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THE CAUCHY PROBLEM FOR CERTAIN SYSTEMS WITH DOUBLE CHARACTERISTICS

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1. Introduction and statement of the result

A natural application of recent results about lower bounds for systems of pseudo-differential operators (ψdo's) with double characteristics (see, e.g., [1], [2], [9]) is the study of the C^∞-well posedness of the Cauchy problem for weakly hyperbolic systems of *differential* operators. In this paper we will be concerned with systems of the form

$$P = -D_0^2 I_N + A(x, D') \tag{1}$$

in $\mathbb{R}^{1+n}_x = \mathbb{R}_{x_0} \times \mathbb{R}^n_{x'}$, where $D_0 = (1/i)(\partial/\partial x_0)$, $D' = (D_1, \dots, D_n)$, $D_j = (1/i)(\partial/\partial x_j)$, $1 \le j \le n$, I_N denotes the $N \times N$ identity matrix and $A(x, D')$ is an $N \times N$ matrix of *second order differential operators* with smooth coefficients.

We will make the following strong assumption on the principal symbol $a_2(x, \xi')$, $(x, \xi') \in \mathbb{R}_{x_0} \times (T^*\mathbb{R}^n \setminus 0) =: \mathbb{R}_{x_0} \times \dot{T}^*\mathbb{R}^n_{x'}$, of $A(x, D')$.

ASSUMPTION 1. There exists a smooth conic (closed) connected submanifold $\Sigma \subset \mathbb{R}_{x_0} \times \dot{T}^*\mathbb{R}^n_{x'}$ and an integer l with $1 \le l \le N$ such that

$$a_2(x, \xi') = \left[\begin{array}{ccc|c} \mu_1(x, \xi') & & & \\ & \ddots & & 0 \\ & & \mu_l(x, \xi') & \\ \hline & 0 & & b(x, \xi') \end{array}\right]$$

where:

- $b(x, \xi')$ is a smooth $(N-l) \times (N-l)$ positive-definite Hermitian matrix (of course, in case $l < N$);
- The μ_j, $1 \le j \le l$, are smooth non-negative symbols, vanishing exactly to second order on Σ, that is, with $\mathrm{dist}_\Sigma(x, \xi'/|\xi'|)$ denoting the distance of $(x, \xi'/|\xi'|)$ to Σ,

$$\mu_j(x, \xi') \approx |\xi'|^2 \, \mathrm{dist}_\Sigma\left(x, \frac{\xi'}{|\xi'|}\right)^2. \tag{2}$$

(For definiteness, we will suppose throughout the paper that $1 \leq l < N$.)

Although Assumption 1 is very restrictive, there are nevertheless important examples where the above structure shows up, namely when treating the Cauchy problem for the d'Alambert operator associated with the Kohn-Laplacian of a CR-manifold.

Our aim is to give sufficient conditions on P (namely, on the first order matrix term of the symbol of A) in order for the Cauchy problem to be C^∞-well posed in the following sense (see [3]):

(I) *For any given* $u \in \mathcal{E}'(\mathbb{R}^{1+n}_x; \mathbb{C}^N)$ *with* $\operatorname{supp} u \subset \{x_0 \geq 0\}$ *and such that* $Pu = 0$ *in* $\{x_0 > 0\}$ *then* $u \equiv 0$;

(II) *For any given* $f \in C^\infty(\mathbb{R}^{1+n}_x; \mathbb{C}^N)$ *with* $\operatorname{supp} f \subset \{x_0 \geq 0\}$ *and for any given relatively open set* $\Omega \subset \{x_0 \geq 0\}$ *with compact closure, there exists* $u \in C^\infty(\mathbb{R}^{1+n}_x; \mathbb{C}^N)$ *satisfying* $Pu = f$ *in* $\Omega \cap \{x_0 > 0\}$.

Having well-posedness in the sense of (I) and (II) above, can be considered only as an initial step for fully treating the Cauchy problem. More refined results concerning finite propagation speed of supports and of the C^∞ (polarized) wave-front set will be (hopefully) treated elsewhere.

We decided to deal here only with sufficient conditions for C^∞-well posedness. In the final section we have gathered a few remarks concerning the extent to which our conditions are necessary.

We now make precise the geometrical setting and state the main result. The proof of the well-posedness, that is essentially a vector-valued variant of the approach of Hörmander and Ivriĭ (see [3], [6] and [7]), is given in Section 2.

In the first place, we fix the "hyperbolic character" of the symbols

$$p_j(x, \xi) := -\xi_0^2 + \mu_j(x, \xi'), \quad j = 1, \ldots, l,$$

and the symplectic nature of the double-characteristic manifold of the p_j

$$\Sigma' := \{(x, \xi) \in \dot{T}^*\mathbb{R}^{1+n}_x ; \ \xi_0 = 0, \ a_2(x, \xi') = 0\}.$$

Namely, we make the following

ASSUMPTION 2. Upon denoting by $F_j(\rho)$ the fundamental matrix of p_j at $\rho \in \Sigma'$,

$$\text{(3)} \qquad \begin{cases} \operatorname{Spec}(F_j(\rho)) \subset i\mathbb{R} \\ \operatorname{Ker}(F_j(\rho)^2) \cap \operatorname{Im}(F_j(\rho)^2) = \{0\}, \end{cases}$$

for all $\rho \in \Sigma'$, all $j = 1, \ldots, l$ (note that condition (2) automatically yields $\operatorname{Ker} F_j(\rho) = T_\rho\Sigma'$);

(4) Σ' is *non-involutive* and the standard symplectic 2-form $\sum_{j=0}^n d\xi_j \wedge dx_j$ has constant rank on Σ'.

We explicitly remark that supposing the eigenvalues of the matrices $F_j(\rho)$ to be purely imaginary (i.e., according to the terminology of [3] and [6], that the p_j be *non-effectively hyperbolic*) amounts to requiring a condition that already in the scalar case necessarily imposes restrictions on the first-order terms. Condition (4) on Σ' is chosen for the sake of definiteness, for at the level of energy estimates it involves only Melin's inequality for systems (see Section 2). When Σ' is involutive, one has to use Hörmander's inequality for systems.

The conditions on the lower-order terms of A concern the following $l \times l$ matrix:

$$a_1^s(x,\xi') := (\sigma_1(A)_{j,j'}(x,\xi'))_{j,j'=1,\dots,l} + \frac{i}{2}\,\mathrm{diag}(\langle\partial_x,\partial_\xi\rangle\mu_j(x,\xi'))_{j=1,\dots,l}. \tag{5}$$

First of all, we have a spectral condition, namely, upon denoting

$$L_A(\rho) := \mathrm{diag}(\mathrm{Tr}^+ F_j(\rho))_{j=1,\dots,l} + a_1^s(\rho),\ \ \rho \in \Sigma', \tag{6}$$

we require

(H1) $$\mathrm{Spec}(L_A(\rho)) \subset \mathbb{R}_+,\ \ \forall \rho \in \Sigma'.$$

Recall that $\mathrm{Tr}^+ F_j(\rho) = \sum_{0<\gamma\,;\, i\gamma\in\mathrm{Spec}(F_j(\rho))} \gamma$.

It is important to note that we do not assume the matrix $a_1^s(\rho)$ to be self-adjoint. However, we have to require that $a_1^s(\rho)$ is *symmetrizable* in a suitable sense that we next make precise. Fix any $\bar\rho \in \Sigma$, and consider the *distinct* germs at $\bar\rho$ of the μ_j, that we call $\lambda_1,\dots,\lambda_k$ $(1 \le k \le l)$. Let l_h, $1 \le l_h \le l$, be the multiplicity of λ_h, $h = 1,\dots,k$. It follows that *there exists a conic neighborhood* $\Gamma_{\bar\rho} \subset \mathbb{R}_{x_0} \times \dot T^*\mathbb{R}^n_{x'}$ *of* $\bar\rho$ *and a* constant $l \times l$ *unitary matrix* e *such that on* $\Gamma_{\bar\rho}$ *we have*

$$e^*\,\mathrm{diag}(\mu_j(x,\xi'))_{j=1,\dots,l}e =: \Lambda(x,\xi') = \mathrm{diag}(\lambda_h(x,\xi')I_{l_h})_{h=1,\dots,k}. \tag{7}$$

Notice that any two amongst the λ_h are distinct as functions on $\Gamma_{\bar\rho}$. On $\Gamma_{\bar\rho}$ we hence make the following symmetrizability assumption.

