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COUNTEREXAMPLES FOR BOUNDEDNESS OF PSEUDODIFFERENTIAL OPERATORS

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1. Introduction

The Kohn-Nirenberg correspondence assigns to a symbol $\sigma(x,\omega)$ in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^{2d})$ the operator $\sigma(X,D)\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ defined by

$$\sigma(X,D)f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\hat{f}(\omega)e^{2\pi i x\cdot\omega}\,d\omega.$$

This is the classical version of pseudodifferential operators that is used in the investigation of partial differential operators, cf. [21]. In the language of physics, the Kohn-Nirenberg correspondence and its relatives such as the Weyl correspondence are methods of quantization. In the language of engineering, they are time-varying filters.

The Kohn-Nirenberg correspondence is usually analyzed using methods from hard analysis. The problems arising from the theory of partial differential equations suggest using the classical Hörmander symbol classes $S^m_{\rho,\delta}(\mathbb{R}^{2d})$, which are defined in terms of differentiability conditions [21], [31]. On the other hand, if we introduce the time-frequency shifts

$$M_\omega T_x f(t) = e^{2\pi i\omega\cdot t} f(t-x), \tag{1}$$

then we can write $\sigma(X,D)$ as a formal superposition of time-frequency shifts:

$$\begin{aligned}
\sigma(X,D)f(x) &= \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\eta, y-x)e^{2\pi i\eta\cdot x} f(y)\,d\eta\,dy\\
&= \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\eta,u)e^{2\pi i\eta\cdot x} f(x+u)\,du\,d\eta\\
&= \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\eta,u)\left(M_\eta T_{-u} f\right)(x)\,du\,d\eta. \qquad (2)
\end{aligned}$$

From this perspective, it seems natural to use symbols in function classes that are associated to the time-frequency shifts $M_\eta T_u$. Specifically, this is done by investigating

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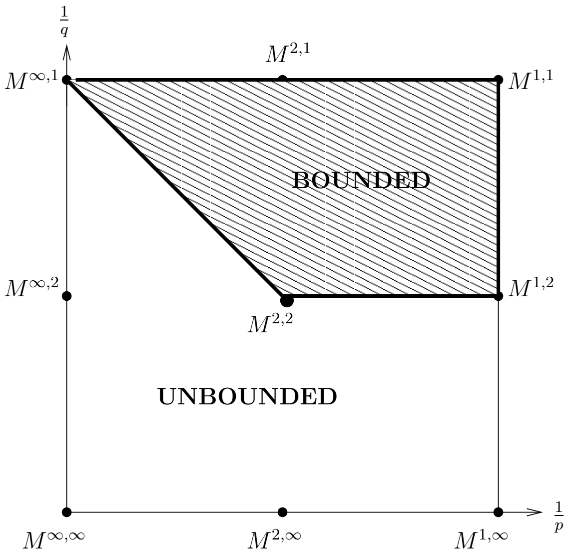

Fig. 1. Set of (p,q) for which $\sigma(x,\omega) \in M^{p,q}(\mathbb{R}^{2d})$ implies $\sigma(X,D)$ is bounded or unbounded on $L^2(\mathbb{R}^d)$.

the class of function spaces known as the *modulation spaces*. Modulation space norms are quantitative measures of the time-frequency concentration of a function or distribution, and have proven useful in the study of many aspects of time-frequency analysis. In these terms, the investigation of pseudodifferential operators amounts to the question of how a pseudodifferential operator affects the time-frequency concentration of a function.

The modulation spaces were invented and extensively investigated by Feichtinger over the period 1980–1995, with some of the main references being [9], [10], [11], [12], [13]. They are now recognized as the appropriate function spaces for time-frequency analysis, and occur naturally in mathematical problems involving time-frequency shifts $M_\omega T_x$. For a detailed development of the theory of modulation spaces and their weighted counterparts, we refer to the original literature mentioned above and to [16, Chapter 11–13].

In this note we will employ the unweighted modulation spaces $M^{p,q}(\mathbb{R}^{2d})$ as symbol classes in the study of pseudodifferential operators. We will completely characterize which of these spaces yield operators $\sigma(X,D)$ that extend to bounded mappings of $L^2(\mathbb{R}^d)$ into itself. In particular, we construct counterexamples demonstrating the sharpness of our conditions. Because of the invariance properties of the modulation spaces, the same results also hold for the Weyl correspondence. Our results are succinctly summarized in the diagram in Fig. 1.

