Kamae, T. and Yu-Mei, X.
Osaka J. Math.
41 (2004), 257–265

TWO DIMENSIONAL WORD WITH $2k$ MAXIMAL PATTERN COMPLEXITY

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(Received October 7, 2002)

1. Introduction

For an infinite 1-dimensional word $\alpha = \alpha_0\alpha_1\alpha_2\cdots$ over a finite alphabet A, Teturo Kamae and Luca Zamboni [1] introduced the maximal pattern complexity as

$$p^*_\alpha(k) := \sup_\tau \sharp\{\alpha_{n+\tau(0)}\alpha_{n+\tau(1)}\cdots\alpha_{n+\tau(k-1)};\ n = 0,\ 1,\ 2, \dots\}$$

where the supremum is taken over all sequences of integers $0 = \tau(0) < \tau(1) < \cdots < \tau(k-1)$ of length k, and $\sharp S$ denotes the cardinality of the set S. They proved that α is eventually periodic if and only if $p^*_\alpha(k)$ is bounded in k, while otherwise, $p^*_\alpha(k) \geq 2k$ $(k = 1,\ 2, \dots)$.

Teturo Kamae, Rao Hui and Xue Yu-Mei [3] considered the maximal pattern complexity for 2-dimensional words defined on $\mathbb{Z}^2$ and proved that either $p^*_\alpha(k)$ is bounded in k or $p^*_\alpha(k) \geq 2k$ $(k = 1,\ 2, \dots)$ if α satisfies a 2-dimensional recurrence condition.

In this paper, we consider the maximal pattern complexity for 2-dimensional words defined on

$$\Omega := \mathbb{N}^2 \setminus \{(0, 0)\}.$$

Let $\alpha = (\alpha(x, y)_{(x,y)\in\Omega}) \in A^\Omega$ be a 2-dimensional word over $\mathbf{A} = \{0, 1\}$ defined on Ω. Let τ be a finite set in $\mathbb{Z}^2$ with $(0, 0) \in \tau$ and $\sharp\tau = k$, which is called a *k-window*. For any $i \in \Omega$ with $i + \tau \subset \Omega$, we denote

$$\alpha[i + \tau] := (\alpha(i + j))_{j\in\tau} \in A^\tau.$$

We also denote

$$F_\tau(\alpha) := \{(\alpha[i + \tau];\ i \in \Omega \text{ with } i + \tau \subset \Omega\}$$
$$p^*_\alpha(k) := \sup\{\sharp F_\alpha(\tau);\ \tau\colon k\text{-window}\}\quad (k = 1,\ 2, \dots).$$

DEFINITION 1. α is called *eventually 2-periodic* if there exist p, $q \in \mathbb{Z}_+$ and a, $b \in \mathbb{N}$ such that for any $(x, y) \in \Omega$, $\alpha(x, y) = \alpha(x + p, y)$ holds if $x \geq a$ and $\alpha(x, y) = \alpha(x, y + q)$ holds if $y \geq b$.

DEFINITION 2. α is called *minimal* if for any positive integer L, there exists N such that for any $(n, m) \in \Omega$ there exists $(n', m') \in \Omega$ with $|n - n'| \le N$, $|m - m'| \le N$ such that $\alpha(x + n', y + m') = \alpha(x, y)$ holds for any $(x, y) \in \Omega$ with $x < L$, $y < L$.

DEFINITION 3. α is called *sectionally periodic* if for any (a, b), $(p, q) \in \Omega$, the word β on $n \in \mathbb{N}$ defined by $\beta(n) = \alpha(a + np, b + nq)$ is periodic.

In this paper, we characterize the words with bounded maximal pattern complexity. We give an example of word α with $p^*_\alpha(k) = 2k$ $(k = 1, 2, \ldots)$ which is minimal and sectionally periodic.

2. Words with bounded maximal pattern complexity

Theorem 1. *α is eventually 2-periodic if and only if $p^*_\alpha(k)$ is bounded in k.*

Proof. Assume that α is eventually 2-periodic. Take $p, q \in \mathbb{Z}_+$ and $a, b \in \mathbb{N}$ such that for any $(x, y) \in \Omega$, $\alpha(x, y) = \alpha(x + p, y)$ holds if $x \ge a$ and $\alpha(x, y) = \alpha(x, y + q)$ holds if $y \ge b$.