(H2)$_{\bar\rho}$: There exists an $l \times l$ smooth, homogeneous of degree zero matrix $t(x,\xi')$, $(x,\xi') \in \Gamma_{\bar\rho}$, such that

(•) $$t(x,\xi') = t(x,\xi')^* > 0;$$

(••) $t(x,\xi')$ is blockwise diagonal with k blocks of size $l_h \times l_h$, $h = 1,\dots,k$, which correspond (in position) to the blocks of Λ;

(• • •) $$(e\,t(\rho)e^*)a_1^s(\rho) = a_1^s(\rho)^*(e\,t(\rho)e^*),\ \ \forall \rho \in \Gamma_{\bar\rho} \cap \Sigma.$$

REMARK 1.1. 1. Condition (••) in **(H2)**$_{\bar\rho}$ implies that

$$[t(x,\xi'),\Lambda(x,\xi')] = 0,\ \ \forall (x,\xi') \in \Gamma_{\bar\rho}.$$

2. Condition (• • •) in **(H2)**$_{\bar\rho}$ shows that the existence of such a matrix t does not depend on the rearrangement of the distinct germs of the μ_j.

Although Condition $(\mathbf{H2})_{\bar\rho}$ $(\bullet\bullet)$ is very restrictive (e.g., when $k = l$ then t must be diagonal), we are unable to avoid it in our approach (of course, if $a_1^s(\rho) = a_1^s(\rho)^*$, then t may be trivially chosen to be the indentity matrix I_l).

We are now ready to state our result.

Theorem 1.2. *Consider the differential system* (1), *satisfying* Assumptions 1 *and* 2. *If* (**H1**) *and* $(\mathbf{H2})_{\bar\rho}$, *for all* $\bar\rho \in \Sigma$, *hold, then the Cauchy problem is* C^∞*-well posed.*

2. Proof of the theorem

The proof uses the nowadays classical approach of Carleman estimates, exactly as in [3] (see also [6]). As usual, one has to distinguish between microlocal estimates near Σ and microlocal estimates away from Σ. In the latter case, $A(x, D')$ is positive elliptic, and Carleman estimates are established by using a positive square root of A. We therefore concentrate on the estimates near Σ. The key point is the following lemma.

Lemma 2.1. *Fix any* $\bar\rho \in \Sigma$ *and let* $\Gamma_{\bar\rho}$ *be a corresponding conic neighborhood as in* $(\mathbf{H2})_{\bar\rho}$. *Let* $\chi(x, D')$ *be a* scalar (*properly supported*) *pseudodifferential operator of order* 0 *with* $\operatorname{supp}\chi \subset \Gamma_{\bar\rho}$, *and* $\chi \equiv 1$ *in a neighborhood of* $\bar\rho$. *Fix any* $N \times N$ *matrix* $Q(x, D)$ *of the form*

$$(8)\quad Q(x,D) = \left[\begin{array}{c|c} q_0(x,D')D_0 I_l + q_1(x,D') & 0 \\ \hline 0 & \tilde q_0(x,D')D_0 I_{N-l} + \tilde q_1(x,D') \end{array}\right] + Q_0(x,D'),$$

where q_0, $\tilde q_0$ *are scalar* ψ*do's of order* 0, q_1 *is an* $l \times l$ *matrix of first-order* ψ*do's with principal symbol* vanishing on Σ, $\tilde q_1$ *is an* $(N-l)\times(N-l)$ *matrix of first-order* ψ*do's, and* Q_0 *is an* $N \times N$ *matrix of* 0*th-order* ψ*do's* (*all the* ψ*do being properly supported*). *Then, for any fixed compact* $K \subset \mathbb{R}^{1+n}$ *and any given* $s \in \mathbb{R}$ *there exist constants* C, $\tau_0 > 0$ *such that for every* $u \in C_0^\infty(K; \mathbb{C}^N)$, *upon defining*

$$f := (\chi(x, D')u_j)_{j=1,\dots,l},\quad g := (\chi(x, D')u_j)_{j=l+1,\dots,N},$$

the following a-priori estimate holds for all $\tau \ge \tau_0$:

$$\begin{aligned} &\tau^4 \int_0^{+\infty} e^{2\tau x_0}\left[\|f(x_0,\cdot)\|_s^2 + \|g(x_0,\cdot)\|_s^2\right] dx_0 \\ &\quad + \tau^2 \int_0^{+\infty} e^{2\tau x_0}\left[\|f(x_0,\cdot)\|_{s+1/2}^2 + \|g(x_0,\cdot)\|_{s+1}^2 + \left\|\left(Q\begin{bmatrix} f \\ g\end{bmatrix}\right)(x_0,\cdot)\right\|_s^2\right] dx_0 \\ &\le C\int_0^{+\infty} e^{2\tau x_0}\left\|\left(P\begin{bmatrix} f \\ g\end{bmatrix}\right)(x_0,\cdot)\right\|_s^2 dx_0, \end{aligned} \tag{9}$$

where $\|\phi(x_0, \cdot)\|_r$ *denotes the Sobolev norm of order* r *in the* x'*-variables.*

The core of the section is the proof of the above lemma, that we postpone by first showing how the lemma yields the proof of the theorem.

The main point consists in proving that the microlocal estimates (9) can be glued together into local ones.

Lemma 2.2. *Same hypotheses of* Theorem 1.2. *Given any compact* $K \subset \mathbb{R}^{1+n}$ *and any* $s \in \mathbb{R}$ *there exist constants* C, $\tau_0 > 0$ *such that*

$$\int_0^{+\infty} e^{2\tau x_0} \|u(x_0, \cdot)\|^2_{s+1/2}\, dx_0 \le C \int_0^{+\infty} e^{2\tau x_0} \|(Pu)(x_0, \cdot)\|^2_s\, dx_0, \tag{10}$$

for all $u \in C_0^\infty(K; \mathbb{C}^N)$ *and all* $\tau \ge \tau_0$.

Now notice that also $P^* = -D_0^2 I_N + A(x, D')^*$ satisfies the same Assumptions and hypotheses fulfilled by P (simply because $L_{A^*}(\rho) = L_A(\rho)^*$, and $t(x, \xi')^{-1}$ satisfies hypothesis $(\mathbf{H2})_{\bar\rho}$ relative to P^*). Hence estimates (10) hold true for P^*, whence it follows by a Hahn-Banach argument (see [4], Thm. 9.3.2, and [3], Thm. 4.4.3), that the Cauchy problem for P is well-posed.

Proof of Lemma 2.2. Let $\chi = \chi(x, D')$ be a 0th-order *scalar* microlocalizer near some fixed point of Σ (supported in a neighborhood on which (**H2**) holds). Write

$$P\chi u = \chi P u + \sum_{j=0}^{n} (c_j P^{(j)} u + d_j P_{(j)} u) + \mathrm{junk}(x, D')u, \tag{11}$$

where the operators $P^{(j)}$, resp. $P_{(j)}$, have principal symbol $\partial_{\xi_j}\sigma_2(P)$, resp. $\partial_{x_j}\sigma_2(P)$, the c_j and $\mathrm{junk}(x, D')$ are suitable operators of order 0, and the d_j have order -1. Set

$$N_s(\phi; \tau) := \left(\int_0^{+\infty} e^{2\tau x_0} \|\phi(x_0, \cdot)\|^2_s\, dx_0 \right)^{1/2}.$$

Then, for a suitable $C > 0$ independent of τ and u,

$$N_s(P\chi u; \tau) \le C \left(N_s(Pu; \tau) + \sum_{j=0}^{n} \left[N_s(P^{(j)} u; \tau) + N_s(|D'|^{-1} P_{(j)} u; \tau) \right] + N_s(u; \tau) \right). \tag{12}$$

Since $P^{(j)}$ and $|D'|^{-1} P_{(j)}$ are "admissible" perturbations of type $Q(x, D)$ as in Lemma 2.1, we may use inequality (9) to obtain, for all τ sufficiently large, and with

new constants, all again denoted by C,

$$\begin{aligned}(13)\qquad &\tau^4 N_s(\chi u;\tau)^2 + \tau^2 N_{s+1/2}(\chi u;\tau)^2 + \tau^2 \sum_{j=0}^{n} \left[N_s(P^{(j)}\chi u;\tau)^2 + N_s(|D'|^{-1}P_{(j)}\chi u;\tau)^2\right] \\ &\leq C N_s(P\chi u;\tau)^2.\end{aligned}$$

Since inequality (13) holds true also when $\chi(x,\xi')$ is supported away from Σ, using a microlocal partition of unity near K gives

$$\begin{aligned}(14)\qquad &\tau^4 N_s(u;\tau)^2 + \tau^2 N_{s+1/2}(u;\tau)^2 + \tau^2 \sum_{j=0}^{n} \left[N_s(P^{(j)}u;\tau)^2 + N_s(|D'|^{-1}P_{(j)}u;\tau)^2\right] \\ &\leq C\left(N_s(Pu;\tau)^2 + \sum_{j=0}^{n}\left[N_s(P^{(j)}u;\tau)^2 + N_s(|D'|^{-1}P_{(j)}u;\tau)^2\right] + N_s(u;\tau)^2\right).\end{aligned}$$