2. Time-frequency representations and modulation spaces

The modulation space norms provide a quantitative measure of time-frequency concentration. We will use the short-time Fourier transform as an appropriate definition of the time-frequency content of a function f at "time" t and frequency ω, but we could just as well use any time-frequency representation, such as the ambiguity function or the Wigner distribution [16, Chapter 4].

DEFINITION 1. Fix a nonzero window $g \in L^2(\mathbb{R}^d)$. Then the *short-time Fourier transform* (STFT) of f with respect to g is

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega}\,\mathrm{d}t, \quad x,\omega \in \mathbb{R}^d.$$

The STFT can be written in a number of equivalent ways, for example:

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle = (f \cdot T_x\overline{g})\widehat{\ }(\omega) = e^{-2\pi i x\cdot\omega} V_{\hat{g}}\hat{f}(\omega,-x).$$

Clearly, in this formulation, the STFT can be extended to many dual pairs. In particular, if $g \in \mathcal{S}(\mathbb{R}^d)$, then $V_g f$ is defined for any tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$. In this way the STFT becomes an instrument to measure the time-frequency concentration of distributions.

DEFINITION 2. Fix a nonzero *window function* g in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, and let $1 \le p,q \le \infty$. Then the modulation space $M^{p,q}(\mathbb{R}^d)$ is the subspace of the tempered distributions consisting of all $f \in \mathcal{S}'(\mathbb{R}^d)$ for which

$$\|f\|_{M^{p,q}} = \|V_g f\|_{L^{p,q}(\mathbb{R}^{2d})} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_g f(x,\omega)|^p\,\mathrm{d}x\right)^{q/p}\mathrm{d}\omega\right)^{1/q} < \infty.$$

We define $M^p = M^{p,p}$. In particular, $\|f\|_{M^p} = \|V_g f\|_{L^p}$.

The definition of $M^{p,q}$ is independent of the choice of the window $g \in \mathcal{S}(\mathbb{R}^d)$, and different windows g yield equivalent norms on $M^{p,q}$ [16, Proposition 11.3.2]. We will employ both the modulation spaces $M^{p,q}(\mathbb{R}^d)$ and $M^{p,q}(\mathbb{R}^{2d})$ in our analysis, the domain being clear from context if not explicitly specified.

The modulation spaces have an elegant structure theory and possess atomic decompositions similar to the Besov spaces. The space M^1 serves as an important Banach space of test functions in time-frequency analysis. This space is invariant under the Fourier transform and is an algebra under both convolution and pointwise multiplication. Any compactly supported function g such that $\hat{g} \in L^1$ belongs automatically to M^1 [9].

An important property of the modulation spaces is that they are invariant under the operator which transforms a Kohn-Nirenberg symbol to a Weyl symbol. In particular, given a symbol $\sigma(x, \omega)$, the symbol $\tau(x, \omega)$ whose Weyl transform equals the Kohn-Nirenberg transform of $\sigma(x, \omega)$ is given by $\hat{\tau}(\xi, u) = e^{-\pi i u \cdot \xi} \hat{\sigma}(\xi, u)$. By [17, Lemma 2.1],

$$\sigma \in M^{p,q}(\mathbb{R}^{2d}) \iff \tau \in M^{p,q}(\mathbb{R}^{2d}).$$

Consequently, all of our results are unchanged if the Kohn-Nirenberg correspondence is replaced by the Weyl correspondence, or equivalently, the operator $\sigma(X, D)$ can be interpreted as being either the Kohn-Nirenberg or Weyl transform of $\sigma(x, \omega)$.

3. Pseudodifferential operators on $L^2(\mathbb{R}^d)$

In the literature on pseudodifferential operators, the modulation spaces figure implicitly in [3], [19], [28], [30], [32], and enter explicitly in [17], where $M^{\infty,1}(\mathbb{R}^{2d})$ in particular is used as a symbol class to establish the boundedness of $\sigma(X, D)$ on $M^p(\mathbb{R}^d)$, $1 \le p \le \infty$, including $M^2 = L^2$ as a special case. Further developments using modulation spaces have been obtained in [1], [23], [24], [33].

