 + \alpha\beta' - \beta\alpha' + \beta - \beta'}{\alpha\alpha'}, \\ K_2(z) &= \frac{f(\alpha' z_1 + \beta') f'(z_1) z_2}{f((\alpha\alpha' z_1 + \alpha\beta' - \beta\alpha' + \beta - \beta')/\alpha\alpha') f'((\alpha\alpha' z_1 + \alpha\beta' + \beta - \beta')/\alpha')}. \end{aligned}$$

As a consequence, considering the case of $(\beta, \beta') = (0, 0)$, we have

$$(**) \qquad [K_{\alpha,0,f}, K_{\alpha',0,f'}](z) = \left(z_1, \frac{f(\alpha' z_1) f'(z_1) z_2}{f(z_1) f'(\alpha z_1)} \right).$$

Now it follows immediately from $(**)$ that $[Z_0, Z_0]$ is abelian. On the other hand, consider three elements

$$P(z) = (\alpha z_1 + \beta, z_2), \;\; Q(z) = (z_1, z_2 \exp z_1), \;\text{ and }\; R(z) = (\gamma z_1, z_2 \exp z_1)$$

in Z_1. Then, using the computation result above, we obtain

$$\begin{aligned} [P, Q](z) &= (z_1, z_2 \exp\{(1 - \alpha) z_1 - \beta\}), \\ [P, R](z) &= \left(\frac{\alpha\gamma z_1 + \beta(1 - \gamma)}{\alpha\gamma}, z_2 \exp\left\{ (1 - \alpha) z_1 - \frac{\beta}{\gamma} \right\} \right), \end{aligned}$$

and therefore $[[P, Q], [P, R]]$ is not the identity mapping whenever $\beta(\alpha - 1)(\gamma - 1) \neq 0$. This implies that $[Z_1, Z_1]$ is not abelian, and our assertion that $\mathrm{Aut}(\Omega_1)$ and $\mathrm{Aut}(\Omega_0)$ are not isomorphic is shown.

Summarizing our results obtained so far, we have shown that if M is a connected Stein manifold of dimension n and if the topological groups $\mathrm{Aut}(M)$ and $\mathrm{Aut}(\Omega_k)$ are isomorphic, then M is biholomorphically equivalent to Ω_h with $h \geq k$.

To complete the proof of our Main Theorem, it is sufficient to see $h = k$. Suppose contrarily that $h \neq k$. Then, for the connected Stein manifold Ω_k of dimension n, we have that $\mathrm{Aut}(\Omega_k)$ and $\mathrm{Aut}(\Omega_h)$ are isomorphic. By letting $M = \Omega_k$, an application of what we have shown just above yields that Ω_k is biholomorphically equivalent to Ω_p with $p \geq h$. Since $k < h \leq p$, this contradicts the fact that Ω_s and Ω_t are not homeomorphic when $s \neq t$. We thus obtain $h = k$, and our Main Theorem is proved. □

It remains to prove the corollary to the Main Theorem. If $k + l = k' + l'$, then it is immediate from the Main Theorem that $\mathrm{Aut}(X_{k,l})$ and $\mathrm{Aut}(X_{k',l'})$ are isomorphic precisely when $(k, l) = (k', l')$. To prove the corollary in the case of $k + l \neq k' + l'$, we need the following lemma.

Lemma 2.1. *Let M be a connected Stein manifold of dimension n. If $N > n$, then there is no injective continuous group homomorphism of the torus T^N into the topological group* $\mathrm{Aut}(M)$.

Proof. Suppose contrarily that we have an injective continuous group homomorphism ρ of T^N into $\mathrm{Aut}(M)$. Choose an n-dimensional subtorus T^n of T^N. By the Standardization Theorem, there exists a biholomorphic mapping $F: M \to D$ of M onto a Reinhardt domain D in $\mathbf{C}^n$ such that $F\rho(T^n)F^{-1} = T(D)$. Set $G = F\rho(T^N)F^{-1}$ and take a relatively compact subdomain U of D. Then $D_0 := \{g(z) \in D \mid g \in G, z \in U\}$ is a bounded Reinhardt domain in $\mathbf{C}^n$ and G can be regarded as a connected compact subgroup of the Lie group $\mathrm{Aut}(D_0)$ containing $T(D_0)$. Since G is isomorphic to T^N and $N > n = \dim T(D_0)$, G is a torus in $\mathrm{Aut}(D_0)$ containing $T(D_0)$ properly. But, by [8, Section 4, Proposition 1], $T(D_0)$ is a maximal torus in $\mathrm{Aut}(D_0)$, that is, any torus in $\mathrm{Aut}(D_0)$ containing $T(D_0)$ must coincide with $T(D_0)$. This is a contradiction, and our assertion is proved. □