Hence inequality (10) follows by choosing, once more, τ sufficiently large. □

We now turn to the proof of Lemma 2.1. We will actually prove inequality (9) for the system

$$\tilde{P} = \left[\begin{array}{c|c} e & 0 \\ \hline 0 & I_{N-l}\end{array}\right]^{*} P \left[\begin{array}{c|c} e & 0 \\ \hline 0 & I_{N-l}\end{array}\right],$$

where e is any fixed constant unitary $l\times l$ matrix satisfying (7). Of course, if (9) holds for $\tilde{P}$, it then holds for P too. Hence, we hereon suppose that, on $\Gamma_{\bar{\rho}}$, we already have

$$(15)\qquad \sigma_2(A)(x,\xi') = a_2(x,\xi') = \left[\begin{array}{c|c} \operatorname{diag}(\lambda_h(x,\xi')I_{l_h})_{h=1,\dots,k} = \Lambda(x,\xi') & 0 \\ \hline 0 & b(x,\xi')\end{array}\right],$$

$$(16)\qquad \operatorname{Spec}(L_A(\rho)) \subset \mathbb{R}_+, \quad \forall \rho \in \Sigma',$$

where

$$(17)\qquad L_A(\rho) = \operatorname{diag}((\operatorname{Tr}^+ F_h(\rho))I_{l_h})_{h=1,\dots,k} + a_1^s(\rho),$$

$F_h(\rho)$ being the fundamental matrix of

$$q_h(x,\xi) := -\xi_0^2 + \lambda_h(x,\xi'), \quad h = 1,\dots,k,$$

and

$$(18)\qquad a_1^s(x,\xi') = (\sigma_1(A)_{j,j'}(x,\xi'))_{j,j'=1,\dots,l} + \frac{i}{2}\operatorname{diag}(\langle\partial_x,\partial_\xi\rangle\lambda_h(x,\xi')I_{l_h})_{h=1,\dots,k}.$$

In this setting, $(\mathbf{H2})_{\bar\rho}$ $(\bullet\bullet\bullet)$ reads as

$$t(\rho)a_1^s(\rho) = a_1^s(\rho)^* t(\rho), \quad \forall \rho \in \Gamma_{\bar\rho} \cap \Sigma. \tag{19}$$

The crucial point in the proof of the lemma, is the choice of the appropriate "energy form", and the use of Melin-type lower bounds for systems with double characteristics. Each one of these steps requires some preparations.

2.1. The energy form. Since in general $(\partial/\partial x_0)\lambda_h(x,\xi')$ does not vanish to second order on Σ, we need to replace the D_0-derivative by a suitable 1st-order operator. To this purpose, it is necessary to recall a few well-known facts (see [3] and [6]) concerning the hyperbolic quadratic forms

$$T_\rho T^*\mathbb{R}^{1+n} \ni v \mapsto \sigma(v, F_h(\rho)v), \quad h = 1, \dots, k, \quad \rho \in \Sigma'. \tag{20}$$

Lemma 2.3. *Upon denoting $V_h(\rho) \subset T_\rho T^*\mathbb{R}^{1+n}$ the hyperbolicity cone of the quadratic form* (20) *with respect to the direction* $(\delta x = 0; \delta\xi = e_0 = (1,0,\dots,0))$, *we have the following equivalent assertions*:

(i) $$\begin{cases} \operatorname{Spec}(F_h(\rho)) \subset i\mathbb{R}, \\ \operatorname{Ker}(F_h(\rho)^2) \cap \operatorname{Im}(F_h(\rho)^2) = \{0\}; \end{cases}$$

(ii) $$\operatorname{Ker}(F_h(\rho)^2) \cap V_h(\rho) \neq \emptyset;$$

(iii) *There exists a non-zero vector* $\zeta \in \operatorname{Ker} F_h(\rho) \cap \operatorname{Im} F_h(\rho)$ *for which*

$$\sigma(w, F_h(\rho)w) \geq 0, \quad \forall w \in (\operatorname{Span}\{\zeta\})^\sigma,$$

and

$$w \in (\operatorname{Span}\{\zeta\})^\sigma \text{ and } \sigma(w, F_h(\rho)w) = 0 \implies w \in \operatorname{Ker} F_h(\rho).$$

Observe that the set of non-zero vectors ζ satisfying (iii) of the lemma, is precisely the set

$$F_h(\rho)\left([V_h(\rho) \cup (-V_h(\rho))] \cap \operatorname{Ker}\left(F_h(\rho)^2\right)\right). \tag{21}$$

Since $\sigma|_{\Sigma'}$ has constant rank and q_h vanishes exactly to second order on Σ' (i.e. $\operatorname{Ker} F_h(\rho) = T_\rho\Sigma'$, for all ρ), the family $\rho \mapsto \operatorname{Ker}(F_h(\rho)^2)$ forms a smooth vector bundle. On the other hand, the convex cones $V_h(\rho)$ depend on ρ in an *inner semicontinuous* fashion (i.e. if $C \subset V_h(\rho_0)$ is a compact set, then $C \subset V_h(\rho)$ for all ρ in a suitable neighborhood of ρ_0). It follows that we can construct (microlocally) a smooth vector-field $\rho \mapsto z_h(\rho) \in \operatorname{Ker}(F_h(\rho)^2) \cap V_h(\rho)$, homogeneous of degree 0 in the fibers.

Hence, by virtue of (21), $\rho \mapsto F_h(\rho)z_h(\rho)$ is a smooth vector field which satisfies (iii) of Lemma 2.3 at any given point ρ. We can therefore find a real symbol

$$-\xi_0 - m_h(x,\xi') =: m_h^+(x,\xi), \tag{22}$$

homogeneous of degree 1 in ξ, $m_h(x,\xi')$ defined in $\Gamma_{\bar\rho}^-$ and *vanishing* on Σ, such that, possibly after a suitable normalization of z_h,

$$H_{m_h^+}(\rho) = F_h(\rho)z_h(\rho). \tag{23}$$

Furthermore, upon setting

$$m_h^-(x,\xi) := -\xi_0 + m_h(x,\xi'), \tag{24}$$

we can write (near $\bar\rho$)

$$q_h(x,\xi) = -\xi_0^2 + \lambda_h(x,\xi') = -m_h^-(x,\xi)m_h^+(x,\xi) + r_h(x,\xi'), \tag{25}$$

with

$$r_h(x,\xi') := \lambda_h(x,\xi') - m_h(x,\xi')^2 \geq 0,$$

and *vanishing exactly to second order on* Σ.

It is important to recall also the following consequences of the above construction (see, once more, [3] and [6]).

Lemma 2.4. *We have, for all* $h = 1,\dots,k$,

$$\begin{aligned}
&\sigma\left(H_{m_h^+}(\rho), H_{m_h^-}(\rho)\right) = -2\{\xi_0, m_h\}(\rho) = 0,\\
&H_{\{r_h, m_h^+\}}(\rho) = -2F_{r_h}(\rho)H_{m_h^+}(\rho) = 0,\\
&\mathrm{Tr}^+ F_h(\rho) = \mathrm{Tr}^+ F_{r_h}(\rho).
\end{aligned}$$

The time-slices $\Sigma_c := \Sigma \cap \{x_0 = c\}$ *are smooth conic submanifolds of* $\dot T^*\mathbb{R}^n_{x'}$, *such that* $\mathrm{rank}\left(\sum_{j=1}^n d\xi_j \wedge dx_j|_{\Sigma_c}\right)$ *is constant and* $r_h(x_0 = c, x', \xi')$ *vanishes exactly to second order on* Σ_c.

We finally arrive at the following microlocal factorization of P near $\bar\rho$:

$$\begin{aligned}
P = &\left[\begin{array}{c|c} -M^-(x,D)M^+(x,D) & 0 \\ \hline 0 & -D_0^2 I_{N-l} \end{array}\right] +\\
&+ \left[\begin{array}{c|c} R(x,D') & 0 \\ \hline 0 & B(x,D') \end{array}\right] + \left[\begin{array}{c|c} 0 & \gamma(x,D') \\ \hline \delta(x,D') & 0 \end{array}\right],
\end{aligned} \tag{26}$$

where:

• $M^{\pm}(x, D) := \operatorname{diag}\left((-D_0 \mp M_h(x, D'))I_{l_h}\right)_{h=1,\dots,k} =: -D_0 I_l \mp M(x, D')$, for 1st-order *scalar* ψdo's $M_h(x, D') = M_h(x, D')^*$ such that $\sigma_1(M_h)(x, \xi') = m_h(x, \xi')$;

• R is an $l \times l$ matrix of 2nd-order ψdo's with $\sigma_2(R)(x, \xi') = \operatorname{diag}\left(r_h(x, \xi')I_{l_h}\right)_{h=1,\dots,k}$;

• B is an $(N - l) \times (N - l)$ matrix of 2nd-order ψdo's with $\sigma_2(B)(x, \xi') = b(x, \xi')$ (recall that $b = b^* > 0$);

• γ, resp. δ, is an $l \times (N - l)$, resp. $(N - l) \times l$, matrix of 1st-order ψdo's.

