KURAMOCHI BOUNDARY AND HARMONIC FUNCTIONS WITH FINITE DIRICHLET INTEGRALS ON UNLIMITED COVERING SURFACES

Dedicated to Professor Hiroki Sato on the occasion of his sixtieth birthday

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0. Introduction

Let W be an open Riemann surface with the Green function and $\tilde{W}$ an m-sheeted $(1 < m < \infty)$ unlimited covering surface of W. Denote by $\pi = \pi_{\tilde{W}}$ the projection of $\tilde{W}$ onto W. Consider the Kuramochi compactification W^* (resp. $\tilde{W}^*$) of W (resp. $\tilde{W}$). Denote by $\Delta = \Delta^W$ (resp. $\tilde{\Delta} = \Delta^{\tilde{W}}$) the Kuramochi boundary of W (resp. $\tilde{W}$). We also denote by $\Delta_1 = \Delta_1^W$ (resp. $\tilde{\Delta}_1 = \Delta_1^{\tilde{W}}$) the set of all minimal points in Δ (resp. $\tilde{\Delta}$). It is known that π naturally has a unique continuous extension π^* to $\tilde{W}^*$ (see [7, 2) in Proposition 2.1]). For $\zeta \in \Delta$, we set $\tilde{\Delta}_1(\zeta) = (\pi^*)^{-1}(\zeta) \cap \tilde{\Delta}_1$. Denote by $\nu(\zeta) = \nu_{\tilde{W}}(\zeta)$ the cardinal number of $\tilde{\Delta}_1(\zeta)$. Let $HD(W)$ (resp. $HD(\tilde{W})$) be the set of harmonic functions with finite Dirichlet integrals on W (resp. $\tilde{W}$). Suppose that $HD(W)$ contains a non-constant element in the sequel. Set $HD(W) \circ \pi = \{h \circ \pi : h \in HD(W)\}$. It is easily seen that $HD(W) \circ \pi \subset HD(\tilde{W})$. Then, we give necessary and sufficient conditions for the property that $HD(\tilde{W}) = HD(W) \circ \pi$ in terms of the Kuramochi compactification as follows.

Main Theorem. *The following three conditions are equivalent.*
(i) $HD(\tilde{W}) = HD(W) \circ \pi$;
(ii) *for all $\zeta \in \Delta_1$ except possibly for a full-polar subset of Δ_1, $\nu(\zeta) = 1$;*
(iii) *for almost every $\zeta \in \Delta_1$ with respect to the harmonic measure μ_z^W ($z \in W$) on Δ, $\nu(\zeta) = 1$.*

By [7] we know that $1 \leq \nu(\zeta) \leq m$. According to the above theorem the property that $HD(\tilde{W}) = HD(W) \circ \pi$ is a necessary and sufficient condition to minimize $\nu(\zeta)$ for almost every $\zeta \in \Delta_1$ with respect to the harmonic measure μ_z^W ($z \in W$) on Δ. Thus we are interested in a necessary and sufficient condition to maximize $\nu(\zeta)$, that is, $\nu(\zeta) = m$ for almost every $\zeta \in \Delta_1$ with respect to the harmonic measure μ_z^W ($z \in W$) on Δ. We shall give a sufficient condition for the condition that $\nu(\zeta) = m$ for almost every $\zeta \in \Delta_1$ with respect to the harmonic measure μ_z^W ($z \in W$) on Δ in the case that

W is the unit disc. Consider the unit disc D as the base surface of W. Let $\tilde{D}$ be an m-sheeted unlimited covering surface of D with the projection $\pi_{\tilde{D}}$ and $\{z_n\}$ the set of the projection of branch points of $\tilde{D}$. It is well-known that the Kuramochi compactification D^* of D is homeomorphic to the closure $\bar{D}$ of D in $\mathbf{C}$ with respect to the Euclidean topology and that Δ^D consists of only minimal points and is homeomorphic to the boundary ∂D in $\mathbf{C}$ with respect to the Euclidean topology. Recall the following condition ($\sharp$) which is considered in [5]

$$(\sharp) \qquad\qquad \sum_{z_n \neq 0} \frac{1}{\log\{1/(1-|z_n|)\}} < +\infty.$$

Then, the following holds.

Proposition.   *Suppose that $\{z_n\}$ satisfies the condition ($\sharp$). Then, for almost every $e^{i\theta} \in \partial D$ with respect to the harmonic measure μ_z^D ($z \in D$) on ∂D, $\nu(e^{i\theta}) = m$.*

We organize this article as follows. After preliminaries (§1), we give the relation for potential theoretic notions between the base surface and its covering surface in §2. Main Theorem and Proposition are proved in §3 and §4, respectively. Finally, in case that $W \in O_{HD} \setminus O_G$, we give necessary and sufficient conditions for an m-sheeted unlimited covering surfaces $\tilde{W}$ of W to belong to the same class in §5.

1. Preliminaries

In this section we prepare some notations, definitions, and lemmas from potential theory.

1.1.   Let R be an open Riemann surface with the Green function, $K = K_R$ a closed parametric disc or a disjoint union of finitely many closed parametric discs in R, and $R_0 = R \setminus K$. Set

$\mathcal{S}(R) = \{s\colon s \text{ is a non-negative superharmonic function on } R\}$,
$H(R) = \{h\colon h \text{ is harmonic on } R\}$,
$HP_+(R) = \{h \in H(R)\colon h \geq 0 \text{ on } R\}$,
$HP(R) = \left\{ h \in H(R)\colon \begin{array}{l} \text{there exist } h_j \in HP_+(R) \ (j = 1, 2) \text{ such that} \\ h = h_1 - h_2 \text{ on } R \end{array} \right\}$,
$\mathcal{HP}_+(R_0) = \left\{ v \in HP_+(R_0)\colon \lim_{z \to \eta} v(z) = 0 \text{ for every } \eta \in \partial K \right\}$,
$\mathcal{HP}(R_0) = \left\{ v \in H(R_0)\colon \begin{array}{l} \text{there exist } v_j \in \mathcal{HP}_+(R_0) \ (j = 1, 2) \text{ such that} \\ v = v_1 - v_2 \text{ on } R_0 \end{array} \right\}$.

For every $h \in HP_+(R)$, denote by $H_h^{R_0}$ the generalized Dirichlet solution of h

on R_0 in the sense of Perron-Wiener-Brelot (cf. [2, pp. 20–21]), that is,

$$H_h^{R_0}(z) = \inf \left\{ u(z) \colon u \in \mathcal{S}(R_0), \ \liminf_{z \to \eta} u(z) \geq h(\eta) \text{ for all } \eta \in \partial K \right\}.$$

For a general $h \in HP(R)$, taking $h_j \in HP_+(R)$ ($j = 1, 2$) with $h = h_1 - h_2$ on R, we define $H_h^{R_0} = H_{h_1}^{R_0} - H_{h_2}^{R_0}$. It is well-known that the mapping $h \mapsto H_h^{R_0}$ is additive, that is, for any $h, h' \in HP_+(R)$, $H_{h+h'}^{R_0} = H_h^{R_0} + H_{h'}^{R_0}$ on R_0. From additivity of the mapping $h \mapsto H_h^{R_0}$ it is easily seen that $H_h^{R_0}$ does not depend on the choice of $h_j \in HP_+(R)$ ($j = 1, 2$) such that $h = h_1 + h_2$ on R. Note that, for every $h \in HP(R)$, $H_h^{R_0} \in HP(R_0)$ and $h - H_h^{R_0} \in \mathcal{HP}(R_0)$.