In this section we present sufficient conditions for the boundedness of pseudodifferential operators on $L^2(\mathbb{R}^d)$ when the symbol is taken in a modulation space $M^{p,q}$. These results follow from known endpoint results. In the following section we will show that these conditions are sharp.

For $1 \le p < \infty$ we let $\mathcal{I}_p$ denote the p-Schatten class, which is the Banach space of all compact operators on $L^2(\mathbb{R}^d)$ whose singular values lie in l^p [2], [7], [29]. Although not a standard notation, for convenience we will denote the Banach space of all bounded operators on $L^2(\mathbb{R}^d)$ by $\mathcal{I}_\infty$, with norm $\|A\|_{\mathcal{I}_\infty} = \|A\|_{\mathrm{op}}$.

Theorem 3. (a) *If* $\sigma \in L^2(\mathbb{R}^{2d}) = M^2(\mathbb{R}^{2d})$, *then* $\sigma(X, D) \in \mathcal{I}_2$ *and* $\|\sigma(X, D)\|_{\mathcal{I}_2} = \|\sigma\|_{L^2}$.
(b) *If* $\sigma \in M^1(\mathbb{R}^{2d})$, *then* $\sigma(X, D) \in \mathcal{I}_1$.

Theorem 3 (a) is due to Pool [27]. Statement (b) was stated independently by Feichtinger and Sjöstrand, with the first proof published in [15]; see also [17, Proposition 4.1] or [18, Proposition 6.1]. As discussed in [17], Theorem 3 improves the trace-class results of Daubechies [6] and Hörmander [20]. So far the best result using a weighted modulation space as a symbol class seems to be found in [19] (see [18] for the formulation in modulation space terms), with a related result in [26].

The following is [16, Theorem 14.5.2], and extends the results of [17] to all the unweighted modulation spaces.

Theorem 4. *If $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$, then $\sigma(X, D)$ is a bounded mapping of $M^{p,q}(\mathbb{R}^d)$ into itself for each $1 \le p, q \le \infty$, with a uniform estimate*

$$\|\sigma(X, D)\|_{\mathrm{op}} \le \|\sigma\|_{M^{\infty,1}}.$$

In particular, $\sigma(X, D)$ is bounded on $L^2(\mathbb{R}^d)$.

It can be shown that $C^{d+1}(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^d)$ [16, Theorem 14.5.3], and thus Theorem 4 implies the following corollary in the spirit of the celebrated Calderòn-Vaillancourt theorem: *if $\sigma \in C^{2d+1}(\mathbb{R}^{2d})$, then $\sigma(X, D)$ is bounded on $M^{p,q}$ for every $1 \le p, q \le \infty$*, cf. [4] and [14, Theorem 2.73]. In fact, the more involved arguments of [19] or [25] show that the Hölder-Zygmund class $C^{d+\epsilon}(\mathbb{R}^d)$ is contained in $M^{\infty,1}(\mathbb{R}^d)$ for all $\epsilon > 0$. However, $M^{\infty,1}$ is not defined by a smoothness criterion, and includes non-differentiable functions.

A special case of Theorem 4 was proved by Sjöstrand [30], who was apparently unaware of the extended theory of modulation spaces that was available. Among hard analysts, the space $M^{\infty,1}$ is sometimes known as Sjöstrand's class. Further investigations were done by Boulkhemair [3], who rediscovered a decomposition of $M^{\infty,1}$ of Feichtinger [10], and more recently by Toft [32], [33].

To extend the above endpoint results, we use the basic inclusion and interpolation properties of modulation spaces. In particular, recall the following facts.

(a) Inclusion Theorem [16, Theorem 12.2.2]:

$$M^{p_1,q_1} \subset M^{p_2,q_2} \iff p_1 \le p_2,\ q_1 \le q_2. \tag{3}$$

(b) Complex interpolation [8], [11]: $\left[M^{1,1}, M^{2,2}\right]_\theta = M^{p,p}$ for $1 \le p \le 2$, and $\left[M^{2,2}, M^{\infty,1}\right]_\theta = M^{p,p'}$ for $2 \le p \le \infty$.