Let τ be a k-window. Let

$$\begin{aligned}
\Omega_1 &:= \{i = (x, y) \in \Omega;\ i + \tau \subset \Omega \cap [a, \infty) \times [b, \infty)\}\\
\Omega_2 &:= \{i = (x, y) \in \Omega \setminus \Omega_1;\ i + \tau \subset \Omega \cap [a, \infty) \times [0, \infty)\}\\
\Omega_3 &:= \{i = (x, y) \in \Omega \setminus \Omega_1;\ i + \tau \subset \Omega \cap [0, \infty) \times [b, \infty)\}\\
\Omega_4 &:= \{i = (x, y) \in \Omega \setminus (\Omega_1 \cup \Omega_2 \cup \Omega_3);\quad i + \tau \subset \Omega\}.
\end{aligned}$$

For any $i = (x, y) \in \Omega_1$, we have

$$\alpha[i + (np, mq) + \tau] = \alpha[i + \tau] \quad (\forall n,\ m = 0, 1, 2, \ldots).$$

Therefore, there exist at most pq different elements among $\alpha[i + \tau]$ with $i = \Omega_1$.

For any $i = (x, y) \in \Omega_2$, we have

$$\alpha[i + (np, 0) + \tau] = \alpha[i + \tau] \quad (\forall n = 0, 1, 2, \ldots).$$

Hence, there exist at most pb different elements among $\alpha[i + \tau]$ with $i = \Omega_2$.

In the same way, there exist at most qa different elements among $\alpha[i + \tau]$ with $i = \Omega_2$. Finally, there exist at most ab elements in Ω_4.

Therefore, we have

$$\sharp F_\alpha(\tau) \le pq + pb + qa + ab = (p + a)(q + b).$$

Thus, $p^*_\alpha(k) \le (p + a)(q + b)$ for $k = 1, 2, \ldots$, and hence, $p^*_\alpha(k)$ is bounded in k.

Conversely, assume that $\sup_{k=1,\ 2,\dots} p^*_\alpha(k) = C < \infty$. There exist $k = 1,\ 2,\dots$ and a k-window τ such that $\sharp F_\alpha(\tau) = C$. Take a positive integer L such that τ is contained in a square of size $L \times L$. Let σ be the $(L+1)^2$-window such that

$$\sigma = \{(x, y) \in \Omega;\ 0 \le x \le L,\ 0 \le y \le L\}$$

and σ' be the $(L+2)^2$-window such that

$$\sigma' = \{(x, y) \in \Omega;\ 0 \le x \le L+1,\ 0 \le y \le L+1\}.$$

Since

$$C = \sharp F_\alpha(\tau) \le \sharp F_\alpha(\sigma) \le \sharp F_\alpha(\sigma') \le C,$$

we have $\sharp F_\alpha(\sigma) = \sharp F_\alpha(\sigma') = C$. This implies that each element $\xi \in F_\alpha(\sigma)$ has a unique extension in $F_\alpha(\sigma')$. Therefore, there exists a function $h\colon F_\alpha(\sigma) \to F_\alpha(\sigma')$ such that $h(\alpha[i+\sigma]) = \alpha[i+\sigma']$ for any $i \in \Omega$.

In particular, there exist functions $f,\ g\colon F_\alpha(\sigma) \to F_\alpha(\sigma)$ such that

$$\begin{aligned} f(\alpha[i+\sigma]) &= \alpha[i+(1,0)+\sigma] \\ g(\alpha[i+\sigma]) &= \alpha[i+(0,1)+\sigma] \end{aligned} \tag{1}$$

for any $i \in \Omega$.

Since f is a transformation on a finite set, there exist $a \in \mathbb{N}$ and a period $p \in \mathbb{Z}_+$ such that

$$f^{n+p} = f^n \tag{2}$$

any $n = a,\ a+1,\ a+2,\dots$. Since

$$\alpha[(x, y)+\sigma] = f^x(\alpha[(0, y)+\sigma])$$

by (1), it follows from (2) that

$$\alpha[(x, y)+\sigma] = \alpha[(x+p, y)+\sigma]$$

for any $(x, y) \in \Omega$ with $x \ge a$.