We define a mapping T^R from $HP(R)$ to $\mathcal{HP}(R_0)$ by the following.

$$T^R(h) = h|_{R_0} - H_h^{R_0} \quad (h \in HP(R)).$$

Next we give the definition of a mapping $S^R \colon \mathcal{HP}(R_0) \to HP(R)$ as follows. First, for every $v \in \mathcal{HP}_+(R_0)$, we define a mapping S^R by

$$S^R(v)(z) = \inf\{s(z) \colon s \in \mathcal{S}(R), \ s \geq v \text{ on } R_0\}.$$

Then, we note that, for $v \in \mathcal{HP}_+(R_0)$, $S^R(v) \in HP_+(R)$ by the well-known Perron-Wiener-Brelot method, and have the following.

Lemma 1.1. *The mapping S^R is additive, that is, for any $v_j \in \mathcal{HP}_+(R_0)$ ($j = 1, 2$), $S^R(v_1 + v_2) = S^R(v_1) + S^R(v_2)$ on R.*

Proof. Let v_j ($j = 1, 2$) be elements of $\mathcal{HP}_+(R_0)$. Since, for every $s_j \in \mathcal{S}(R)$ with $s_j \geq v_j$ on R_0 ($j = 1, 2$), $s_1 + s_2 \in \mathcal{S}(R)$ and $s_1 + s_2 \geq v_1 + v_2$ on R_0, from the definition of S^R it is easily seen that $S^R(v_1 + v_2) \leq S^R(v_1) + S^R(v_2)$ on R.

To see the inverse inequality take $s \in \mathcal{S}(R)$ with $s \geq v_1 + v_2$ on R_0. To prove that $S^R(v_1 + v_2) \geq S^R(v_1) + S^R(v_2)$ on R it is sufficient to prove that $s - S^R(v_1) \geq v_2$ on R_0 because $s - S^R(v_1) \in \mathcal{S}(R)$. Set

$$V_2 = \begin{cases} v_2 & \text{on } R_0 \\ 0 & \text{on } K \end{cases}.$$

Then, V_2 is a non-negative continuous subharmonic function on R. Since $s - V_2 \in \mathcal{S}(R)$ and $s - V_2 \geq v_1$ on R_0, by the definition of $S^R(v_1)$ we find that $s - V_2 \geq S^R(v_1)$ on R. Therefore we have the desired result. $\qquad \square$

For a general $v \in \mathcal{HP}(R_0)$, taking $v_j \in \mathcal{HP}_+(R_0)$ ($j = 1, 2$) with $v = v_1 - v_2$ on R_0, we define $S^R(v) = S^R(v_1) - S^R(v_2)$ on R. From Lemma 1.1 it is easily seen that $S^R(v)$ does not depend on the choice of v_j ($j = 1, 2$). Then, we have the following.

Lemma 1.2. *If* $v \in \mathcal{HP}(R_0)$, *then* $S^R(v) - v = H^{R_0}_{S^R(v)}$ *on* R_0.

Proof. We may assume $v \in \mathcal{HP}_+(R_0)$. Since $S^R(v) \geq v$ on R_0, $S^R(v) - v \geq H^{R_0}_{S^R(v)}$ on R_0. Set

$$s = \begin{cases} S^R(v) & \text{on } K \\ v + H^{R_0}_{S^R(v)} & \text{on } R_0 \end{cases}.$$

Then, $s \in \mathcal{S}(R)$ ([2, Hilfssatz 1.2]) and $s \geq v$ on R_0. Therefore $v + H^{R_0}_{S^R(v)} \geq S^R(v)$ on R_0. $\qquad\square$

Corollary 1.1. $T^R(S^R(v)) = v$ *for* $v \in \mathcal{HP}(R_0)$.

Proof. By the above lemma, we have $T^R(S^R(v)) = S^R(v) - H^{R_0}_{S^R(v)} = v$ on R_0. $\qquad\square$

Lemma 1.3. $S^R(T^R(h)) = h$ *for* $h \in HP(R)$.

Proof. We may assume $h \in HP_+(R)$. Obviously $h \geq T^R(h)$, so that $h \geq S^R(T^R(h))$ on R. Set

$$P_h = \begin{cases} h & \text{on } K \\ H^{R_0}_h & \text{on } R_0 \end{cases}.$$

By [2, Satz 4.5] and [2, Satz 4.8], P_h is a Green potential on R. If $s \in \mathcal{S}(R)$ and $s \geq T^R(h)$ on R_0, then $s \geq h - H^{R_0}_h$ on R_0, and hence $P_h \geq h - s$ on R. Since P_h is a Green potential and $h - s$ is subharmonic on R, it follows that $h - s \leq 0$ ([2, Satz 4.6]). Hence $h \leq s$ and thus $h \leq S^R(T^R(h))$. $\qquad\square$

By Corollary 1.1 and Lemma 1.3, we have

Corollary 1.2 ([3]). T^R *is a bijective mapping from* $HP(R)$ *onto* $\mathcal{HP}(R_0)$.

Set

$$HD(R) = \left\{ h \in H(R) \colon \iint_R |\operatorname{grad} h|^2 \, dx dy < +\infty \right\};$$

$$\mathcal{HD}(R_0) = \left\{ h \in HD(R_0) \colon \lim_{z \to \eta} h(z) = 0 \text{ for every } \eta \in \partial K \right\}.$$

Lemma 1.4. $H^{R_0}_h \in \mathcal{HD}(R_0)$ *for any* $h \in HP(R)$.

Proof. We may assume $h \in HP_+(R)$. Consider P_h as in the proof of Lemma 1.3. Since the associated measure of P_h is supported by the compact set ∂K on which h is

bounded, it has finite energy, so that $H_h^{R_0}$ has finite Dirichlet integral ([2, Satz 7.2]). □

Corollary 1.3. T^R *is a bijective mapping from* $HD(R)$ *onto* $\mathcal{HD}(R_0)$.

Proof. We know (e.g., [2, p. 83]) that $HD(R) \subset HP(R)$. Thus by Corollary 1.2, it suffices to show that $T^R(HD(R)) = \mathcal{HD}(R_0)$. Lemma 1.4 implies $T^R(HD(R)) \subset \mathcal{HD}(R_0)$. On the other hand, if $v \in \mathcal{HD}(R_0)$, then $S^R(v) \in HD(R)$ by Lemmas 2.1 and 1.4. Hence, in view of Corollary 1.1, $v = T^R(S^R(v)) \in T^R(HD(R))$. □

1.2. In this subsection we shortly recall the notion of the Kuramochi compactification.

Let z and ξ be points of R_0. Denote by $g_\xi(z) = g_\xi^{R_0}(z)$ the Green function on R_0 with pole at ξ. A function $N(z,\xi) = N^{R_0}(z,\xi)$ on $R_0 \times R_0$ is called the Kuramochi kernel if it has the following properties:

(i) $z \mapsto N(z,\xi) - g_\xi(z)$ is harmonic on R_0 for every point $\xi \in R_0$;

(ii) for every point $\eta \in \partial K$, $\lim_{z \to \eta} N(z,\xi) = 0$;

(iii) if K_1 is a compact subset of R with $K_1^\circ \supset K$, and ξ a point of $K_1^\circ \setminus K$, then, for every connected component G of $R \setminus K_1$, and for every point $z \in G$, $\int N(\zeta,\xi)\,d\omega_z^G(\zeta) = N(z,\xi)$, where K_1° is the interior of K_1 in R and ω_z^G is the full-harmonic measure at z with respect to G; see [2, p. 158] or [7] for the definition of the full-harmonic measure.

We remark that the Kuramochi kernel has symmetric property, that is, $N(z,\xi) = N(\xi,z)$ for $(z,\xi) \in R_0 \times R_0$.

Set $N_\xi(z) = N_\xi^{R_0}(z) = N^{R_0}(z,\xi)$ and call it the Kuramochi function on R_0 with pole at ξ. Define $N_\xi = 0$ on K. Then N_ξ is a continuous function on R. Set $\mathcal{K} = \{N_\xi(\cdot) : \xi \in R_0\} \cup C_0^\infty(R)$. It is well-known that there exists a compactification R^* of R satisfying the following conditions:

(i) each $f \in \mathcal{K}$ has a continuous extension f^* to R^*;

(ii) the functions f^* ($f \in \mathcal{K}$) separate the points of R^*.