The p-Schatten classes interpolate like L^p-spaces, namely, $[\mathcal{I}_1, \mathcal{I}_\infty]_\theta = \mathcal{I}_p$ for $1 \le p \le \infty$, cf. [22, Theorem 2.c.6]. The following statements therefore follow immediately.

Theorem 5. (a) *If $1 \le p, q \le 2$ and $\sigma \in M^{p,q}(\mathbb{R}^{2d})$, then $\sigma(X, D) \in \mathcal{I}_{\max\{p,q\}}$.*
(b) *If $2 \le p \le \infty$ and $1 \le q \le p'$, and if $\sigma \in M^{p,q}(\mathbb{R}^{2d})$, then $\sigma(X, D) \in \mathcal{I}_p$.*
(c) *In particular, if $1 \le q \le 2$ and $1 \le p \le q'$, then $\sigma(X, D)$ is a bounded operator on $L^2(\mathbb{R}^d)$.*

Proof. (a) Let $1 \le p, q \le 2$, and set $\mu = \max\{p, q\}$. If $\sigma \in [M^{1,1}, M^{2,2}]_\theta = M^{p,p}$, then $\sigma(X, D) \in [\mathcal{I}_1, \mathcal{I}_2]_\theta = \mathcal{I}_p$. By (3), we have $M^{p,q} \subset M^{\mu,\mu}$ and therefore $\sigma(X, D) \in \mathcal{I}_\mu$.

(b) If $2 \le p \le \infty$ and $\sigma \in [M^{2,2}, M^{\infty,1}]_\theta = M^{p,p'}$, then $\sigma(X, D) \in [\mathcal{I}_2, \mathcal{I}_\infty]_\theta = \mathcal{I}_p$. Since $q \le p'$, we have $M^{p,q} \subset M^{p,p'}$, and consequently $\sigma(X, D) \in \mathcal{I}_p$. □

4. Counterexamples

Our main goal is to show that Theorem 5 is sharp. We will prove the following statement.

Theorem 6. (a) *If $q > 2$, then for any $1 \le p \le \infty$ there exists $\sigma \in M^{p,q}(\mathbb{R}^{2d})$ such that $\sigma(X, D)$ is unbounded on $L^2(\mathbb{R}^d)$.*
(b) *If $p \ge 2$ and $p > q'$, then there exists $\sigma \in M^{p,q}(\mathbb{R}^{2d})$ such that $\sigma(X, D)$ is unbounded on $L^2(\mathbb{R}^d)$.*

4.1. Proof of Theorem 6 (a). For this portion of the proof of Theorem 6, it will be more convenient to work in the setting of the Weyl correspondence. Hence in this part we let $\sigma(X, D)$ denote the Weyl transform of $\sigma(x, \omega)$. The Wigner distribution

$$W(f, g)(x, \omega) = \int f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i \omega \cdot t}\, \mathrm{d}t$$

will play an important role because of the fact that $\langle \sigma(X, D) f, g\rangle = \langle \sigma, W(g, f)\rangle$. In particular, if σ is chosen to have the form $\sigma = W(\varphi, \psi)$, then $\sigma(X, D)$ is the rank-one operator $\sigma(X, D) f = \langle f, \psi\rangle \varphi$. This motivates the following lemma. A different proof of this lemma has been independently obtained by Toft in [33], and a variety of related results can be found in [5].

Lemma 7. *Let $1 \le p \le q \le \infty$ be given. If $\psi \in M^p(\mathbb{R}^d)$ and $\varphi \in M^q(\mathbb{R}^d)$, then $W(\varphi, \psi) \in M^{p,q}(\mathbb{R}^{2d})$.*

Proof. Fix any nonzero window function $g \in \mathcal{S}(\mathbb{R}^d)$. Then $G = W(g, g) \in \mathcal{S}(\mathbb{R}^{2d})$, and for $z = (z_1, z_2)$ and $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$ we have by [16, Lemma 14.5.1] that

$$|V_{W(g,g)} W(\varphi, \psi)(z, \zeta)| = \left| V_g\psi\left(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2}\right) V_g\varphi\left(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2}\right)\right|.$$