In particular, we have

$$\alpha(x, y) = \alpha(x+p, y)$$

for any $(x, y) \in \Omega$ with $x \ge a$. In the same way, we have

$$\alpha(x, y) = \alpha(x, y+q)$$

for any $(x, y) \in \Omega$ with $y \ge b$. Thus, α is eventually 2-periodic. $\square$

3. A word with $2k$ maximal pattern complexity

A window τ' is said to be an *immediate extension* of a window τ if $\tau' \supset \tau$ and $\sharp\tau' = \sharp\tau + 1$.

The following Lemma 1 is proved in [2, Theorem 3] for words defined on $\mathbb{N}$. It remains true for words defined on Ω.

Lemma 1. *Let $\alpha \in \{0,1\}^\Omega$ be such that $p^*_\alpha(2) = 4$. Assume that for any 2-window τ and for any immediate extension τ' of τ, it holds that $\sharp F_\alpha(\tau') \leq \sharp F_\alpha(\tau) + 2$. Then, we have $p^*_\alpha(k) \leq 2k$ $(k = 1,\ 2, \ldots)$.*

Define a 2-dimensional word $\alpha \in \{0,1\}^\Omega$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } e_2(x) = e_2(y) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

for any $(x, y) \in \Omega$, where for $x \in \mathbb{N}$, $e_2(x) = n$ if and only if $2^n \mid x$ and $2^{n+1} \nmid x$. We also define $e_2(0) = \infty$.

REMARK 1. The word α defined by (3) together with $\alpha((0,0)) = 0$ is the fixed point of the 2-dimensional substitution

$$\sigma :\ 0 \to \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \quad \text{and} \quad 1 \to \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}, \tag{4}$$

so that $\alpha = \sigma^\infty(0)$.

Theorem 2. *For α defined by* (3)*, we have $p^*_\alpha(k) = 2k$ for any $k = 1,\ 2, \ldots$.*

Proof. First we prove that $p^*_\alpha(k) \geq 2k$ $(k = 1,\ 2, \ldots)$. It is clear that $p^*_\alpha(1) = 2$. For any $k = 2,\ 3, \ldots$, take a k-window $\tau := \{(0,0), (1,1), \ldots, (k-1, k-1)\}$. Then, since

$$\begin{aligned} \alpha[(1,1) + \tau] &= (1, 1, \ldots, 1) \\ \alpha[(2^k - n,\ 2^{k+1} - n) + \tau] &= (1, \ldots, 1, \overset{(n)}{0}, 1, \ldots, 1) \\ &\quad (n = 0, 1, \ldots, k-1), \end{aligned}$$

$F_\alpha(\tau)$ contains $k+1$ elements containing the letter 0 at most once.

Now, let us consider the elements in $F_\alpha(\tau)$ containing the letter 0 at least twice. They are determined by $a \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $0 \leq a < 2^n$ and $a + 2^n < k$ since there exists a unique element in $F_\alpha(\tau)$ of the form

$$(1, \ldots, 1, \overset{(a)}{0}, 1, \ldots, 1, \overset{(a+2^n)}{0}, *\,*\,*)$$

which is realized as $\alpha[(2^n - a,\ 2^{n+1} - a) + \tau]$. There are exactly

$$L := \sum_{n=0}^{\lfloor \log_2 k \rfloor} \min\{2^n, k - 2^n\}$$

number of elements of this type. Since

$$\begin{aligned} L &= \sum_{n=0}^{\lfloor \log_2 k \rfloor - 1} 2^n + k - 2^{\lfloor \log_2 k \rfloor} \\ &= 2^{\lfloor \log_2 k \rfloor} - 1 + k - 2^{\lfloor \log_2 k \rfloor} = k - 1, \end{aligned}$$

we have $\sharp F_\alpha(\tau) = k + 1 + k - 1 = 2k$. Thus, $p^*_\alpha(k) \geq 2k$ $(k = 1,\ 2, \ldots)$.

To prove that $p^*_\alpha(k) \leq 2k$ $(k = 1,\ 2, \ldots)$, it is sufficient by Lemma 1 to prove that for any 2-window τ and for any immediate extension τ' of τ, it holds that

(5) $$\sharp F_\alpha(\tau') \leq \sharp F_\alpha(\tau) + 2.$$

Take an arbitrary 2-window $\tau = \{(0, 0) = \tau_0,\ \tau_1\}$ and an arbitrary immediate extension $\tau' = \{(0, 0) = \tau_0,\ \tau_1,\ \tau_2\}$ of τ.