Set $\Delta^R = R^* \setminus R$. We call R^* (resp. Δ^R) the *Kuramochi compactification* of R (resp. the *Kuramochi boundary* of R). We note that R^* does not depend on the choice of K, that is, if $R^{*\prime}$ is the Kuramochi compactification of R which is constructed by using another closed parametric disc (or a disjoint union of a finite number of closed parametric discs) K' then the identity mapping ι of R onto itself is extended to be a homeomorphism of R^* onto $R^{*\prime}$. For a point $\zeta \in \Delta^R$, we define the Kuramochi function $N_\zeta = N_\zeta^{R_0}$ with pole at ζ by $\lim_{R_0 \ni \xi \to \zeta} N_\xi$. Then we remark that N_ζ is a non-negative full-superharmonic function on R_0 (see [2, p. 159]) which is harmonic on R_0, and that $R_0 \cup \Delta^R$ is metrizable by the following distance $d(\cdot,\cdot) = d^{R_0 \cup \Delta^R}(\cdot,\cdot)$

(which we call the Kuramochi distance):

$$d(\xi, \eta) = \sum_{n=1}^{+\infty} \frac{1}{2^n} \left| \frac{N_\xi(p_n)}{1 + N_\xi(p_n)} - \frac{N_\eta(p_n)}{1 + N_\eta(p_n)} \right|,$$

where $\{p_n\}_{n=1}^{+\infty}$ is a sequence in R_0 such that $\{p_n\}_{n=1}^{+\infty}$ is dense in R_0 with respect to the usual topology.

DEFINITION 1.1. Let ζ be a point of Δ^R. Then, we call ζ a *minimal point* if $N_\zeta^{R_0} = h_1 + h_2$ on R_0 with positive full-superharmonic functions h_j on R_0 implies that each h_j is proportional to $N_\zeta^{R_0}$.

$N_\zeta^{R_0}$ ($\zeta \in \Delta^R$) is said to be a *minimal Kuramochi function* with pole at ζ if ζ is a minimal point. We call the set of all minimal points of Δ^R the *minimal Kuramochi boundary* of R which is denoted by Δ_1^R. We refer for the details of the Kuramochi compactification to [2], [8], [11] etc.

1.3. In this subsection we shall study some properties of full-polar sets. Set $\Delta = \Delta^R$. We begin with the definition of full-polar sets (see [2, pp. 188–189]).

DEFINITION 1.2. A subset E of $R_0 \cup \Delta$ is said to be *full-polar* if there exists a positive full-superharmonic function s on R_0 such that $\lim_{z \to \xi} s(z) = +\infty$ for every point $\xi \in E$.

REMARK. In the above definition, under the condition that E is a subset of Δ, we may suppose that the above s has a finite Dirichlet integral and satisfies $\lim_{z \to \eta} s(z) = 0$ for every $\eta \in \partial K$.

Denote by $\Delta_r = \Delta_r^R$ the set of all regular points of Δ with respect to the Dirichlet problem on Δ (cf. [2, p. 93]). Set $\Delta_{r,1} = \Delta_{r,1}^R = \Delta_r \cap \Delta_1$ ($\Delta_1 = \Delta_1^R$). The next proposition holds.

Proposition 1.1 (cf. [2, Satz 16.4 and Folgesatz 17.26]). *The sets $\Delta \setminus \Delta_1$ and $\Delta \setminus \Delta_{r,1}$ are full-polar.*

Let f be a generalized real valued function on Δ which is resolutive with respect to the Dirichlet problem on the Kuramochi compactification R^* of R, and denote by H_f^* the generalized Dirichlet solution of f on R in the sense of Perron-Wiener-Brelot (cf. [2, pp. 85–86]), that is,

$$H_f^*(z) = \inf \left\{ u(z) : \begin{array}{l} u \text{ is superharmonic and bounded bellow on } R \\ \text{and } \liminf_{\xi \to \zeta} u(\xi) \ge f(\zeta) \text{ for all } \zeta \in \Delta \end{array} \right\}.$$

We denote by μ_z^R the harmonic measure on Δ relative to $z(\in R)$ and R, so that

$$H_f^*(z) = \int_\Delta f(\zeta)\,d\mu_z^R(\zeta).$$

We recall that a subset E of Δ is a null set with respect to μ_z^R if for any $\varepsilon > 0$ there is an open subset O of Δ such that $O \supset E$ and $\mu_z^R(O) < \varepsilon$, or, equivalently, there exists a G_δ-set E' such that $E \subset E'$ and $\mu_z^R(E') = 0$.

Lemma 1.5. *Any full-polar subset of Δ is a null set with respect to μ_z^R for every* $z \in R$.

Proof. Let E be a full-polar subset of Δ. By Definition 1.2 and its remark there exists a positive full-superharmonic function u on R_0 such that $\lim_{z \to \zeta} u(z) = +\infty$ for every point $\zeta \in E$, $\lim_{z \to \eta} u(z) = 0$ for every $\eta \in \partial K$, and u has a finite Dirichlet integral. Set

$$U = \begin{cases} 0 \text{ on } K \\ u \text{ on } R_0 \end{cases}.$$

Then U is a Dirichlet function in the sense of [2]. By the Royden decomposition ([2, Satz 7.6]) we find that U is uniquely represented as the sum of an element h_U of $HD(R)$ and a Dirichlet potential P_U on R. By [2, Hilfssatz 7.7] there is a Green potential ϕ_U on R such that $|P_U| \leq \phi_U$ on R and hence, there is an element $s = h_U + \phi_U$ of $\mathcal{S}(R)$ such that $s \geq U$ on R. Set $E' = \{\zeta \in \Delta \colon \lim_{z \to \zeta} u(z) = +\infty\}$. Then, E' is a G_δ-set containing E and $\varepsilon s \geq H_{\chi_{E'}}^*$ for any $\varepsilon > 0$, where $\chi_{E'}$ is the defining function of E'. Hence $H_{\chi_{E'}}^* = 0$, which means that E is null with respect to μ_z^R for every $z \in R$. $\qquad\square$

1.4. In this subsection we give a definition of thinness. Set $\Delta = \Delta^R$ and $\Delta_1 = \Delta_1^R$.

DEFINITION 1.3 (cf. [2, p. 206]). Let ζ be a point of Δ and E a closed subset of R. We say that E is *thin* at ζ if there exists a polar subset N of E satisfying one of the following conditions:

i) ζ does not belong to the closure $\mathrm{Cl}(E \setminus N)$ of $E \setminus N$ in R^*;
ii) ζ belongs to $\mathrm{Cl}(E \setminus N)$ and there exists a non-negative full-superharmonic function s on R_0 such that

$$\liminf_{z(\in (E \cap R_0) \setminus N) \to \zeta} s(z) > s(\zeta),$$

where we refer to [2, p. 177] for the definition of the value $s(\zeta)$ of s at ζ.

This fact means that, for every closed subset E of R, E is thin at ζ if and only

if $E \cap R_0$ is thin at ζ.

Let ζ be a point of Δ_1 and ρ a positive number. Set $B_\rho(\zeta) = \{z \in R_0 \cup \Delta: d^{R_0 \cup \Delta}(z, \zeta) < \rho\}$. From the definition of thinness we see that $R_0 \setminus B_\rho(\zeta)$ is thin at ζ.