Writing $\mathcal{I}\Phi(z) = \Phi(-z)$ and $\tilde{\zeta} = (\zeta_2, -\zeta_1)$, we therefore have for $q < \infty$ that

$$\begin{aligned}
&\|W(\varphi, \psi)\|_{M^{p,q}} \\
&= \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} |V_{W(g,g)} W(\varphi, \psi)(z, \zeta)|^p\, \mathrm{d}z\right)^{q/p} \mathrm{d}\zeta\right)^{1/q} \\
&= \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} \left|V_g\psi\left(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2}\right)\right|^p \left|V_g\varphi\left(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2}\right)\right|^p \mathrm{d}z\right)^{q/p} \mathrm{d}\zeta\right)^{1/q} \\
&= \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} |V_g\psi(z)|^p \left|\mathcal{I} V_g\varphi\left(\tilde{\zeta} - z\right)\right|^p \mathrm{d}z\right)^{q/p} \mathrm{d}\zeta\right)^{1/q}
\end{aligned}$$

$$= \left(\int_{\mathbb{R}^{2d}} \left(|V_g\psi|^p * |\mathcal{I}V_g\varphi|^p \left(\tilde{\zeta}\right) \right)^{q/p} \, \mathrm{d}\zeta \right)^{1/q}$$
$$= \| |V_g\psi|^p * |\mathcal{I}V_g\varphi|^p \|_{L^{q/p}}^{1/p}$$
$$\leq \| |V_g\psi|^p \|_{L^1}^{1/p} \, \| |\mathcal{I}V_g\varphi|^p \|_{L^{q/p}}^{1/p}$$
$$= \|V_g\psi\|_{L^p} \|V_g\varphi\|_{L^q}$$
$$= \|\psi\|_{M^p} \|\varphi\|_{M^q}.$$

Young's convolution inequality is applicable above since $q/p \geq 1$. The case $q = \infty$ is similar. □

Now we can prove Theorem 6 (a) for the case $q > 2$ and $1 \leq p \leq q$. The case $p > q$ will be covered by the proof of part (b).

We construct a counterexample in the form of a rank-one operator. Since $q > 2$, we have that $L^2(\mathbb{R}^d)$ is a proper subspace of $M^q(\mathbb{R}^d)$. Choose any $\psi \in M^q \setminus L^2$, and any nonzero $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then $\sigma = W(\varphi, \psi) \in M^{p,q}(\mathbb{R}^{2d})$ by Lemma 7, yet $\sigma(X, D)$ is the rank-one operator $\sigma(X, D)f = \langle f, \psi \rangle \varphi$, which is unbounded on $L^2(\mathbb{R}^d)$.

4.2. Preparation for the proof of Theorem 6 (b). For the remainder of the proof of Theorem 6 it will most convenient to work in the setting of the Kohn-Nirenberg correspondence. We will seek a counterexample of the form $\sigma(x, \omega) = m(x)\mu(\omega) = (m \otimes \mu)(x)$. For such a separable symbol, the Kohn-Nirenberg transform $\sigma(X, D)$ coincides with the product-convolution operator

$$\sigma(X, D)f = m \cdot (\check{\mu} * f),$$

where $\check{\mu} = \mathcal{F}^{-1}\mu$ is the inverse Fourier transform of μ. For further simplification, we will try to find functions m, μ, f of the form $m = \sum_{k\in\mathbb{Z}^d} \alpha_k T_k g$, $\check{\mu} = \sum_{k\in\mathbb{Z}^d} \beta_k T_k g$, and $f = \sum_{k\in\mathbb{Z}^d} \gamma_k T_k g$. However, before constructing this counterexample we require some preparation.

Lemma 8. *Assume that $\sigma = m \otimes \mu \in \mathcal{S}'(\mathbb{R}^{2d})$. Then $\sigma \in M^{p,q}(\mathbb{R}^{2d})$ if and only if both m, $\mu \in M^{p,q}(\mathbb{R}^d)$.*

Proof. Choose a window $g \in \mathcal{S}(\mathbb{R}^{2d})$ of the form $g = g_1 \otimes g_2$ with g_1, $g_2 \in \mathcal{S}(\mathbb{R}^d)$. Then the STFT factors as $V_g\sigma = V_{g_1}m \otimes V_{g_2}\mu$, and the result immediately follows from Definition 2. □

Next we estimate the modulation space norms of several Gabor sums. The following is a special case of [16, Theorem 12.2.4].