Since the principal symbol of R vanishes on Σ to second order, it makes sense to consider, for $\rho \in \Sigma$,

$$\operatorname{sub}(R)(\rho) := \sigma_1(R)(\rho) + \frac{i}{2}\langle \partial_x, \partial_\xi \rangle \sigma_2(R)(\rho). \tag{27}$$

We claim that

$$\operatorname{sub}(R)(\rho) = a_1^s(\rho), \tag{28}$$

where $a_1^s(\rho)$ was defined in (18).

In fact, since

$$-M^- M^+ = -D_0^2 I_l - \operatorname{diag}\left([D_0, M_h]I_{l_h}\right)_{h=1,\dots,k} + \operatorname{diag}\left(M_h^2 I_{l_h}\right)_{h=1,\dots,k},$$

one readily computes

$$\sigma_1(R) = (\sigma_1(A)_{j,j'})_{j,j'=1,\dots,l} + \operatorname{diag}\left(\left(\frac{1}{i}\{\xi_0, m_h\} - \frac{1}{i}\sum_{j=0}^{n} \partial_{\xi_j} m_h \partial_{x_j} m_h\right) I_{l_h}\right)_{h=1,\dots,k},$$

so that, by Lemma 2.4, we have on Σ

$$\sigma_1(R)(\rho) = (\sigma_1(A)_{j,j'}(\rho))_{j,j'=1,\dots,l} + \frac{i}{2}\operatorname{diag}\left((\langle \partial_x, \partial_\xi \rangle m_h^2)(\rho) I_{l_h}\right)_{h=1,\dots,k},$$

and the claim follows.

REMARK 2.5. Factorization (26) above can be greatly simplified when the fundamental matrices $F_h(\rho)$, $h = 1, \dots, k$, commute for all $\rho \in \Sigma'$. In fact, in this case, since all $\operatorname{Ker}(F_h(\rho)^2)$ are equal, we may choose just one *scalar* symbol $m(x, \xi')$, and hence use the operators $M^{\pm}(x, D) = (-D_0 \mp M(x, D'))I_l$.

We have now to pull hypothesis $(\mathbf{H2})_{\bar\rho}$ into play. We hence suppose, as we may, that there exists an $l \times l$ matrix of 0th-order ψdo's $T(x, D')$ such that

$$T(x, D') = T(x, D')^* > 0, \text{ and } \sigma_0(T)(x, \xi') = t(x, \xi') \text{ in } \Gamma_{\bar\rho}. \tag{29}$$

At this point, using the same notation as in Lemma 2.1, we define the energy form as follows:

$$E\left(x_0; \begin{bmatrix} f \\ g \end{bmatrix}\right) := ((M^+ f)(x_0, \cdot), (TM^+ f)(x_0, \cdot)) + \|D_0 g(x_0, \cdot)\|_0^2 + \mathrm{Re}((Rf)(x_0, \cdot), (Tf)(x_0, \cdot)) + \mathrm{Re}((Bg)(x_0, \cdot), g(x_0, \cdot)), \tag{30}$$

where $(\cdot, \cdot)$ denotes the usual inner-product of $L^2(\mathbb{R}^n_{x'}; \mathbb{C}^l)$ or $L^2(\mathbb{R}^n_{x'}; \mathbb{C}^{N-l})$, according to the needs. Following the classical approach, we will estimate

$$-\int_0^{+\infty} e^{2\tau x_0} \frac{d}{dx_0} E\left(x_0; \begin{bmatrix} f \\ g \end{bmatrix}\right) dx_0, \tag{31}$$

for τ positive large. This will yield inequality (9) of Lemma 2.1 when $s = 0$ and $Q(x, D) = 0$. Afterwards, we will show how to obtain (9) in the general case.

In estimating (31) a crucial role is played, as already mentioned, by a Melin-type lower bound for systems with double characteristics. We now make this precise, in a form which is directly related to our situation (for a more general setting, see [1], [2] and [9]).

2.2. Melin's inequality. Suppose we have an $l \times l$ system $H(y, D_y) = H(y, D_y)^*$ of 2nd-order (properly supported) ψdo's in $\mathbb{R}^n_y$, and suppose

(32) All entries of the Hermitian matrix $\sigma_2(H)(y, \eta)$ vanish to second order on S,

and $\sigma_2(H)(y, \eta)$ is *positive transversally elliptic with respect to* S, that is

$$\langle \sigma_2(H)(y, \eta)v, v\rangle_{\mathbb{C}^l} \gtrsim |\eta|^2 \,\mathrm{dist}_S\left(y, \frac{\eta}{|\eta|}\right)^2 |v|^2_{\mathbb{C}^l}, \quad \forall v \in \mathbb{C}^l, \tag{33}$$

where $S \subset \dot{T}^*\mathbb{R}^n_y$ is a smooth *non-involutive* conic (closed) submanifold of codimension $2\nu + \mu$,

$$\dim(T_\rho S \cap T_\rho S^\sigma) = \mu \geq 0, \quad \dim(T_\rho S^\sigma / (T_\rho S \cap T_\rho S^\sigma)) = 2\nu \geq 2, \quad \forall \rho \in S.$$

We want to attach to H a suitable *symplectic invariant*, namely a continuous real-valued function λ_H defined on the dual bundle $(TS \cap TS^\sigma)'$ of $TS \cap TS^\sigma$ (bundle that, in case $\mu = 0$, is identified with S).

Fix any $\rho_0 \in S$, and consider a *canonical flattening* χ *of* S *near* ρ_0, that is a symplectomorphism

$$\chi : \Gamma_{\rho_0} \subset \dot{T}^*\mathbb{R}^n_y \to \tilde{\Gamma} \subset \dot{T}^*\mathbb{R}^n_z = \dot{T}^*\left(\mathbb{R}^\nu_{z'} \times \mathbb{R}^\mu_{z''} \times \mathbb{R}^{n-(\nu+\mu)}_{z'''}\right),$$

defined in a conic neighborhood Γ_{ρ_0} of ρ_0 onto an open conic set $\tilde{\Gamma}$, such that

$$\chi(\Gamma_{\rho_0} \cap \Sigma) = \{(z', z'', z'''; \zeta', \zeta'', \zeta''') \in \tilde{\Gamma};\ z' = \zeta' = 0,\ \zeta'' = 0\}. \tag{34}$$

Such a χ exists, see [5], III, Thm. 21.2.4.

Put, for $\rho \in \Gamma_{\rho_0} \cap S$,

$$h_{\rho,\chi}(\zeta''; z', \zeta') := \sum_{|\alpha|+|\beta|+|\gamma|=2} \frac{1}{\alpha!\,\beta!\,\gamma!} \left(\partial_{z'}^{\alpha}\partial_{\zeta'}^{\beta}\partial_{\zeta''}^{\gamma}(\sigma_2(H)\circ\chi^{-1})\right)(\chi(\rho))\zeta''^{\gamma} z'^{\alpha}\zeta'^{\beta} + \mathrm{sub}(H)(\rho), \tag{35}$$

where, recall,

$$\mathrm{sub}(H)(\rho) = \sigma_1(H)(\rho) + \frac{i}{2}\langle \partial_y, \partial_\eta\rangle \sigma_2(H)(\rho).$$

(Remark that $\mathrm{sub}(H)(\rho)$ is an $l \times l$ self-adjoint matrix because of the self-adjointness of H.)

Next consider the Weyl-quantization

$$H_{\rho,\chi} = H_{\rho,\chi}(\zeta''; \cdot) := \mathrm{Op}^{\mathrm{w}}(h_{\rho,\chi})(\zeta''; z', D_{z'}), \tag{36}$$

as an unbounded operator in $L^2(\mathbb{R}^{\nu}_{z'}; \mathbb{C}^l)$, depending on the parameters $\rho \in \Gamma_{\rho_0} \cap S$ and $\zeta'' \in \mathbb{R}^{\mu}$. By virtue of (33), for all ρ and ζ'', the operator $H_{\rho,\chi}$ has a bounded-from-below discrete spectrum, made of real eigenvalues (with finite multiplicities), diverging to $+\infty$ (see, e.g., [10]). In particular, the lowest eigenvalue $\lambda_\chi(\rho, \zeta'')$ depends continuously on ρ and ζ''. It turns out that using another symplectomorphism χ' with the same property (34), yields an operator $H_{\rho,\chi'}$ which is unitarily equivalent to $H_{\rho,\chi}$ (see [5], III, Thm. 18.5.9). This implies that the local functions $\lambda_\chi(\rho, \zeta'')$ can be glued together into a continuous function $\lambda_H : (TS \cap TS^{\sigma})' \to \mathbb{R}$. We finally have the following theorem.