2. Relation between potential theoretic notions on covering surfaces and potential theoretic notions on base surfaces

Let W, $\tilde{W}$, π, W^*, $\tilde{W}^*$, π^*, $\Delta = \Delta^W$, $\tilde{\Delta} = \Delta^{\tilde{W}}$, $\Delta_1 = \Delta_1^W$, $\tilde{\Delta}_1 = \Delta_1^{\tilde{W}}$, $HD(W)$, and $HD(\tilde{W})$ be as in Introduction. Let $K = K_W$ be a closed parametric disc in W. Set $W_0 = W \setminus K$ and $\tilde{W}_0 = \tilde{W} \setminus \pi^{-1}(K)$. Let $g_\xi = g_\xi^{W_0}$ (resp. $\tilde{g}_{\tilde{\xi}} = \tilde{g}_{\tilde{\xi}}^{\tilde{W}_0}$) be the Green function on W_0 (resp. $\tilde{W}_0$) with pole at $\xi \in W_0$ (resp. $\tilde{\xi} \in \tilde{W}_0$). First, the following lemma gives us a relation between full-polar sets on base surface and full-polar sets on its covering surfaces.

Lemma 2.1 ([7, Lemma 2.3]). *Let E be a subset of $W_0 \cup \Delta$. Then E is full-polar if and only if $(\pi^*)^{-1}(E)$ is full-polar.*

By [10, Lemma 3.1] and [2, Satz 4.8] we have the following.

Lemma 2.2. *Let h be an element of $HP(W)$. Then,*

$$H_{h \circ \pi}^{\tilde{W}_0} = H_h^{W_0} \circ \pi$$

on $\tilde{W}$.

Set $HD(W) \circ \pi = \{h \circ \pi : h \in HD(W)\}$ and $\mathcal{HD}(W_0) \circ \pi = \{h \circ \pi : h \in \mathcal{HD}(W_0)\}$. Since $\tilde{W}$ is a finitely sheeted covering surface of W, $HD(W) \circ \pi$ (resp. $\mathcal{HD}(W_0) \circ \pi$) is contained in $HD(\tilde{W})$ (resp. $\mathcal{HD}(\tilde{W}_0)$). From the above lemma the next lemma follows.

Lemma 2.3. $HD(\tilde{W}) = HD(W) \circ \pi$ *if and only if* $\mathcal{HD}(\tilde{W}_0) = \mathcal{HD}(W_0) \circ \pi$.

Proof. Let $\tilde{T} = T^{\tilde{W}}$ (resp. $T = T^W$) be the mapping from $HD(\tilde{W})$ (resp. $HD(W)$) onto $\mathcal{HD}(\tilde{W}_0)$ (resp. $\mathcal{HD}(W_0)$) as in §1.1. We first remark that

(♭) $\tilde{T}(HD(W) \circ \pi) = T(HD(W)) \circ \pi.$

For, taking any $h \in HD(W)$, by Corollary 1.3 and Lemma 2.2 and by the fact that $HD(W) \subset HP(W)$, we have

$$\tilde{T}(h \circ \pi) = h \circ \pi - H_{h \circ \pi}^{\tilde{W}_0} = h \circ \pi - H_h^{W_0} \circ \pi = (h - H_h^{W_0}) \circ \pi = T(h) \circ \pi$$

on $\tilde{W}_0$.

Suppose that $HD(\tilde{W}) = HD(W) \circ \pi$. By Corollary 1.3 and ($\flat$) we have

$$\mathcal{HD}(\tilde{W}_0) = \tilde{T}(HD(\tilde{W})) = \tilde{T}(HD(W) \circ \pi) = T(HD(W)) \circ \pi = \mathcal{HD}(W_0) \circ \pi.$$

Conversely suppose that $\mathcal{HD}(\tilde{W}_0) = \mathcal{HD}(W_0) \circ \pi$. By Corollary 1.3 and ($\flat$) we have

$$HD(\tilde{W}) = \tilde{T}^{-1}(\mathcal{HD}(\tilde{W}_0)) = \tilde{T}^{-1}(\mathcal{HD}(W_0) \circ \pi) = \tilde{T}^{-1}(T(HD(W)) \circ \pi) = HD(W) \circ \pi. \ \square$$

We recall the functionals φ and ψ defined in [7, Definition 2.1]. Let $\tilde{f}$ be an extended real-valued function on $\tilde{W}$. Set, for $z \in W$,

$$\varphi[\tilde{f}](z) = \sum_{\pi(\tilde{z})=z} m(\tilde{z})\tilde{f}(\tilde{z})$$

and

$$\psi[\tilde{f}](z) = \min\{\tilde{f}(\tilde{z}) \colon \pi(\tilde{z}) = z\},$$

where $m(\tilde{z})$ is the multiplicity of $\tilde{z}$ by π. It is easily seen that, for any superharmonic function $\tilde{s}$ on $\tilde{W}_0$, $\varphi[\tilde{s}]$ and $\psi[\tilde{s}]$ are superharmonic functions on W_0 and that, for any harmonic function $\tilde{h}$ on $\tilde{W}_0$, $\varphi[\tilde{h}]$ is a harmonic function on W_0 (cf. [7, Lemma 2.2]).

Denote by $\Delta_r = \Delta_r^W$ (resp. $\tilde{\Delta}_r = \Delta_r^{\tilde{W}}$) the set of all regular points of Δ (resp. $\tilde{\Delta}$) with respect to the Dirichlet problem on W (resp. $\tilde{W}$). Set $\Delta_{r,1} = \Delta_r \cap \Delta_1$ (resp. $\tilde{\Delta}_{r,1} = \tilde{\Delta}_r \cap \tilde{\Delta}_1$). We say that a non-negative harmonic function h (resp. $\tilde{h}$) on W_0 (resp. $\tilde{W}_0$) is quasi-bounded on W_0 (resp. $\tilde{W}_0$) if there is a monotone increasing sequence $\{h_n\}$ (resp. $\{\tilde{h}_n\}$) of non-negative bounded harmonic functions such that $\lim_{n \to +\infty} h_n = h$ (resp. $\lim_{n \to +\infty} \tilde{h}_n = \tilde{h}$) on W_0 (resp. $\tilde{W}_0$). The following characterization of regular boundary points is useful.

Proposition 2.1 ([2, Satz 17.24 and Satz 17.25]). *The following conditions for* $\zeta \in \Delta_1$ *are equivalent.*
(i) ζ *(resp.* $\tilde{\zeta}$*) is a regular point of* Δ *(resp.* $\tilde{\Delta}$*);*
(ii) *the Kuramochi function* N_ζ *(resp.* $\tilde{N}_{\tilde{\zeta}}$*) is quasi-bounded on* W_0 *(resp.* $\tilde{W}_0$*);*
(iii) $\lim_{z \to \zeta} g_\xi(z) = 0$ *(resp.* $\lim_{\tilde{z} \to \tilde{\zeta}} \tilde{g}_{\tilde{\xi}}(\tilde{z}) = 0$*) for some (equivalently every) point* $\xi \in W_0$ *(resp.* $\tilde{\xi} \in \tilde{W}_0$*).*

Lemma 2.4.
$$(\pi^*)^{-1}(\Delta_{r,1}) \cap \tilde{\Delta}_1 = \tilde{\Delta}_{r,1}.$$

Proof.   First suppose that $\tilde{\zeta} \in (\pi^*)^{-1}(\Delta_{r,1}) \cap \tilde{\Delta}_1$. Setting $\zeta = \pi^*(\tilde{\zeta})$, we find that $\zeta \in \Delta_{r,1}$. Hence, by Proposition 2.1 N_ζ is quasi-bounded on W_0. From this fact it is easily seen that $N_\zeta \circ \pi$ is quasi-bounded on $\tilde{W}_0$. On the other hand, by [7, Proposi-

tion 2.1], we have

$$0 < \tilde{N}_{\tilde{\zeta}}(\tilde{z}) \leq (\varphi[\tilde{N}_{\tilde{\zeta}}] \circ \pi)(\tilde{z}) = (N_{\zeta} \circ \pi)(\tilde{z}),$$

for every $\tilde{z} \in \tilde{W}_0$. By [2, Satz 2.1] $\tilde{N}_{\tilde{\zeta}}$ is also quasi-bounded on $\tilde{W}_0$. By Proposition 2.1 we have $\tilde{\zeta} \in \tilde{\Delta}_{r,1}$, and hence $(\pi^*)^{-1}(\Delta_{r,1}) \cap \tilde{\Delta}_1 \subset \tilde{\Delta}_{r,1}$.