To prove (5), we divide into 3 cases according to the parity of τ_1

Case 1: $\tau_1 \in e \times e$

Case 2: $\tau_1 \in e \times o$

Case 3: $\tau_1 \in o \times o$,

where "e" stands for the set of even numbers, while "o" stands for the set of odd numbers. By symmetry, we can reduce the case $\tau_1 \in o \times e$ to Case 2.

Lemma 2. (i) *In* Case 1, $F_\alpha(\tau) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ *holds*.
(ii) *In* Case 2, $F_\alpha(\tau) = \{(0, 0), (0, 1), (1, 0)\}$ *holds.*
(iii) *In* Case 3, $F_\alpha(\tau) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ *holds.*

Proof. Let $\tau_1 = (u, v)$.
(i) Let $(u, v) \in e \times e$. For $(x, y) \in e \times o$, we have $\alpha[(x, y) + \tau] = (0, 0)$. If $u = v$, then by taking integers N and M with $e_2(u) < N < M$, we have $\alpha[(2^N, 2^M) + \tau] = (0, 1)$. If $u \neq v$, then assuming that $u < v$ without loss of generality, we have $\alpha[(v - u, 0) + \tau] = (0, 1)$. If $u \neq v$, then we have $\alpha[(2^N v - u, 2^N v - u) + \tau] = (1, 0)$ for a sufficiently large integer N. If $u = v$, then by taking integers N and M with $e_2(u) < N < M$, we have $\alpha[(2^N - u, 2^M - v) + \tau] = (1, 0)$. Finally, for $(x, y) \in o \times o$, we have $\alpha[(x, y) + \tau] = (1, 1)$.
(ii) Let $(u, v) \in e \times o$. Then, $\alpha[(2, 4) + \tau] = (0, 0)$, $\alpha[(v, u) + \tau] = (0, 1)$, $\alpha[(1, 1) + \tau] =$

$(1,0)$, while $\alpha[(x,y)+\tau] = (1,1)$ is impossible since either x and y have different parities or $x+u$ and $y+v$ have different parities.
(iii) Let $(u,v) \in o \times o$. For $(x,y) \in e \times o$, we have $\alpha[(x,y)+\tau] = (0,0)$. We also have $\alpha[(2,4)+\tau] = (0,1)$ and $\alpha[(2^N - u, 2^M - v)+\tau] = (1,0)$ for integers N and M such that $u+v < 2^N < 2^M$. Moreover, $\alpha[(2,2)+\tau] = (1,1)$. $\square$

We divide the above 3 cases into the following 10 subcases according to the parity of τ_2

Case 1-1: $\tau_1 \in e \times e,\ \ \tau_2 \in e \times e$
Case 1-2: $\tau_1 \in e \times e,\ \ \tau_2 \in e \times o$
Case 1-3: $\tau_1 \in e \times e,\ \ \tau_2 \in o \times o$
Case 2-1: $\tau_1 \in e \times o,\ \ \tau_2 \in e \times e$
Case 2-2: $\tau_1 \in e \times o,\ \ \tau_2 \in e \times o$
Case 2-3: $\tau_1 \in e \times o,\ \ \tau_2 \in o \times e$
Case 2-4: $\tau_1 \in e \times o,\ \ \tau_2 \in o \times o$
Case 3-1: $\tau_1 \in o \times o,\ \ \tau_2 \in e \times e$
Case 3-2: $\tau_1 \in o \times o,\ \ \tau_2 \in e \times o$
Case 3-3: $\tau_1 \in o \times o,\ \ \tau_2 \in o \times o$.