Theorem 2.6 (Melin's inequality). *For the operator $H(y, D_y)$ above the following conditions are equivalent*:

For any given compact $K \subset \mathbb{R}^n_y$ there exist $c, C > 0$ such that

$$(Hu, u) \geq c\,\|u\|^2_{1/2} - C\,\|u\|^2_0,\ \ \forall u \in C_0^{\infty}(K; \mathbb{C}^l); \tag{M}$$

$$\lambda_H(\rho, v) > 0,\ \ \forall (\rho, v) \in (TS \cap TS^{\sigma})'. \tag{37}$$

For a proof, see [1], [2] and, in the symplectic case, [9].

2.3. Proof of inequality (9) when $s = 0$ and $Q(x, D) = 0$. Write

$$E\left(x_0; \begin{bmatrix} f \\ g \end{bmatrix}\right) = E_1(x_0) + E_2(x_0),$$

where

$$E_1 := \|D_0 g\|_0^2 + \mathrm{Re}(Bg, g),$$
$$E_2 := (M^+ f, TM^+ f) + \mathrm{Re}(Rf, Tf),$$

and recall that

$$P \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} -M^- M^+ f + Rf + \gamma g \\ -D_0^2 g + Bg + \delta f \end{bmatrix}. \tag{38}$$

We next compute $D_0 E$. Using the fact that $D_0 = -M^- + M = -M^+ - M$, one has

$$\begin{aligned} D_0 E_1 = &-2i\,\mathrm{Im}\left(P \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} 0 \\ D_0 g \end{bmatrix}\right) \\ &+ i\,\mathrm{Im}((B - B^*)g, D_0 g) + i\,\mathrm{Im}([D_0, B]g, g) + 2i\,\mathrm{Im}(\delta f, D_0 g), \end{aligned} \tag{39}$$

$$\begin{aligned} D_0 E_2 = &\,2i\,\mathrm{Im}\left(P \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} TM^+ f \\ 0 \end{bmatrix}\right) \\ &+ 2i\,\mathrm{Im}(TM^+ f, \gamma g) - (M^+ f, [D_0, T]M^+ f) + 2i\,\mathrm{Im}(TMM^+ f, M^+ f) \\ &+ i\,\mathrm{Im}((R^* T - TR)f, M^+ f) + i\,\mathrm{Im}(Rf, [T, M]f) \\ &- i\,\mathrm{Im}([M^+, R]f, Tf) - i\,\mathrm{Im}(Rf, [D_0, T]f). \end{aligned} \tag{40}$$

Hence, summation gives

$$\begin{aligned} D_0 E = &\,2i\,\mathrm{Im}\left(P \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} TM^+ f \\ -D_0 g \end{bmatrix}\right) \\ &+ 2i\,\mathrm{Im}[(\delta f, D_0 g) + (TM^+ f, \gamma g)] + (([T, M] - [D_0, T]^*)M^+ f, M^+ f) \\ &+ i\,\mathrm{Im}(\{[T, M]^* R - [D_0, T]^* R - T[M^+, R]\}f, f) + i\,\mathrm{Im}((R^* T - TR)f, M^+ f) \\ &+ i\,\mathrm{Im}((B - B^*)g, D_0 g) + i\,\mathrm{Im}([D_0, B]g, g). \end{aligned} \tag{41}$$

It is important to notice that, by virtue of our hypothesis on $t(x, \xi')$ and by the nature of M, $[T, M]$ is $l \times l$ of order 0. To simplify notation, from now on we denote by $J_j = J_j(x, D')$, $j = 0, 1, 2$, a generic system of j-th order ψdo's, not necessarily the same in each appearance, whose structure does not play any special role. We may therefore rewrite (41) as follows

$$\begin{aligned} D_0 E = &\,2i\,\mathrm{Im}\left(P \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} TM^+ f \\ -D_0 g \end{bmatrix}\right) \\ &+ 2i\,\mathrm{Im}[(\delta f, D_0 g) + (TM^+ f, \gamma g)] + i\,\mathrm{Im}(J_0 M^+ f, M^+ f) \\ &+ i\,\mathrm{Im}((J_0 R - T[M^+, R])f, f) + i\,\mathrm{Im}((R^* T - TR)f, M^+ f) \\ &+ i\,\mathrm{Im}(J_1 g, D_0 g) + i\,\mathrm{Im}(J_2 g, g). \end{aligned} \tag{42}$$

Integrating by parts

$$\int_0^{+\infty} e^{2\tau x_0} \left(\frac{1}{i} D_0 E(x_0) \right) dx_0$$

yields

$$\begin{aligned} E(0) + 2\tau \int_0^{+\infty} e^{2\tau x_0} E(x_0)\, dx_0 &= 2 \int_0^{+\infty} e^{2\tau x_0} \operatorname{Im} \left(P \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} TM^+ f \\ -D_0 g \end{bmatrix} \right) dx_0 \\ &\quad + 2 \int_0^{+\infty} e^{2\tau x_0} \operatorname{Im}(\delta f, D_0 g)\, dx_0 + I_1 + I_2 + I_3, \end{aligned} \tag{43}$$

where

$$\begin{aligned} I_1 := 2 \int_0^{+\infty} e^{2\tau x_0} \operatorname{Im}(TM^+ f, \gamma g)\, dx_0 + \int_0^{+\infty} e^{2\tau x_0} \operatorname{Im}(J_0 M^+ f, M^+ f)\, dx_0 \\ + \int_0^{+\infty} e^{2\tau x_0} \operatorname{Im}[(J_1 g, D_0 g) + (J_2 g, g)]\, dx_0, \end{aligned} \tag{44}$$

$$I_2 := \int_0^{+\infty} e^{2\tau x_0} \operatorname{Im}((R^* T - TR) f, M^+ f)\, dx_0, \tag{45}$$

$$I_3 := \int_0^{+\infty} e^{2\tau x_0} \operatorname{Im}((J_0 R - T[M^+, R]) f, f)\, dx_0. \tag{46}$$

We have to estimate all the terms on the r.h.s. of (43) (we suppose, as we may, $\tau \geq 1$ throughout the sequel). It is convenient to remark that the following inequality holds:

$$\int_0^{+\infty} e^{2\tau x_0} \left(\|TM^+ f\|_0^2 + \|M^+ f\|_0^2 \right) dx_0 \leq C_T \int_0^{+\infty} e^{2\tau x_0} (M^+ f, TM^+ f)\, dx_0, \tag{47}$$

for a constant $C_T \geq 1$ (depending on T and on the compact K, but independent of τ and f). Using (47), we have by the Cauchy-Schwarz inequality:

$$\begin{aligned} &2 \int_0^{+\infty} e^{2\tau x_0} \operatorname{Im} \left(P \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} TM^+ f \\ -D_0 g \end{bmatrix} \right) dx_0 \\ &\leq \frac{C_T}{\tau} \int_0^{+\infty} e^{2\tau x_0} \left\| P \begin{bmatrix} f \\ g \end{bmatrix} \right\|_0^2 dx_0 + \tau \int_0^{+\infty} e^{2\tau x_0} \left[(M^+ f, TM^+ f) + \|D_0 g\|_0^2 \right] dx_0, \end{aligned} \tag{48}$$

$$
\begin{aligned}
I_1 &\le C_T \int_0^{+\infty} e^{2\tau x_0}(M^+ f, TM^+ f)\,dx_0 + \int_0^{+\infty} e^{2\tau x_0}(J_2 g, g)\,dx_0 \\
&\quad + CC_T \int_0^{+\infty} e^{2\tau x_0}(M^+ f, TM^+ f)\,dx_0 + \int_0^{+\infty} e^{2\tau x_0}\,\|D_0 g\|_0^2\,dx_0 \\
&\quad + \int_0^{+\infty} e^{2\tau x_0}(J_2 g, g)\,dx_0 \\
&\le C\left[C_T \int_0^{+\infty} e^{2\tau x_0}(M^+ f, TM^+ f)\,dx_0\right. \\
&\quad \left. + \int_0^{+\infty} e^{2\tau x_0}\,\|D_0 g\|_0^2\,dx_0 + \int_0^{+\infty} e^{2\tau x_0}\,\|g\|_1^2\,dx_0\right],
\end{aligned}
\tag{49}
$$

for a suitable $C > 0$ (here and below, C will stand for a suitable positive constant independent of τ, f and g, not necessarily the same in each appearance),

$$
\begin{aligned}
I_2 &\le \int_0^{+\infty} e^{2\tau x_0}((R^*T - TR)^*(R^*T - TR)f, f)\,dx_0 \\
&\quad + C_T \int_0^{+\infty} e^{2\tau x_0}(M^+ f, TM^+ f)\,dx_0.
\end{aligned}
\tag{50}
$$