Next suppose $\tilde{\zeta} \in \tilde{\Delta}_{r,1}$. From [7, Theorem 1] it is easily seen that $\zeta = \pi^*(\tilde{\zeta}) \in \Delta_1$. By [7, Proposition 2.1] $N_{\zeta}(z) = \varphi[\tilde{N}_{\tilde{\zeta}}](z)$ for every $z \in W_0$. By Proposition 2.1 $\tilde{N}_{\tilde{\zeta}}$ is quasi-bounded on $\tilde{W}_0$ and hence, $N_{\zeta} = \varphi[\tilde{N}_{\tilde{\zeta}}]$ is quasi bounded on W_0. From Proposition 2.1 it follows that $\zeta \in \Delta_{r,1}$. Then $\tilde{\Delta}_{r,1} \subset (\pi^*)^{-1}(\Delta_{r,1}) \cap \tilde{\Delta}_1$. This completes the proof. □

For the harmonic measures the next lemma holds.

Lemma 2.5. *Let E be a Borel subset of Δ, and $\tilde{z}$ a point of $\tilde{W}$. Then,*

$$\mu^W_{\pi(\tilde{z})}(E) = \mu^{\tilde{W}}_{\tilde{z}}((\pi^*)^{-1}(E)).$$

Proof. Consider the generalized Dirichlet solution $H^*_{\chi_E}$ (resp. $H^*_{\chi_{(\pi^*)^{-1}(E)}}$) of the defining function χ_E (resp. $\chi_{(\pi^*)^{-1}(E)}$) of E (resp. $(\pi^*)^{-1}(E)$) on W (resp. $\tilde{W}$) in the sense of Perron-Wiener-Brelot. If $s \in \mathcal{S}(W)$ satisfies $\liminf_{\xi \to \zeta} s(\xi) \geq 1$ for all $\zeta \in E$, then $s \circ \pi \in \mathcal{S}(\tilde{W})$ satisfies $\liminf_{\tilde{\xi} \to \tilde{\zeta}} (s \circ \pi)(\tilde{\xi}) \geq 1$ for all $\tilde{\zeta} \in (\pi^*)^{-1}(E)$. Hence by definition

$$H^*_{\chi_E}(\pi(\tilde{z})) \geq H^*_{\chi_{(\pi^*)^{-1}(E)}}(\tilde{z}).$$

Conversely if $\tilde{s} \in \mathcal{S}(\tilde{W})$ satisfies $\liminf_{\tilde{\xi} \to \tilde{\zeta}} \tilde{s}(\tilde{\xi}) \geq 1$ for all $\tilde{\zeta} \in (\pi^*)^{-1}(E)$, then $\psi[\tilde{s}] \in \mathcal{S}(W)$ satisfies $\liminf_{\xi \to \zeta} \psi[\tilde{s}](\xi) \geq 1$ for all $\zeta \in E$. Hence we have

$$H^*_{\chi_E}(\pi(\tilde{z})) \leq \psi\left[H^*_{\chi_{(\pi^*)^{-1}(E)}}\right](\pi(\tilde{z})) \leq H^*_{\chi_{(\pi^*)^{-1}(E)}}(\tilde{z}).$$

Hence

$$H^*_{\chi_E}(\pi(\tilde{z})) = H^*_{\chi_{(\pi^*)^{-1}(E)}}(\tilde{z}).$$

Therefore we have the desired result.   □

From the above lemma we easily obtain the following.

Lemma 2.6. *Let E be a subset of Δ and z a point of W. Then E is a null set with respect to μ^W_z if and only if $(\pi^*)^{-1}(E)$ is a null set with respect to $\mu^{\tilde{W}}_{\tilde{z}}$ for every $\tilde{z} \in \pi^{-1}(z)$.*

3. Proof of Main Theorem

3.1. Let W, $\tilde{W}$, π, W^*, $\tilde{W}^*$, π^*, $\Delta = \Delta^W$, $\tilde{\Delta} = \Delta^{\tilde{W}}$, $\Delta_1 = \Delta_1^W$, and $\tilde{\Delta}_1 = \Delta_1^{\tilde{W}}$ be as in Introduction. For a point ζ of Δ_1, set $\tilde{\Delta}_1(\zeta) = \tilde{\Delta}_1 \cap (\pi^*)^{-1}(\zeta)$ and let $\nu(\zeta)$ be the cardinal number of $\tilde{\Delta}_1(\zeta)$. First we characterize $\nu(\zeta)$ by thinness. Let $\mathcal{M}_\zeta$ be the class of open connected subsets M of W such that $W \setminus M$ is thin at ζ. For $M \in \mathcal{M}_\zeta$, denote by $n(M) = n_{\tilde{W}}(M)$ the number of connected components of $\pi^{-1}(M)$. Then, in [7] we presented the following.

Theorem 3.1. *Let ζ be a point of Δ_1. Then,*

$$\nu(\zeta) = \max_{M \in \mathcal{M}_\zeta} n(M).$$

Let $K = K_W$ be a closed parametric disc in W. Set $W_0 = W \setminus K$ and $\tilde{W}_0 = \tilde{W} \setminus \pi^{-1}(K)$. We prepare the next lemma to prove Main theorem.

Lemma 3.1. *Let $\zeta \in \Delta_1$ and $\tilde{\Delta}_1(\zeta) = \{\tilde{\zeta}_1, \ldots, \tilde{\zeta}_q\}$, $(q \leq m)$. Then, there exist sequences $\{\tilde{\xi}_n^{(1)}\}, \ldots, \{\tilde{\xi}_n^{(q)}\}$ such that $\pi(\tilde{\xi}_n^{(1)}) = \cdots = \pi(\tilde{\xi}_n^{(q)})$ $(n = 1, 2, \ldots)$ and $\lim_{n \to +\infty} \tilde{\xi}_n^{(j)} = \tilde{\zeta}_j$ $(j = 1, \ldots, q)$.*

Proof. By Theorem 3.1 there is a subregion M of W such that $W \setminus M$ is thin at ζ and $\pi^{-1}(M)$ consists of q components, $\tilde{M}_1, \ldots, \tilde{M}_q$. We may assume that each $\tilde{W} \setminus \tilde{M}_j$ is thin at $\tilde{\zeta}_j$. Set $\tilde{B}_\rho(\tilde{\zeta}_j) = \{\tilde{z} \in \tilde{W}_0 \cup \tilde{\Delta} : \tilde{d}(\tilde{z}, \tilde{\zeta}_j) < \rho\}$ $(\rho > 0, \ j = 1, \ldots, q)$, where $\tilde{d}(\cdot, \cdot) = d^{\tilde{W}_0 \cup \tilde{\Delta}}(\cdot, \cdot)$ is the Kuramochi distance on $\tilde{W}_0 \cup \tilde{\Delta}$. Then, by the definition of thinness each $\tilde{W} \setminus (\tilde{M}_j \cap \tilde{B}_\rho(\tilde{\zeta}_j))$ is also thin at $\tilde{\zeta}_j$. By the same method as in the proof of Theorem 3.1 (cf. [7, Main Theorem]), we can show that there is a subregion M_ρ of W such that $M_\rho \subset M$, $W \setminus M_\rho$ is thin at ζ, and $\pi^{-1}(M_\rho)$ consists of q components $\tilde{M}_{\rho,1}, \ldots, \tilde{M}_{\rho,q}$ with $\tilde{M}_{\rho,j} \subset \tilde{M}_j \cap \tilde{B}_\rho(\tilde{\zeta}_j)$ $(j = 1, \ldots, q)$.