Suppose $k+l \neq k'+l'$, say, $k+l < k'+l'$, and write $n = k+l$, $n' = k'+l'$. If there exists an isomorphism $\Phi\colon \mathrm{Aut}(X_{k',l'}) \to \mathrm{Aut}(X_{k,l})$, then we have an injective continuous group homomorphism $\Phi \circ \rho_{X_{k',l'}}$ of $T^{n'}$ into $\mathrm{Aut}(X_{k,l})$. Since $X_{k,l}$ is a connected Stein manifold of dimension $n < n'$, this contradicts the above lemma. Therefore, $\mathrm{Aut}(X_{k,l})$ and $\mathrm{Aut}(X_{k',l'})$ are not isomorphic, and the proof of the corollary is completed. □

3. $U(n)$-actions on a Stein manifold of dimension n

The method used in the preceding section can be applied to the study of $U(n)$-actions on a complex manifold M of dimension n. The following theorem gives a different approach from Kaup [5], Isaev and Kruzhilin [4]. In the case where $\mathrm{Aut}(M)$ is not a Lie group, we cannot obtain various results on the conjugacy of subgroups of $\mathrm{Aut}(M)$ by applying the conjugacy theorems in the Lie group theory, in general. However, even when $\mathrm{Aut}(M)$ is not a Lie group, we have a conjugacy result on $\mathrm{Aut}(M)$ in a case, as is shown in our theorem below.

Theorem. *Let M be a connected Stein manifold of dimension $n \geq 2$. Assume that $U(n)$ acts effectively on M as a Lie transformation group through ρ. Then M is biholomorphically equivalent to either B^n or $\mathbf{C}^n$, where B^n denotes the unit ball in $\mathbf{C}^n$. Moreover, if we identify M with B^n or $\mathbf{C}^n$, then there exists an element ψ of* $\mathrm{Aut}(M)$ *such that $\psi\rho(U(n))\psi^{-1} = U(n)$.*

Proof. Choose a maximal torus T^n in $U(n)$. By the Standardization Theorem, there exists a biholomorphic mapping $F\colon M \to D$ of M onto a Reinhardt domain D in $\mathbf{C}^n$ such that $F\rho(T^n)F^{-1} = T(D)$. Set $G = F\rho(U(n))F^{-1}$ and take a relatively compact subdomain U of D. Then $D_0 := \{g(z) \in D \mid g \in G, z \in U\}$ is a bounded Reinhardt domain in $\mathbf{C}^n$ and G can be regarded as a connected compact subgroup of the Lie group $\mathrm{Aut}(D_0)$ containing $T(D_0)$. Recalling that G is isomorphic to $U(n)$ and $n \geq 2$, we can apply Proposition 1.1 and its corollary to D_0 and $G \subset \mathrm{Aut}(D_0)$. Therefore there exists a transformation

$$\begin{aligned} \varphi\colon \mathbf{C}^n \ni (z_1, \dots, z_n) &\longmapsto (w_1, \dots, w_n) \in \mathbf{C}^n, \\ w_i = r_i z_{\sigma(i)}, &\quad 1 \leq i \leq n, \end{aligned}$$

such that, for $\tilde{D}_0 = \varphi(D_0)$ and $\tilde{G} = \varphi G \varphi^{-1} \subset \mathrm{Aut}(\tilde{D}_0)$, we have $\tilde{G} = U(n)$, where $r_1, \dots, r_n$ are positive constants and σ is a permutation of $\{1, \dots, n\}$. Putting $\tilde{D} = \varphi(D)$, we see by the uniqueness theorem on holomorphic functions that $U(n) = \tilde{G} \subset \mathrm{Aut}(\tilde{D})$, or $g(\tilde{D}) = \tilde{D}$ for all $g \in U(n)$. Since $\tilde{D}$ is a Stein manifold, it follows that $\tilde{D}$ has the form

$$\tilde{D} = \left\{ (z_1, \dots, z_n) \in \mathbf{C}^n \;\middle|\; \sum_{i=1}^{n} |z_i|^2 < r \right\},$$

where $0 < r \leq +\infty$. This shows that $\tilde{D}$, and hence M is biholomorphically equivalent to either B^n or $\mathbf{C}^n$, proving the first assertion.

Now, let us identify M with B^n or $\mathbf{C}^n$. When $M = B^n$, the existence of $\psi \in \mathrm{Aut}(M)$ satisfying the relation $\psi\rho(U(n))\psi^{-1} = U(n)$ is a consequence of the conjugacy of maximal compact subgroups of the simple Lie group $\mathrm{Aut}(B^n)$. So, consider the case of $M = \mathbf{C}^n$. Then, by the same reasoning as above, there exist biholomorphic mappings $F\colon M = \mathbf{C}^n \to D = \mathbf{C}^n$ and $\varphi\colon \mathbf{C}^n \to \mathbf{C}^n$ such that $(\varphi \circ F)\rho(U(n))(\varphi \circ F)^{-1} = U(n)$. Therefore, the composition $\psi = \varphi \circ F$ is an element of $\mathrm{Aut}(\mathbf{C}^n)$ required in the theorem. □

Added in proof. After the submission of this paper, the authors learned in the letter of August 21, 2002, from Professor A. Isaev that, in the special case of $k = n$, the same result as our Main Theorem had been obtained independently by him (Proc. Steklov Inst. Math. **235** (2001), 103–106).

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