Note that $R^*T - TR$ is an $l \times l$ 1st-order system because $\sigma_2(R)$ is *blockwise scalar* and $\sigma_0(T) = t$ is *blockwise diagonal*. We next have

$$
2\int_0^{+\infty} e^{2\tau x_0}\,\mathrm{Im}(\delta f, D_0 g)\,dx_0 = 2\int_0^{+\infty} e^{2\tau x_0}\,\mathrm{Re}(\delta f, \partial_0 g)\,dx_0
$$

by integration by parts

$$
\begin{aligned}
&= -2\,\mathrm{Re}(\delta(0, x', D')f(0,\cdot), g(0,\cdot)) - 4\tau \int_0^{+\infty} e^{2\tau x_0}\,\mathrm{Re}(\delta f, g)\,dx_0 \\
&\qquad - 2\int_0^{+\infty} e^{2\tau x_0}\,\mathrm{Re}\,[((\partial_0\delta)f, g) + (\delta(\partial_0 f), g)]\,dx_0 \\
&= -2\,\mathrm{Re}(\delta(0)f(0), g(0)) - 4\tau \int_0^{+\infty} e^{2\tau x_0}\,\mathrm{Re}(f, \delta^* g)\,dx_0 \\
&\quad - 2\int_0^{+\infty} e^{2\tau x_0}\,\mathrm{Re}(f, (\partial_0\delta)^* g)\,dx_0 - 2\int_0^{+\infty} e^{2\tau x_0}\,\mathrm{Im}(-D_0 f, \delta^* g)\,dx_0 \\
&=: I - 2\int_0^{+\infty} e^{2\tau x_0}\,\mathrm{Im}(-D_0 f, \delta^* g)\,dx_0 \\
&= I - 2\int_0^{+\infty} e^{2\tau x_0}\,\mathrm{Im}(M^+ f, \delta^* g)\,dx_0 - 2\int_0^{+\infty} e^{2\tau x_0}\,\mathrm{Im}(Mf, \delta^* g)\,dx_0.
\end{aligned}
$$

It follows, again by (47), for $\varepsilon > 0$ to be picked later on,

$$
\begin{aligned}
&2\int_0^{+\infty} e^{2\tau x_0}\,\mathrm{Im}(\delta f, D_0 g)\,dx_0\\
&\le C\left[\frac{1}{\varepsilon}\,\|f(0)\|_0^2 + \varepsilon\,\|g(0)\|_1^2\right] + \tau^2\int_0^{+\infty} e^{2\tau x_0}\,\|f\|_0^2\,dx_0 + C\int_0^{+\infty} e^{2\tau x_0}\,\|g\|_1^2\,dx_0\\
&\quad + C_T\int_0^{+\infty} e^{2\tau x_0}(M^+ f, TM^+ f)\,dx_0 + \int_0^{+\infty} e^{2\tau x_0}(M^2 f, f)\,dx_0.
\end{aligned}
\tag{51}
$$

At this point, we have gotten the following inequality:

$$
\begin{aligned}
&E(0) + 2\tau\int_0^{+\infty} e^{2\tau x_0}E(x_0)\,dx_0\\
&\le \frac{C_T}{\tau}\int_0^{+\infty} e^{2\tau x_0}\left\|P\begin{bmatrix} f\\ g\end{bmatrix}\right\|_0^2 dx_0\\
&\quad + (\tau + CC_T)\int_0^{+\infty} e^{2\tau x_0}(M^+ f, TM^+ f)\,dx_0 + (\tau + C)\int_0^{+\infty} e^{2\tau x_0}\,\|D_0 g\|_0^2\,dx_0\\
&\quad + \tau^2\int_0^{+\infty} e^{2\tau x_0}\,\|f\|_0^2\,dx_0 + C\int_0^{+\infty} e^{2\tau x_0}\,\|g\|_1^2\,dx_0 + C\left[\frac{1}{\varepsilon}\,\|f(0)\|_0^2 + \varepsilon\,\|g(0)\|_1^2\right]\\
&\quad + \int_0^{+\infty} e^{2\tau x_0}\,\mathrm{Im}(\{J_0 R - T[M^+, R] + (R^*T - TR)^*(R^*T - TR) + M^2\}f, f)\,dx_0.
\end{aligned}
\tag{52}
$$

Taking into account the definition of the energy form (30) we have

$$
\begin{aligned}
&E(0) - C\left[\frac{1}{\varepsilon}\,\|f(0)\|_0^2 + \varepsilon\,\|g(0)\|_1^2\right]\\
&\quad + (\tau - CC_T)\int_0^{+\infty} e^{2\tau x_0}(M^+ f, TM^+ f)\,dx_0 + (\tau - C)\int_0^{+\infty} e^{2\tau x_0}\,\|D_0 g\|_0^2\,dx_0\\
&\quad + 2\tau\int_0^{+\infty} e^{2\tau x_0}\left[\mathrm{Re}(Rf, Tf) - \frac{1}{2\tau}\,\mathrm{Im}\big(\{J_0 R - T[M^+, R] + (R^*T - TR)^*(R^*T - TR)\right.\\
&\qquad\qquad \left. + M^2\}f, f\big)\right]dx_0\\
&\quad + 2\tau\int_0^{+\infty} e^{2\tau x_0}\left[\mathrm{Re}(Bg, g) - \frac{C}{2\tau}\,\|g\|_1^2\right]dx_0 - \tau^2\int_0^{+\infty} e^{2\tau x_0}\,\|f\|_0^2\,dx_0\\
&\le \frac{C_T}{\tau}\int_0^{+\infty} e^{2\tau x_0}\left\|P\begin{bmatrix} f\\ g\end{bmatrix}\right\|_0^2 dx_0.
\end{aligned}
\tag{53}
$$

By choosing τ sufficiently large, we may get rid of some constants in (53), obtaining

$$
\begin{aligned}
&E(0) - C\left[\frac{1}{\varepsilon}\,\|f(0)\|_0^2 + \varepsilon\,\|g(0)\|_1^2\right] \\
&\quad + \frac{\tau}{2}\int_0^{+\infty} e^{2\tau x_0}\left[(M^+ f, TM^+ f) + \|D_0 g\|_0^2\right] dx_0 - \tau^2\int_0^{+\infty} e^{2\tau x_0}\,\|f\|_0^2\,dx_0 \\
&\quad + 2\tau\int_0^{+\infty} e^{2\tau x_0}(H_\tau f, f)\,dx_0 + 2\tau\int_0^{+\infty} e^{2\tau x_0}\left[\mathrm{Re}(Bg, g) - \frac{C}{2\tau}\,\|g\|_1^2\right] dx_0 \\
&\quad \le \frac{C_T}{\tau}\int_0^{+\infty} e^{2\tau x_0}\left\|P\begin{bmatrix} f \\ g\end{bmatrix}\right\|_0^2 dx_0,
\end{aligned}
\tag{54}
$$

where the operator $H_\tau = H_\tau(x, D')$ is defined by

$$
H_\tau := \mathrm{Re}(TR) - \frac{1}{2\tau}\,\mathrm{Im}(J_0 R - T[M^+, R] + (R^*T - TR)^*(R^*T - TR) + M^2),
\tag{55}
$$

with

$$
\mathrm{Re}(\Psi) = \frac{\Psi + \Psi^*}{2}, \quad \mathrm{Im}(\Psi) = \frac{\Psi - \Psi^*}{2i}.
$$

We next use the following lemma (whose proof is exactly as in [3], Section 4.3).

Lemma 2.7. *For all τ sufficiently large, the following estimates hold*:

$$
\int_0^{+\infty} e^{2\tau x_0}(M^+ f, TM^+ f)\,dx_0 \ge C\left[\tau\,\|f(0)\|_0^2 + \tau^2\int_0^{+\infty} e^{2\tau x_0}\,\|f\|_0^2\,dx_0\right];
\tag{56}
$$

$$
\int_0^{+\infty} e^{2\tau x_0}\,\|D_0 g\|_0^2\,dx_0 \ge C\left[\tau\,\|g(0)\|_0^2 + \tau^2\int_0^{+\infty} e^{2\tau x_0}\,\|g\|_0^2\,dx_0\right].
\tag{57}
$$

From (54) it follows, by using the above lemma and the definition (30) of $E(0)$,

$$
\begin{aligned}
&(\mathrm{Re}(TR)(0) f(0), f(0)) + \left(C\tau^2 - \frac{C}{\varepsilon}\right)\|f(0)\|_0^2 \\
&\quad + \mathrm{Re}(B(0)g(0), g(0)) + C\tau^2\,\|g(0)\|_0^2 - C\varepsilon\,\|g(0)\|_1^2 \\
&\quad + (C\tau^3 - \tau^2)\int_0^{+\infty} e^{2\tau x_0}\,\|f\|_0^2\,dx_0 + C\tau^3\int_0^{+\infty} e^{2\tau x_0}\,\|g\|_0^2\,dx_0 \\
&\quad + 2\tau\int_0^{+\infty} e^{2\tau x_0}(H_\tau f, f)\,dx_0 + 2\tau\int_0^{+\infty} e^{2\tau x_0}\left[\mathrm{Re}(Bg, g) - \frac{C}{2\tau}\,\|g\|_1^2\right] dx_0 \\
&\quad \le \frac{C_T}{\tau}\int_0^{+\infty} e^{2\tau x_0}\left\|P\begin{bmatrix} f \\ g\end{bmatrix}\right\|_0^2 dx_0.
\end{aligned}
\tag{58}
$$