Let $\{\rho_n\}$ be a monotone decreasing sequence of positive real numbers converging to 0. We can choose sequences $\{\xi_n\}$ and $\{\tilde{\xi}_n^{(j)}\}$ $(j = 1, \ldots, q)$ such that $\xi_n \in M_{\rho_n}$ and $\tilde{\xi}_n^{(j)} \in \pi^{-1}(\xi_n) \cap (\tilde{M}_j \cap \tilde{B}_{\rho_n}(\tilde{\zeta}_j))$ $(j = 1, \ldots, q)$. Then the sequences $\{\tilde{\xi}_n^{(j)}\}$ have the required property. $\square$

3.2. Let $g_\xi = g_\xi^{W_0}$ (resp. $\tilde{g}_{\tilde{\xi}} = \tilde{g}_{\tilde{\xi}}^{\tilde{W}_0}$) be the Green function on W_0 (resp. $\tilde{W}_0$) with pole at $\xi \in W_0$ (resp. $\tilde{\xi} \in \tilde{W}_0$). Denote by $\Delta_r = \Delta_r^W$ (resp. $\tilde{\Delta}_r = \Delta_r^{\tilde{W}}$) the set of all regular points of Δ (resp. $\tilde{\Delta}$) with respect to the Dirichlet problem on W (resp. $\tilde{W}$). Set $\Delta_{r,1} = \Delta_r \cap \Delta_1$ (resp. $\tilde{\Delta}_{r,1} = \tilde{\Delta}_r \cap \tilde{\Delta}_1$). Denote by $N_\xi = N_\xi^{W_0}$ (resp. $\tilde{N}_{\tilde{\xi}} = \tilde{N}_{\tilde{\xi}}^{\tilde{W}_0}$) the Kuramochi function on W_0 (resp. $\tilde{W}_0$) with pole at $\xi \in W_0$ (resp. $\tilde{\xi} \in \tilde{W}_0$).

We obtain the following theorem which implies Main theorem.

Theorem 3.2. *The following conditions are equivalent.*

(i) $HD(\tilde{W}) = HD(W) \circ \pi$;

(ii) *for every* $\zeta \in \Delta_{r,1}$, $\nu(\zeta) = 1$;

(iii) *for all* $\zeta \in \Delta_1$ *except possibly for a full-polar subset of* Δ_1, $\nu(\zeta) = 1$;

(iv) *for almost every* $\zeta \in \Delta_1$ *with respect to the harmonic measure* μ_z^W $(z \in W)$ *on* Δ, $\nu(\zeta) = 1$.

Proof.   (i) $\Longrightarrow$ (ii): Suppose that (i) holds. By Lemma 2.3 we find that $HD(\tilde{W}) = HD(W) \circ \pi$ if and only if $\mathcal{HD}(\tilde{W}_0) = \mathcal{HD}(W_0) \circ \pi$. Fix $\tilde{\xi} \in \tilde{W}_0$ and set $\xi = \pi(\tilde{\xi})$. It is well-known that $\tilde{N}_{\tilde{\xi}} - \tilde{g}_{\tilde{\xi}} \in \mathcal{HD}(\tilde{W}_0)$ (cf. [2, p. 160]). By assumption there exists a function $g \in \mathcal{HD}(W_0)$ such that $\tilde{N}_{\tilde{\xi}} - \tilde{g}_{\tilde{\xi}} = g \circ \pi$ on $\tilde{W}_0$. We have, for every $z \in W_0$,

$$N_\xi(z) - g_\xi(z) = \sum_{\tilde{z} \in \pi^{-1}(z)} m(\tilde{z}) \left(\tilde{N}_{\tilde{\xi}}(\tilde{z}) - \tilde{g}_{\tilde{\xi}}(\tilde{z}) \right) = m \cdot g(z),$$

where $m(\tilde{z})$ is the multiplicity of $\tilde{z}$ by π, and hence $\tilde{N}_{\tilde{\xi}}(\tilde{z}) - \tilde{g}_{\tilde{\xi}}(\tilde{z}) = (1/m)(N_\xi - g_\xi) \circ \pi(\tilde{z})$ holds for every $\tilde{z} \in \tilde{W}_0$ and for every $\tilde{\xi} \in \tilde{W}_0$ with $\xi = \pi(\tilde{\xi})$. Thus $\pi(\tilde{\xi}) = \pi(\tilde{\xi}')$ implies $\tilde{N}_{\tilde{\xi}}(\tilde{z}) - \tilde{g}_{\tilde{\xi}}(\tilde{z}) = \tilde{N}_{\tilde{\xi}'}(\tilde{z}) - \tilde{g}_{\tilde{\xi}'}(\tilde{z})$ for every $\tilde{z} \in \tilde{W}_0$.

Fix $\zeta \in \Delta_{r,1}$ and $\tilde{z} \in \tilde{W}_0$. By Lemma 2.4 we note that $\tilde{\Delta}_1(\zeta) \subset \tilde{\Delta}_{r,1}$. By Proposition 2.1 and the symmetry of the Green function, for any $\tilde{\zeta} \in \tilde{\Delta}_1(\zeta)$, $\lim_{\tilde{\xi} \to \tilde{\zeta}} \tilde{g}_{\tilde{\xi}}(\tilde{z}) = 0$. Suppose that $\tilde{\zeta}'$ is another point of $\tilde{\Delta}_1(\zeta)$. By Lemma 3.1 there exist sequences $\{\tilde{\xi}_n\}$ and $\{\tilde{\xi}'_n\}$ with $\pi(\tilde{\xi}_n) = \pi(\tilde{\xi}'_n)$ $(n = 1, 2, \ldots)$ such that $\lim_{n \to +\infty} \tilde{\xi}_n = \tilde{\zeta}$ and $\lim_{n \to +\infty} \tilde{\xi}'_n = \tilde{\zeta}'$. Then we have

$$\lim_{n \to +\infty} \left(\tilde{N}_{\tilde{\xi}_n}(\tilde{z}) - \tilde{g}_{\tilde{\xi}_n}(\tilde{z}) \right) = \lim_{n \to +\infty} \left(\tilde{N}_{\tilde{\xi}'_n}(\tilde{z}) - \tilde{g}_{\tilde{\xi}'_n}(\tilde{z}) \right)$$

and hence $\tilde{N}_{\tilde{\zeta}}(\tilde{z}) = \tilde{N}_{\tilde{\zeta}'}(\tilde{z})$. Thus $\tilde{\zeta} = \tilde{\zeta}'$. This shows $\nu(\zeta) = 1$. Therefore (ii) follows.

(ii) $\Longrightarrow$ (iii): Suppose that (ii) holds. By Proposition 1.1, for all $\zeta \in \Delta_1$ except possibly for a full-polar subset $\Delta_1 \setminus \Delta_{r,1}$ of Δ_1, $\nu(\zeta) = 1$.

(iii) $\Longrightarrow$ (iv): Suppose that (iii) holds. By Lemma 1.5, for almost every $\zeta \in \Delta_1$ with respect to the harmonic measure μ_z^W $(z \in W)$ on Δ, $\nu(\zeta) = 1$.