Lemma 9. *Assume that $g \in M^1$ and $1 \le p, q \le \infty$. Then there exists $C > 0$ such that for every $\alpha \in l^p(\mathbb{Z}^d)$ and $\beta \in l^q(\mathbb{Z}^d)$ we have*

$$\Big\| \sum_{n \in \mathbb{Z}^d} \alpha_n T_n g \Big\|_{M^{p,q}} \le C \|\alpha\|_{l^p} \text{ and } \Big\| \sum_{n \in \mathbb{Z}^d} \beta_n M_n g \Big\|_{M^{p,q}} \le C \|\beta\|_{l^q}.$$

If $p, q < \infty$, then both sums converge unconditionally in $M^{p,q}$. If $p = \infty$ or $q = \infty$ with $(p,q) \ne (1,\infty), (\infty, 1)$, then both sums converge weak in $M^{p,q}$, otherwise weak* in M^∞.*

Lemma 10. *Assume that $2 < p \le \infty$ and $p' < q < 2$. Let $g \in M^1$ be given with compact support. Let $\alpha \in l^p(\mathbb{Z}^d)$, $\beta \in l^q(\mathbb{Z}^d)$, and $\gamma \in l^2(\mathbb{Z}^d)$ be given. Define*

$$m = \sum_{k \in \mathbb{Z}^d} \alpha_k T_k g \in M^{p,q}, \quad \mu = \sum_{k \in \mathbb{Z}^d} \beta_k M_{-k} \hat{g} \in M^{p,q}, \quad f = \sum_{k \in \mathbb{Z}^d} \gamma_k T_k g \in L^2. \tag{4}$$

Then $m \cdot (\check{\mu} * f) \in M^{s,t}$ for some $2 < s < \infty$ and all $1 \le t \le \infty$. Furthermore, $m \cdot (\check{\mu} * f)$ is given explicitly as

$$m \cdot (\check{\mu} * f) = \sum_{|k| \le K} \sum_{l \in \mathbb{Z}^d} \alpha_{l+k} (\beta * \gamma)_l T_l (g * g) \cdot T_{l+k} g, \tag{5}$$

for some $K > 0$ depending only on the size of the support of g, with convergence of the series in $M^{s,t}$ (weak if $t = \infty$).*

Proof. Define $1/u = 1/q - 1/2 = 1/q + 1/2 - 1$. Then $2 < u < \infty$, and by Young's inequality we have $\beta * \gamma \in l^q * l^2 \subset l^u$. Since $g * g \in M^1$, we have by Lemma 9 that the series $\sum_l (\beta * \gamma)_l T_l (g * g)$ converges in $M^{u,t}$ (weak* if $t = \infty$). Further,

$$\begin{aligned} \check{\mu} * f &= \sum_{n \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \gamma_n \beta_k (T_n g * T_k g) \\ &= \sum_{l \in \mathbb{Z}^d} \left(\sum_{n \in \mathbb{Z}^d} \gamma_n \beta_{l-n} \right) T_l (g * g) \\ &= \sum_{l \in \mathbb{Z}^d} (\beta * \gamma)_l T_l (g * g). \end{aligned}$$

Let $j \in \mathbb{Z}^d$ be fixed, and define $(\tau_j \alpha)_k = \alpha_{j+k}$. Then by Hölder's inequality, $\tau_j \alpha \cdot (\beta * \gamma) \in l^p \cdot l^u \subset l^s$ where $1/s = 1/p + 1/u = 1/p + 1/q - 1/2$ (note that $2 < s < \infty$). Since $(g * g) \cdot T_j g \in M^1$, we have by Lemma 9 that the series $\sum_l \alpha_{l+j} \cdot (\beta * \gamma)_l T_l \big((g * g) \cdot T_j g \big)$ converges in $M^{s,t}$ (weak* if $t = \infty$).