By Gårding's inequality we have

$$\mathrm{Re}(B(0)g(0), g(0)) + C\tau^2 \|g(0)\|_0^2 - C\varepsilon \|g(0)\|_1^2 \geq 0, \tag{59}$$

provided ε is picked sufficiently small, and for all τ large enough, and

$$\begin{aligned} &2\tau \int_0^{+\infty} e^{2\tau x_0} \left[\mathrm{Re}(Bg, g) - \frac{C}{2\tau} \|g\|_1^2 \right] dx_0 + C\tau^3 \int_0^{+\infty} e^{2\tau x_0} \|g\|_0^2 \, dx_0 \\ &\geq C \left[\tau \int_0^{+\infty} e^{2\tau x_0} \|g\|_1^2 \, dx_0 + \tau^3 \int_0^{+\infty} e^{2\tau x_0} \|g\|_0^2 \, dx_0 \right], \end{aligned} \tag{60}$$

again for all τ large enough.

Now, the crucial point consists in showing that Melin's inequality (M) of Theorem 2.6 holds for the self-adjoint system $\mathrm{Re}(TR)(x_0, x', D')$, x_0 being treated here as a parameter varying in a compact interval. We have to show that (37) holds in this case.

Recall that, by **(H2)**, $t(x, \xi') = \sigma_0(T)(x, \xi')$ is blockwise diagonal with k blocks of size $l_h \times l_h$, $h = 1, \ldots, k$. Since $\sigma_2(R)(x, \xi') = \mathrm{diag}(r_h(x, \xi') I_{l_h})_{h=1,\ldots,k}$, it follows that t and $\sigma_2(R)$ commute, so that

$$\sigma_2(\mathrm{Re}(TR))(x, \xi') = t(x, \xi')\sigma_2(R)(x, \xi') = \sigma_2(R)(x, \xi')t(x, \xi').$$

Next, for all $\rho \in \Sigma$, we have

$$\mathrm{sub}(\mathrm{Re}(TR))(\rho) = \frac{1}{2}\left(t(\rho)\mathrm{sub}(R)(\rho) + \mathrm{sub}(R)(\rho)^* t(\rho)\right)$$

by (28)

$$= \frac{1}{2}\left(t(\rho)a_1^s(\rho) + a_1^s(\rho)^* t(\rho)\right)$$

by (19)

$$= t(\rho)a_1^s(\rho) = a_1^s(\rho)^* t(\rho).$$

On using the notation of (35) and (36), we now prove that $(\mathrm{Re}(TR))_{\rho,\chi}(\zeta''; z', D_{z'})$ is positive as an unbounded operator in $L^2(\mathbb{R}^\nu_{z'}; \mathbb{C}^l)$ (χ being a canonical flattening of Σ' near ρ). We have

$$(\mathrm{Re}(TR))_{\rho,\chi}(\zeta''; z', D_{z'}) = t(\rho)\,\mathrm{diag}(\mathrm{Op}^{\mathrm{w}}(r_{h,\rho,\chi})(\zeta''; z', D_{z'})I_{l_h})_{h=1,\ldots,k} + t(\rho)a_1^s(\rho).$$

Consider any amongst the $\mathrm{Op}^{\mathrm{w}}(r_{h,\rho,\chi})$, and choose linear symplectic coordinates (t, τ) in $T^*\mathbb{R}^n$ (see [5], III, Thm. 21.5.3), in such a way that

$$r_{h,\rho,\chi} = \sum_{j=1}^{\nu} \gamma_j(\tau_j'^2 + t_j'^2) + |\tau''|^2, \quad t', \tau' \in \mathbb{R}^\nu, \quad \tau'' \in \mathbb{R}^\mu,$$

where $0 < \gamma_j$ and $\mathrm{Spec}(F_{r_h}(\rho)) \setminus \{0\} = \{\pm i\gamma_j\,;\ j = 1, \dots, \nu\}$.

It trivially follows that

$$(\mathrm{Op}^{\mathrm{w}}(r_{h,\rho,\chi})\phi, \phi) \geq (\mathrm{Tr}^{+}\, F_{r_h}(\rho))\, \|\phi\|_0^2\,,\ \ \forall \phi \in \mathcal{S}(\mathbb{R}^{\nu}_{z'}).$$

Hence, for any given $\vec{\phi} = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_l \end{bmatrix} \in \mathcal{S}(\mathbb{R}^{\nu}_{z'}; \mathbb{C}^l)$,

$$\begin{aligned} &((\mathrm{Re}(TR))_{\rho,\chi}\vec{\phi}, \vec{\phi}) \\ &= (\mathrm{diag}(\mathrm{Op}^{\mathrm{w}}(r_{h,\rho,\chi})I_{l_h})_{h=1,\dots,k}\sqrt{t(\rho)}\,\vec{\phi}, \sqrt{t(\rho)}\,\vec{\phi}) + (t(\rho)a_1^s(\rho)\vec{\phi}, \vec{\phi}) \\ &\geq (\mathrm{diag}((\mathrm{Tr}^{+}\, F_{r_h}(\rho)I_{l_h})\sqrt{t(\rho)}\,\vec{\phi}, \sqrt{t(\rho)}\,\vec{\phi}) + (t(\rho)a_1^s(\rho)\vec{\phi}, \vec{\phi}) \\ &= ([t(\rho)\,\mathrm{diag}((\mathrm{Tr}^{+}\, F_{r_h}(\rho))I_{l_h}) + t(\rho)a_1^s(\rho)]\vec{\phi}, \vec{\phi}). \end{aligned} \tag{61}$$

From Lemma 2.4 we have

$$\mathrm{diag}((\mathrm{Tr}^{+}\, F_{r_h}(\rho))I_{l_h})_{h=1,\dots,k} + a_1^s(\rho) = L_A(\rho),$$

$L_A(\rho)$ being as in (17).

As a consequence of the elementary Lemma 2.8 below, we can conclude that

$$\mathrm{Spec}(t(\rho)L_A(\rho)) \subset \mathbb{R}_+,$$

whence the positivity of $(\mathrm{Re}(TR))_{\rho,\chi}$ follows.

Lemma 2.8. *Let A be an $l \times l$ complex matrix, and suppose that for some $l \times l$ matrix $B = B^* > 0$ we have $BA = A^*B$. Then*

$$\mathrm{Spec}(BA) \subset \mathbb{R}_+ \iff \mathrm{Spec}(A) \subset \mathbb{R}_+.$$

Proof of the elementary lemma. Suppose that $\mathrm{Spec}(A) \subset \mathbb{R}_+$. We start by observing that A is diagonalizable. In fact, if for some $\mu > 0$ and $v \in \mathbb{C}^l \setminus \{0\}$ we have $(A - \mu)v \neq 0$ and $(A - \mu)^2 v = 0$, then $0 = B(A - \mu)^2 v = (A^* - \mu)B(A - \mu)v$, and hence $\langle B(A - \mu)v, (A - \mu)v\rangle_{\mathbb{C}^l} = 0$, which is a contradiction.

As a consequence $\mathbb{C}^l = \bigoplus_j \mathrm{Ker}(A - \mu_j)$. On writing any $v \in \mathbb{C}^l$ as $v = \sum_j v_j$, $v_j \in \mathrm{Ker}(A - \mu_j)$, we have

$$\langle BAv, v\rangle_{\mathbb{C}^l} = \sum_{j,k}\langle BAv_j, v_k\rangle_{\mathbb{C}^l} = \sum_{j,k}\mu_j\langle Bv_j, v_k\rangle_{\mathbb{C}^l}.$$

Since $B\colon \mathrm{Ker}(A - \mu_j) \to \mathrm{Im}(A - \mu_j)^{\perp}$ for all j, we conclude that $\langle Bv_j, v_k\rangle_{\mathbb{C}^l} = 0$ if $j \neq k$, which proves the positivity of BA.