(iv) $\Longrightarrow$ (i): Suppose that (iv) holds. Let $\tilde{h} \in HD(\tilde{W})$. By [2, Hilfssatz 16.1], there exists a Borel function $\tilde{h}^*$ on $\tilde{\Delta}$ such that $\tilde{h}(\tilde{z}) = \int \tilde{h}^* \, d\mu_{\tilde{z}}^{\tilde{W}}$. By assumption, Proposition 1.1 and Lemma 1.5 the set $N = \{ \zeta \in \Delta \colon \zeta \in \Delta \setminus \Delta_1 \text{ or } \nu(\zeta) \geq 2 \}$ is a null set. Then there exists a G_δ-set N_δ $(\subset \Delta)$ such that $N_\delta \supset N$ and $\mu_z^W(N_\delta) = 0$. Since $\nu(\zeta) = 1$ for every $\zeta \in \Delta \setminus N_\delta$, by [7, Corollary 2.2], we find that the inverse image $(\pi^*)^{-1}(\zeta)$ of ζ by π^* consists of just one minimal point. Then

$$h^*(\zeta) = \begin{cases} \tilde{h}^*((\pi^*)^{-1}(\zeta)) & \text{for } \zeta \in \Delta \setminus N_\delta \\ 0 & \text{for } \zeta \in N_\delta \end{cases}$$

is well-defined. We give the following claim:

CLAIM. h^* is a Borel measurable function on Δ.

Proof of the claim.   By the definition of G_δ-set there exist a sequence $\{O_n\}$ of open subsets of Δ with $N_\delta = \bigcap_{n=1}^{+\infty} O_n$. Set $F_n = \Delta \setminus O_n$. Then F_n is a closed subset of Δ and $\Delta = N_\delta \cup \left(\bigcup_{n=1}^{+\infty} F_n \right)$ with $N_\delta \cap \left(\bigcup_{n=1}^{+\infty} F_n \right) = \emptyset$. In order to prove this claim we shall show that for each $\alpha \in \mathbf{R}$ the set $E_\alpha = \{\zeta \in \Delta : h^*(\zeta) < \alpha\}$ is Borel measurable. Since $E_\alpha = \Delta \cap E_\alpha = (N_\delta \cap E_\alpha) \cup \left(\bigcup_{n=1}^{+\infty} (F_n \cap E_\alpha) \right)$, by the fact that $N_\delta \cap E_\alpha = N_\delta$ or $\emptyset$, it is sufficient to show that $F_n \cap E_\alpha$ is Borel measurable. Since $F_n \subset \Delta \setminus N_\delta$, the inverse image $(\pi^*)^{-1}(\zeta)$ consists of just one minimal point for every $\zeta \in F_n$. Then π^* is a bijection of $\tilde{F}_n = (\pi^*)^{-1}(F_n)$ onto F_n. Since $\tilde{F}_n$ is compact and π^* is continuous on $\tilde{F}_n$, $\pi^*|_{\tilde{F}_n}$ is a homeomorphism. Hence $F_n \cap E_\alpha = \pi^*(\tilde{F}_n \cap \tilde{E}_\alpha)$ is Borel measurable, where $\tilde{E}_\alpha = \{\tilde{\zeta} \in \tilde{\Delta} : \tilde{h}^*(\tilde{\zeta}) < \alpha\}$. The claim is proved.   □

By the above claim h^* is a Borel function on Δ. Then

$$h(z) = \int h^* \, d\mu_z^W$$

is a harmonic function in W. By Proposition 1.1, and Lemmas 1.5, 2.5 and 2.6,

$$\tilde{h}(\tilde{z}) = \int_{\tilde{\Delta}} \tilde{h}^* \, d\mu_{\tilde{z}}^{\tilde{W}} = \int_{\tilde{\Delta} \setminus (\pi^*)^{-1}(N)} h^* \circ \pi^* \, d\mu_{\tilde{z}}^{\tilde{W}} = \int_{\Delta \setminus N} h^* \, d\mu_{\pi(\tilde{z})}^W = (h \circ \pi)(\tilde{z}).$$

It is easily seen that h has a finite Dirichlet integral. Therefore we have (i).   □

4.   In the case that W is the unit disc

Throughout this section we shall consider the case that W is the unit disc $D = \{z \in \mathbf{C} : |z| < 1\}$. Let $\tilde{W} = \tilde{D}$, $\Delta = \Delta^D$, $\Delta_1 = \Delta_1^D$, and ν be as in Introduction.

4.1.   In this subsection we set $D_0 = \{z \in \mathbf{C} : r < |z| < 1\}$ $(0 < r < 1)$. We know that the Kuramochi compactification D^* of D is homeomorphic to the closure $\bar{D} = \{z \in \mathbf{C} : |z| \leq 1\}$ of D in $\mathbf{C}$ with respect to the Euclidean topology and the Kuramochi boundary $\Delta = \Delta^D$ of D is homeomorphic to the boundary $\partial D = \{z \in \mathbf{C} : |z| = 1\}$ of D in $\mathbf{C}$ with respect to the Euclidean topology, which consists of only minimal points. Set $\hat{D}_0 = \{z \in \mathbf{C} : r < |z| < 1/r\}$. We note that every non-negative full-superharmonic function on D_0 is the restriction of a non-negative superharmonic function on $\hat{D}_0$ to D_0 which is symmetric with respect to ∂D and vice versa (cf. [2, p. 234]). By this note we see that
(i)   thinness for the minimal Kuramochi boundary point coincides with thinness in the usual sense (cf. [7, Lemma 1.4]);
(ii)   a full-polar subset of Δ is identified with a polar subset of ∂D (cf. [2, p. 234]).
    The next well-known lemma gives a relation between the harmonic measures $\{\mu_z^D : z \in D\}$ on Δ and the linear measure on ∂D.

Lemma 4.1. *The harmonic measure* μ_z^D $(z \in D)$ *on* Δ *and the linear measure on* ∂D *are mutually absolutely continuous.*

The following is an immediate result from Theorem 3.2.

Theorem 4.1. *Suppose that* $W = D$. *Then, the following conditions are equivalent.*

(i) $HD(\tilde{D}) = HD(D) \circ \pi$;

(ii) *for every* $e^{i\theta} \in \partial D$, $\nu(e^{i\theta}) = 1$;

(iii) *for all* $e^{i\theta} \in \partial D$ *except possibly for a polar subset of* ∂D, $\nu(e^{i\theta}) = 1$;

(iv) *for almost every* $e^{i\theta} \in \partial D$ *with respect to the harmonic measure* μ_z^D $(z \in D)$ *on* ∂D, $\nu(e^{i\theta}) = 1$;

(v) *for almost every* $e^{i\theta} \in \partial D$ *with respect to the linear measure on* ∂D, $\nu(e^{i\theta}) = 1$.

4.2.   Our aim of this subsection is to prove Proposition in Introduction.

Let $\{w_n\}$ be a sequence with $|\Re w_n| < 1/3$ and $0 < \Im w_n < 1/3$ for every n $(\in \mathbf{N})$. Set $I = \bigcup_{n=1}^{+\infty} I_n$ $(I_n = \{w \in \mathbf{C}: \Re w = \Re w_n, \ |\Im w| \le \Im w_n\})$. Let $M > 2$ and $B_{\pm} = \{w \in \mathbf{C}: |w \pm iM| \le 1\}$. Set $G = \hat{\mathbf{C}} \setminus (B_- \cup B_+)$. We denote by $g_\xi(w) = g_\xi^G(w)$ the Green function on G with pole at $\xi \in G$. Then the following lemma holds.

Lemma 4.2. *Suppose that* w_n *satisfies*

$$\sum_{n=1}^{+\infty} \frac{1}{\log(1/\Im w_n)} < +\infty.$$

Then, I *is thin in the usual sense at almost every* $x \in [-1/6, 1/6]$ *with respect to the linear measure on* $\mathbf{R}$, *where* $[-1/6, 1/6] = \{x \in \mathbf{R}: -1/6 \le x \le 1/6\}$.