Lemma 3. (i) *In* Case 1-2, $F_\alpha(\tau') \subset \{0,1\}^3 \setminus \{(0,1,1),(1,0,1),(1,1,1)\}$.
(ii) *In* Case 1-3, $F_\alpha(\tau') \subset \{0,1\}^3 \setminus \{(0,1,0),(1,0,0)\}$.
(iii) *In* Case 2-1, $F_\alpha(\tau') \subset F_\alpha(\tau) \times \{0,1\} \setminus \{(0,1,1)\}$.
(iv) *In* Case 2-2, $F_\alpha(\tau') \subset F_\alpha(\tau) \times \{0,1\} \setminus \{(1,0,1)\}$.
(v) *In* Case 2-3, $F_\alpha(\tau') \subset F_\alpha(\tau) \times \{0,1\} \setminus \{(1,0,1)\}$.
(vi) *In* Case 2-4, $F_\alpha(\tau') \subset F_\alpha(\tau) \times \{0,1\} \setminus \{(0,1,1)\}$.
(vii) *In* Case 3-1, $F_\alpha(\tau') \subset \{0,1\}^3 \setminus \{(0,0,1),(1,0,0)\}$.
(viii) *In* Case 3-2, $F_\alpha(\tau') \subset \{0,1\}^3 \setminus \{(0,1,1),(1,0,1),(1,1,1)\}$.
(ix) *In* Case 3-3, $F_\alpha(\tau') \subset \{0,1\}^3 \setminus \{(0,0,1),(0,1,0)\}$.

Proof. Let $\tau_1 = (u,v)$, $\tau_2 = (u',v')$ and $(x,y) \in \Omega$.

(i) Since either x and y have different parities or $x+u'$ and $y+v'$ have different parities, $(1,0,1)$, $(1,1,1)$ do not belong to $F_\alpha(\tau')$. Moreover, since either $x+u$ and $y+v$ have different parities or $x+u'$ and $y+v'$ have different parities, $(0,1,1)$ does not belong to $F_\alpha(\tau')$.

(ii) Note that $\alpha[(x,y)+\tau] \in \{(1,0),(0,1)\}$ implies $(x,y) \in e \times e$. Since $(x,y) \in e \times e$ implies $\alpha((x,y)+(u',v')) = 1$, $(0,1,0)$ and $(1,0,0)$ do not belong to $F_\alpha(\tau')$.

(iii)(iv)(v)(vi)(viii) They follow by applying the parity argument in the proof of (i).

(vii) It follows by the same argument as in the proof of (ii).

(ix) Note that $\alpha((x, y) + (u, v)) \neq \alpha((x, y) + (u', v'))$ implies $(x, y) \in o \times o$. Since $(x, y) \in o \times o$ implies that $\alpha((x, y)) = 1$, $(0, 0, 1)$, $(0, 1, 0)$ does not belong to $F_\alpha(\tau')$. $\square$

Lemma 4. (i) *For any subcase except for* Case 1-1, *we have* (5).
(ii) *For any subcase except for* Case 1-1, *we have*

$$\sharp(F_\alpha(\tau') \setminus \{(0, 0, 0), (1, 1, 1)\}) \leq 4. \tag{6}$$

Proof. Clear from Lemma 2 and Lemma 3. $\square$

Now we consider Case 1-1. Assume that $\tau_1 \in e \times e$, $\tau_2 \in e \times e$. Then, we have $\alpha[(x, y) + \tau'] = (1, 1, 1)$ if $(x, y) \in o \times o$ and $\alpha[(x, y) + \tau'] = (0, 0, 0)$ if $(x, y) \in e \times o \cup o \times e$. Hence we have

$$F_\alpha(\tau') = \{\alpha[(x, y) + \tau'];\ (x, y) \in e \times e\} \cup \{(0, 0, 0), (1, 1, 1)\}.$$

Let $\tau'/2 := \{0, \tau_1/2, \tau_2/2\}$. Since $e_2(x) = e_2(y)$ is equivalent to $e_2(2x) = e_2(2y)$, we have $\alpha[(x, y)+\tau'] = \alpha[(x/2, y/2)+\tau'/2]$ for any $(x, y) \in e\times e$. Therefore, we have

$$F_\alpha(\tau') = F_\alpha(\tau'/2) \cup \{(0, 0, 0), (1, 1, 1)\}. \tag{7}$$

If $\tau'/2$ is of Case 1-1, we can apply (7) again.