Now, since g has compact support centered at 0, there exists $K > 0$ such that

$T_l(g * g) \cdot T_k g = 0$ whenever $|l - k| > K$. Consequently,

$$\begin{aligned} m \cdot (\check{\mu} * f) &= \sum_{k \in \mathbb{Z}^d} \sum_{l \in \mathbb{Z}^d} \alpha_k (\beta * \gamma)_l T_l(g * g) T_k g \\ &= \sum_{|j| \le K} \sum_{l \in \mathbb{Z}^d} \alpha_{l+j} (\beta * \gamma)_l T_l(g * g) \cdot T_{l+j} g, \end{aligned}$$

with convergence of the series in $M^{s,t}$ (weak* if $t = \infty$). □

4.3. Proof of Theorem 6 (b): Construction of a counterexample. The case $p = 2$ is covered by part (a), so it suffices to assume that $p > 2$ and $q > p'$. Further, by the inclusion properties of the modulation spaces, it suffices to consider the case $2 < p < \infty$ and $p' < q < \infty$.

We choose a window g that will allow us to compute a lower estimate. In particular, we take $g \in M^1$ compactly supported and with $g \ge 0$. To be specific, let $g = \chi_{[-1/2,1/2]^d} * \chi_{[-1/2,1/2]^d}$. Then for some constants a, $C > 0$ we have

$$(g * g) \cdot g \ge C \chi_{[-a,a]^d}. \tag{6}$$

Suppose that α, β, $\gamma \ge 0$ satisfy the hypotheses of Lemma 10, and let m, μ, and f be defined by (4). Then by Lemma 8, we have $\sigma = m \otimes \mu \in M^{p,q}(\mathbb{R}^{2d})$. Further, $m \cdot (\check{\mu} * f)$ is an element of $M^s = M^{s,s}$ for some $2 < s < \infty$, which is a strict superset of L^2.

Since all terms in the series (5) representing $m \cdot (\check{\mu} * f)$ are non-negative, we have $m \cdot (\check{\mu} * f) \ge 0$. Therefore, using the $j = 0$ term in (5) and applying (6), we can estimate the L^2-norm of the product-convolution from below as

$$\begin{aligned} \|\sigma(X, D) f\|_{L^2} &= \|m \cdot (\check{\mu} * f)\|_{L^2} \\ &\ge \left\| \sum_{l \in \mathbb{Z}^d} \alpha_l (\beta * \gamma)_l T_l \left((g * g) \cdot g\right) \right\|_{L^2} \\ &\ge C' \|\alpha \cdot (\beta * \gamma)\|_{l^2}, \end{aligned}$$

for some appropriate constant C'. Consequently, to show that $\sigma(X, D)$ is unbounded on L^2, it suffices to construct nonnegative sequences $\alpha \in l^p$, $\beta \in l^q$, and $\gamma \in l^2$ such that $\alpha \cdot (\beta * \gamma) \notin l^2$.

Since $1/p + 1/q + 1/2 < 3/2$, we may choose $\delta > 0$ so that

$$\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{2}\right)(d + \delta) < \frac{3d}{2}.$$

Define

$$\rho = \frac{d + \delta}{p}, \quad \sigma = \frac{d + \delta}{q}, \quad \tau = \frac{d + \delta}{2}.$$

Set $\alpha_0 = \beta_0 = \gamma_0 = 1$, and for $n \neq 0$ define

$$\alpha_n = |n|^{-\rho}, \ \ \beta_n = |n|^{-\sigma}, \ \ \gamma_n = |n|^{-\tau}.$$

Then $\alpha \in l^p$, $\beta \in l^q$, $\gamma \in l^2$, and each sequence is positive. Further, given $n \in \mathbb{Z}^d$ we have

$$(\beta * \gamma)_n = \sum_{k \in \mathbb{Z}^d} \beta_k \gamma_{n-k} \geq \sum_{|n|/4 \leq |k| \leq 3|n|/4} |k|^{-\sigma} |n-k|^{-\tau} \geq C|n|^d |n|^{-\sigma-\tau}.$$

Hence

$$\alpha_n \cdot (\beta * \gamma)_n \geq C|n|^{d-\rho-\sigma-\tau} \geq C|n|^{-d/2},$$

because $\rho + \sigma + \tau < 3d/2$. Consequently $\alpha \cdot (\beta * \gamma) \notin l^2$. Thus $\sigma \in M^{p,q}(\mathbb{R}^{2d})$ and $f \in L^2(\mathbb{R}^d)$, yet $\sigma(X, D)f \in M^s \setminus L^2$.

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