To prove the converse, suppose $\mathrm{Spec}(BA) \subset \mathbb{R}_+$. Since $\langle Bv, v\rangle_{\mathbb{C}^l} > 0$ for all non-zero $v \in \mathbb{C}^l$, then from $Av = \mu v$ $(v \neq 0)$ we get $0 < \langle BAv, v\rangle = \mu\langle Bv, v\rangle_{\mathbb{C}^l}$, whence $\mu > 0$. □

Now, from Melin's inequality we have

$$\mathrm{Re}((TR)(0)f(0), f(0)) + \left(C\tau^2 - \frac{C}{\varepsilon}\right)\|f(0)\|_0^2 \geq 0, \tag{62}$$

for all τ sufficiently large.

Moreover, by a straightforward perturbation argument, we also have

$$\begin{aligned} &2\tau\int_0^{+\infty} e^{2\tau x_0}(H_\tau f, f)\,dx_0 + (C\tau^3 - \tau^2)\int_0^{+\infty} e^{2\tau x_0}\|f\|_0^2\,dx_0 \\ &\geq C\left[\tau\int_0^{+\infty} e^{2\tau x_0}\|f\|_{1/2}^2\,dx_0 + \tau^3\int_0^{+\infty} e^{2\tau x_0}\|f\|_0^2\,dx_0\right], \end{aligned} \tag{63}$$

for all τ sufficiently large.

Finally, by using inequalities (59), (60), (62), (63), from (58) we obtain inequality (9) when $s = 0$ and $Q(x, D) = 0$. □

2.4. Proof of inequality (9) when $s = 0$ and $Q(x, D) \neq 0$. We consider a perturbation Q of P as given in (8). Because of the already proved inequality (9) when $Q = 0$, any term of the kind $\tau^2\int_0^{+\infty} e^{2\tau x_0}\left\|Q_0\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]\right\|_0^2 dx_0$ can be immediately reabsorbed (for τ large) by the term $\tau^4\int_0^{+\infty} e^{2\tau x_0}\left[\|f\|_0^2 + \|g\|_0^2\right]dx_0$. We may hence suppose $Q_0 = 0$ and

$$Q\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} q_0(x, D')M^+ f + q_1(x, D')f \\ \tilde{q}_0(x, D')D_0 g + \tilde{q}_1(x, D')g \end{bmatrix},$$

with the q and the $\tilde{q}$ as in (8). Now, with $\varepsilon > 0$ to be picked sufficiently small, we add and subtract the term $\varepsilon\tau\int_0^{+\infty} e^{2\tau x_0}\left\|Q\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]\right\|_0^2 dx_0$ in the l.h.s. of inequality (53). The subtracted off contribution gives rise to a term of the form

$$\tau O(\varepsilon)\int_0^{+\infty} e^{2\tau x_0}\left[(M^+ f, TM^+ f) + \|D_0 g\|_0^2 + (q_1^* q_1 f, f) + (\tilde{q}_1^*\tilde{q}_1 g, g)\right]dx_0.$$

Since the principal symbol of $q_1^* q_1$ vanishes to second order on Σ, all the above terms can be handled as before. □

2.5. Proof of inequality (9) in the general case. To prove (9) for any $s \in \mathbb{R}$ and $Q(x, D)$ as in (8), we denote by $\langle D'\rangle^s$ a properly supported, *scalar* ψdo of

order s with principal symbol $|\xi'|^s$, and note that

$$\langle D'\rangle^s P \begin{bmatrix} f \\ g \end{bmatrix} = P\langle D'\rangle^s \begin{bmatrix} f \\ g \end{bmatrix} + [\langle D'\rangle^s, A]\langle D'\rangle^{-s}\langle D'\rangle^s \begin{bmatrix} f \\ g \end{bmatrix} + J_0(x, D')\langle D'\rangle^s \begin{bmatrix} f \\ g \end{bmatrix}.$$

Since $[\langle D'\rangle^s, A]\langle D'\rangle^{-s}$ is a *first-order* ψdo with principal symbol

$$\frac{1}{i}\begin{bmatrix} \{|\xi'|^s, \Lambda(x,\xi')\}|\xi'|^{-s} & 0 \\ 0 & \{|\xi'|^s, b(x,\xi')\}|\xi'|^{-s} \end{bmatrix}$$

and $\{|\xi'|^s, \Lambda(x,\xi')\}$ vanishes on Σ, $[\langle D'\rangle^s, A]\langle D'\rangle^{-s}$ is a perturbation of the same type as in (8). Moreover, since

$$\langle D'\rangle^s Q(x,D) \begin{bmatrix} f \\ g \end{bmatrix} = \langle D'\rangle^s Q(x,D)\langle D'\rangle^{-s}\langle D'\rangle^s \begin{bmatrix} f \\ g \end{bmatrix} + J_0(x, D')\langle D'\rangle^s \begin{bmatrix} f \\ g \end{bmatrix}$$

and $\langle D'\rangle^s Q(x,D)\langle D'\rangle^{-s}$ is again a perturbation of the same type as in (8), estimate (9) follows from the previous cases treated above. □

This completes the proof of Theorem 1.2. □

3. Concluding remarks

First of all, we observe that when $l = 1$ hypothesis (**H1**) reads

$$\begin{cases} \operatorname{Im}(a_1^s(\rho)) = 0 \\ |\operatorname{Re}(a_1^s(\rho))| < \operatorname{Tr}^+ F_1(\rho) \end{cases}, \quad \forall \rho \in \Sigma, \tag{64}$$

where now $a_1^s(x,\xi') = \sigma_1(A)_{1,1}(x,\xi') + (i/2)\langle \partial_x, \partial_\xi\rangle \mu_1(x,\xi')$. Condition (64) is exactly the Ivriĭ-Petkov-Hörmander condition of the scalar case. It is hence conceivable that when $l = 1$ the necessary condition for the well posedness of the Cauchy problem is (64) with the weaker inequality $|\operatorname{Re}(a_1^s(\rho))| \le \operatorname{Tr}^+ F_1(\rho)$ replacing the strict one.

When $l > 1$ and $a_1^s(\rho) = a_1^s(\rho)^*$, $\rho \in \Sigma$, condition (**H1**) can be written as

$$-\operatorname{diag}(\operatorname{Tr}^+ F_j(\rho))_{j=1,\dots,l} < a_1^s(\rho) < \operatorname{diag}(\operatorname{Tr}^+ F_j(\rho))_{j=1,\dots,l} \tag{65}$$

in the sense of Hermitian matrices, for all $\rho \in \Sigma$.

We conjecture that, at least in the case of commuting fundamental matrices F_j, the necessary condition is the weak form of (65) (where $\le$ replaces $<$). For, as shown in the proof of inequality (9), hypothesis (**H1**) ensures that the lowest eigenvalue of $(\operatorname{Re}(T R))_{\rho,\chi}$ is positive (see inequality (61)). When the fundamental matrices F_j all commute on Σ, hypothesis (**H1**) is readily seen to be also necessary to the positivity of the lowest eigenvalue.

Our main concern was to handle the case where $a_1^s(\rho)$ is *not* necessarily Hermitian. A reasonable assumption, in order to avoid the nilpotents in the first-order part, is to

suppose that $a_1^s(\rho)$ is (smoothly) symmetrizable, which is the content of (•) and (•••) of hypothesis (**H2**). Unfortunately, we have been forced to add the requirement that the symmetrizer $t(x,\xi')$ is blockwise diagonal, the blocks corresponding in size and position to those of the principal part $\Lambda(x,\xi')$. The extent to which this "blockwise" condition is caused by our approach through Carleman estimates or is more intrinsically linked to the nature of the problem, we do not know.

Apart from these considerations, the problem of nilpotents in the first-order term of the system still remains to be understood. As a very simple example, consider (with $N = l = 2$) the following system

$$\begin{cases} -D_0^2 u_1 + \mu_1(x,D')u_1 + \gamma_{11}(x,D')u_1 + \gamma_{12}(x,D')u_2 = f_1 \\ -D_0^2 u_2 + \mu_2(x,D')u_2 + \gamma_{22}(x,D')u_2 = f_2, \end{cases} \tag{66}$$

where the γ are first-order differential operators. When (with standard notation)

$$\begin{cases} \operatorname{Im}(\gamma_{jj}^s(\rho)) = 0 \\ |\operatorname{Re}(\gamma_{jj}^s(\rho))| < \operatorname{Tr}^+ F_j(\rho) \end{cases}, \quad \rho \in \Sigma, \quad j = 1,2, \tag{67}$$

the Cauchy problem associated with (66) is C^∞-well posed, regardless the choice of γ_{12}. In fact, one has the following a-priori inequality

$$\int_0^{+\infty} e^{2\tau x_0} \left[\|u_1\|_{s+1/2}^2 + \|u_2\|_{s+1}^2 \right] dx_0 \le C \int_0^{+\infty} e^{2\tau x_0} \left[\|f_1\|_s^2 + \|f_2\|_{s+1/2}^2 \right] dx_0, \tag{68}$$

from which one concludes as usual.

Unfortunately, we are not yet able to cast a "triangularity" condition on $a_1^s(\rho)$ into an invariant framework.

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