Proof.   We show that the balayage $\hat{R}_{g_x}^I = {}^G\hat{R}_{g_x}^I$ of g_x relative to I on G is not equal to g_x. We refer to [1] for informations of balayage. If we prove

$$\int_{-1/6}^{1/6} \hat{R}_{g_x}^I(x) \, dx < +\infty,$$

then $\hat{R}_{g_x}^I(x) < +\infty$ for almost every $x \in [-1/6, 1/6]$ with respect to the linear measure on $\mathbf{R}$ and hence $\hat{R}_{g_x}^I \neq g_x$ for almost every $x \in [-1/6, 1/6]$ with respect to the linear measure on $\mathbf{R}$. Set $F_n = \{w \in \mathbf{C}: |w - \Re w_n| \le \Im w_n\}$. Let φ_n be a continuous superharmonic function on G such that

(i) $\varphi_n(w) = 1$ on F_n;

(ii) $0 < \varphi_n(w) < 1$ on $G \setminus F_n$;

(iii) φ_n is harmonic in $G \setminus F_n$;

(iv) $\varphi_n(w) = 0$ on ∂G.

Then we have the following

$$\varphi_n(w) \le \frac{g_{\Re w_n}(w)}{\min\limits_{\xi \in \partial F_n} g_{\Re w_n}(\xi)}$$

for all $w \in G$. We compare the Green function with $\log(1/|w - \zeta|)$. It is easily seen that there exist positive constants c_1 and c_2 such that

$$c_1 \log \frac{1}{|w - x|} \le g_x(w) \le c_2 \log \frac{1}{|w - x|}$$

for all $x \in [-1/6, 1/6]$ and $w \in \{w \in \mathbf{C} : |\Re w| \le 1/3, \ |\Im w| \le 1/3\}$. By the fact that $\hat{R}_{g_x}^{I_n}(w) = H_{g_x}^{G \setminus I_n}(w)$ for every $w \in G \setminus I_n$ and maximum principle, we have

$$
\begin{aligned}
\int_{-1/6}^{1/6} \hat{R}_{g_x}^{I}(x)\,dx &\le \int_{-1/6}^{1/6} \sum_{n=1}^{+\infty} \hat{R}_{g_x}^{I_n}(x)\,dx \\
&\le \sum_{n=1}^{+\infty} \int_{-1/6}^{1/6} c_2 \log \frac{1}{|x - \Re w_n|} \varphi_n(x)\,dx \\
&\le \sum_{n=1}^{+\infty} \int_{-1/6}^{1/6} c_2 \log \frac{1}{|x - \Re w_n|} \frac{g_{\Re w_n}(x)}{\min\limits_{w \in \partial F_n} g_{\Re w_n}(w)}\,dx \\
&\le \sum_{n=1}^{+\infty} \int_{-1/6}^{1/6} \left(c_2 \log \frac{1}{|x - \Re w_n|}\right)^2 \frac{1}{\min\limits_{w \in \partial F_n} c_1 \log(1/|w - \Re w_n|)}\,dx \\
&= \frac{c_2^2}{c_1} \sum_{n=1}^{+\infty} \frac{1}{\log(1/\Im w_n)} \int_{-1/6}^{1/6} \left(\log \frac{1}{|x - \Re w_n|}\right)^2 dx \\
&< +\infty.
\end{aligned}
$$

This completes the proof. $\qquad\qquad\square$

Proof of Proposition in Introduction. We may assume that $0 \notin \{z_n\}$. Fix $e^{i\theta} \in \Delta$. Then $T_\theta(z) = iM(e^{i\theta} - z)/(e^{i\theta} + z)$ maps D conformally onto the upper half plane $\mathbf{H}$ with $T_\theta(e^{i\theta}) = 0$ and $T_\theta(0) = iM$, where $M > 0$. Then $\tilde{D}$ is an m-sheeted unlimited covering surface of $\mathbf{H}$ with the projection mapping $T_\theta \circ \pi$ and $\{w_n = T_\theta(z_n)\}$ is the set of the projection of branch points. We choose $M \ (> 2)$ so that $B_- = \{w \in \mathbf{C} : |w - iM| \le 1\}$ does not contain the projection of branch points. Set

$$S = \left\{w \in \mathbf{C} : |\Re w| \le \frac{1}{3}, \ |\Im w| \le \frac{1}{3}\right\}$$

and

$$I' = \bigcup_{n=1}^{+\infty} I_n' \left(I_n' = \begin{cases} \{w \in \mathbf{C} \colon \Re w = \Re w_n, \ |\Im w| \leq \Im w_n\}, \text{ if } \Im w_n < \dfrac{1}{3} \\ \{w \in \mathbf{C} \colon \Re w = \Re w_n, \ |\Im w| \geq \Im w_n\}, \text{ if } \Im w_n \geq \dfrac{1}{3} \end{cases} \right).$$

We denote $\mathbf{H} \setminus I'$ by Ω. Since Ω is a simply connected subregion of $\mathbf{H}$ without the set of the projection of branch points, $(T_\theta \circ \pi)^{-1}(\Omega)$ consists of just m connected components. Thus, by [7, Main Theorem] we find that, if $\mathbf{H} \setminus \Omega = \mathbf{H} \cap I'$ is thin at $x \in \mathbf{R}$, then $\nu(T_\theta^{-1}(x)) = m$. To prove the statement of this proposition it is sufficient to prove that $\mathbf{H} \cap I'$ is thin in the usual sense at almost every $x \in \mathbf{R}$ with respect to the linear measure on $\mathbf{R}$ because thinness for the minimal Kuramochi boundary point coincides with thinness in the usual sense. To see this we set $J_1 = \bigcup_{w_n \notin S} I_n'$ and $J_2 = \bigcup_{w_n \in S} I_n'$. Since thinness for the minimal Kuramochi boundary point coincides with thinness in the usual sense, by i) of Definition 1.3, we find that J_1 is thin in the usual sense at every $x \in [-1/6, 1/6]$. Since the condition ($\sharp$) implies

$$\sum_{w_n \in S} \frac{1}{\log(1/\Im w_n)} < +\infty,$$

by Lemma 4.2, we find that J_2 is thin in the usual sense at almost every $x \in [-1/6, 1/6]$ with respect to the linear measure on $\mathbf{R}$. Hence, we conclude that $\mathbf{H} \cap I' = J_1 \cup J_2$ is thin in the usual sense at almost every $x \in [-1/6, 1/6]$ with respect to the linear measure on $\mathbf{R}$. Therefore, since the Möbius transformation T_θ is chosen arbitrarily, we have shown that for almost every $e^{i\theta} \in \partial D$ with respect to the linear measure on ∂D, $\nu(e^{i\theta}) = m$. □

5. Remark

We have proved Main theorem under the assumption that $HD(W)$ contains a non-constant element, or equivalently $W \notin O_{HD}$. In this section we give some necessary and sufficient conditions for $\tilde{W} \in O_{HD}$ under the condition that $W \in O_{HD}$. We say that W belongs to O_G if the Green function does not exist on W. We know that Δ is a full-polar set if and only if $W \in O_G$ (see [2, p. 189]). It is well-known that $W \in O_G$ implies $W \in O_{HD}$. By [2, Folgesatz 16.1] and [2, Folgesatz 16.7], $W \in O_{HD} \setminus O_G$ if and only if $\Delta_1 = \Delta_1^W$ is the union of a singleton and a null set with respect to the harmonic measure on Δ. Then the following lemma is easily obtained.

Lemma 5.1. (1) *The following conditions are equivalent.*
(i) $W \in O_G$;
(ii) $\tilde{W} \in O_G$;

    (iii) Δ *is a full-polar set*;

    (iv) $\tilde{\Delta}$ *is a full-polar set*.

(2) *Suppose that* $W \in O_{HD} \setminus O_G$. *Then, the following conditions are equivalent.*

    (i) $HD(\tilde{W})$ *consists of only constant functions*;

    (ii) $\tilde{W} \in O_{HD} \setminus O_G$;

    (iii) *for the only one minimal point* $\zeta \in \Delta$ *with the harmonic measure* $\mu_z^W(\zeta) = 1$
$(z \in W)$, $\nu(\zeta) = 1$.

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