By applying (7) repeatedly, we have

$$F_\alpha(\tau') = F_\alpha(\tau'/2^e) \cup \{(0, 0, 0), (1, 1, 1)\}$$

with $\tau'/2^e$ not of Case 1-1. Then, by (ii) of Lemma 4, we have $\sharp F_\alpha(\tau') \leq 6$. Thus, we have (5) by Lemma 2, which complete the proof of Theorem 2. $\square$

Theorem 3. *The word α defined by* (3) *is minimal and sectionally periodic.*

Proof. Take any positive integer L. Let N be a positive integer such that $L < 2^N$. Take any $(n, m) \in \Omega$. Then, there exists $(n', m') \in \Omega$ with $|n - n'| \leq 2^N$ and $|m-m'| \leq 2^N$ such that $e_2(n') \geq N$ and $e_2(m') \geq N$. Then, since $e_2(x+n') = e_2(x)$ and $e_2(y+m') = e_2(y)$ for any $(x, y) \in \Omega$ with $x < L$ and $y < L$, we have $\alpha(x+n', y+m') = \alpha(x, y)$ for any $(x, y) \in \Omega$ with $x < L$ and $y < L$. Thus, α is minimal.

Take any (a, b), $(p, q) \in \Omega$. Let β be a word on $n \in \mathbb{N}$ defined by $\beta(n) = \alpha(a + np, b + nq)$.

Let us consider the case where $a + p = 0$ or $b + q = 0$. Without loss of generality, assume $a + p = 0$. Then, we have $a = p = 0$ and $b > 0$, $q > 0$. Hence, β is periodic since $\beta(n) = 0$ $(n = 0,\ 1,\ 2, \ldots)$.

Now assume that $a+p > 0$ and $b+q > 0$. Let us consider the case where $aq-bp = 0$. Suppose that $p = 0$. Then, $a > 0$ and $q > 0$ since $a+p > 0$ and $p+q > 0$. This contradicts with $aq-bp = 0$. Therefore, $p > 0$. By the same reason, $q > 0$. Since $q(a+np) = p(b+nq)$ for $n = 0, 1, 2, \ldots$, we have $e_2(q)+e_2(a+np) = e_2(p)+e_2(b+nq)$ ($n = 0, 1, 2, \ldots$). Therefore, either $\beta(n) = 1$ ($n = 0, 1, 2, \ldots$) or $\beta(n) = 0$ ($n = 0, 1, 2, \ldots$) holds according as $e_2(q) = e_2(p)$ or not, and hence, β is periodic.

Now assume that $aq - bp \neq 0$. Let N be a positive integer such that $N > e_2(|aq-bp|)$. Then, since $q(a+np) - p(b+nq) = aq - bp$ ($n = 0, 1, 2, \ldots$), we have $e_2(|q(a+np) - p(b+nq)|) < N$ ($n = 0, 1, 2, \ldots$). This implies that $\min\{e_2(q(a+np)), e_2(p(b+nq))\} < N$, and hence, $\min\{e_2(a+np), e_2(b+nq)\} < N$ ($n = 0, 1, 2, \ldots$). Therefore, if $e_2(a+np) = e_2(b+nq)$, then $e_2(a+np) = e_2(b+nq) < N$ holds, and hence, we have $e_2(a+(n+2^N)p) = e_2(a+np) = e_2(b+nq) = e_2(b+(n+2^N)q)$.

If $e_2(a+np) < e_2(b+nq)$, then either $e_2(a+np) < e_2(b+nq) \leq N$ or $e_2(a+np) < N \leq e_2(b+nq)$ holds, and hence, we have $e_2(a+(n+2^N)p) = e_2(a+np) < \min\{e_2(b+nq), N\} \leq e_2(b+(n+2^N)q)$. In the same way, if $e_2(a+np) > e_2(b+nq)$, then $e_2(a+(n+2^N)p) > e_2(b+(n+2^N)q)$.

Hence, we proved that $e_2(a+np) = e_2(b+nq)$ holds if and only if $e_2(a+(n+2^N)p) = e_2(b+(n+2^N)q)$ holds, so that $\beta(n) = \beta(n+2^N)$ ($n = 0, 1, 2, \ldots$) and β is periodic.

Thus, α is sectionally periodic. $\square$

ACKNOWLEDGEMENT. The authors thank Prof. Luca Zamboni (Univ. of North Texas), Prof. Masamichi Yosida (Osaka City Univ.), Dr. Rao Hui (Wuhan Univ.) and Dr. Gjini Nertila (Tirana Univ.) for their useful discussions with the authors.

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