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TRIANGLE FUCHSIAN DIFFERENTIAL EQUATIONS WITH APPARENT SINGULARITIES

HIRONORI SHIGA, TORU TSUTSUI and JÜRGEN WOLFART
with an appendix by PAULA B. COHEN

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0. Introduction

In this article we study the Fuchsian differential equation of order 2 with monodromy group a triangle group and having three regular non-apparent singularities and several apparent singularities. To each such differential equation there corresponds a differential operator $L \in \mathbf{C}(z)[d/dz]$. We will give an integral representation of its solution and discuss the algebraicity of the value of the Schwarz map $D(z)$ for $z \in \overline{\mathbf{Q}}$ in the case $L \in \overline{\mathbf{Q}}(z)[d/dz]$.

In Sections 1 and 2 we reconstruct a classical argument of Ritter [14] and thereby obtain a differential equation $L(t)u = 0$ of the shape (1.3) when we have three regular singularities $z = 0, 1, \infty$ and n apparent singularities at $z = t_1, \dots, t_n$, with the Riemann scheme

$$\begin{pmatrix} 0 & 1 & t_1 & \cdots & t_n & \infty \\ 0 & 0 & 0 & \cdots & 0 & \mu' \\ \nu_0 & \nu_1 & 2 & \cdots & 2 & \mu'' \end{pmatrix}. \tag{0.1}$$

In general we get 2^n differential equations $L(t)u = 0$ belonging to the same Riemann scheme. This is deduced from the non-logarithmicity condition for (1.3).

In each differential equation the solution $f(z)$ that is holomorphic at $z = 0$ with $f(0) = 1$ is expressed as a linear combination of Gauss hypergeometric functions, as follows

$$c_0(t)F(\mu', \mu'', 1 - n - \nu_0; z) + \cdots + c_n(t)F(\mu', \mu'', 1 - \nu_0; z), \quad \sum_{i=0}^{n} c_i(t) = 1$$

where the coefficients $c_i(t)$'s are 2^n-valued analytic functions of $t = (t_1, \dots, t_n)$.

In Section 3 we restrict our study to the case where $\nu_0, \nu_1, \mu', \mu'' \in \mathbf{Q}$ with some non-integral condition given by (3.1). We assume $L(t) \in \overline{\mathbf{Q}}(z)[d/dz]$, in particular the apparent singularities $t_1, \dots, t_n$ are algebraic numbers. The above solution $f(z)$ can be regarded as a period $\int_{\gamma_1} \eta(t, z)$ of an abelian differential of the second kind on the hypergeometric curve $X(k,z)$ along a 1-cycle γ_1, where

$$X(k,z)\colon\ y^k = u^{k(a-c)}(u-1)^{k(c+b-1)}(u-z)^{ka}$$

we changed the signatures of the exponents with $(a,b,c) = (\mu', \mu'', 1-n-\nu_0)$ and k is the least common denominator of a, b, c. We note here that the differential $\eta(t,z)$ depends on t (and z), but the curve $X(k,z)$ and the 1-cycle $\gamma_1 = \gamma_1(z)$ do not depend on t.

Putting $\zeta = e^{2\pi i/k}$ the first homology group $H_1(X(k,z),\mathbf{Z})$ can be regarded as a $\mathbf{Z}[\zeta]$-module. By using another generator cycle γ_2 not belonging to $\mathbf{Z}[\zeta]\gamma_1$, we get the second solution $\int_{\gamma_2}\eta(t,z)$ of $L(t)u = 0$. Our Schwarz map is therefore given by $D(z) = \int_{\gamma_1}\eta(t,z) \Big/ \int_{\gamma_2}\eta(t,z)$.

We consider the Prym variety $T(k,z)$ of the covering Riemann surface $X(k,z)$, that is the abelian variety induced from the differentials of the first kind that is not coming from an intermediate covering between $X(k,z)$ and $\mathbf{P}^1$. The dimension of $T(k,z)$ is $\varphi(k) = \sharp(\mathbf{Z}/k\mathbf{Z})^*$ and the extended endomorphism algebra $\mathrm{End}(T(k,z))\otimes\mathbf{Q}$ contains the field $\mathbf{Q}(\zeta)$.

By inspection of $\eta(t,z)$ and $T(k,z)$, we get the following:

(1) For any $z \in U = \mathbf{C} - \{0,1\}$ the cohomology group $H^1_{\mathrm{DR}}(X(k,z))$, the space of differentials of the second kind modulo exact differentials, has a natural ζ-action. We always have a 2-dimensional eigenspace for every eigenvalue $e^{2\pi i l/k}$ with $l \in (\mathbf{Z}/k\mathbf{Z})^*$.

(2) The relation between t and $c_i(t)$ $(i = 0,\ldots,n)$ is stated in Theorem 3.1 and Proposition 3.2 in an explicit way.

In Section 4 we discuss the algebraicity of the value $D(z)$ for $z \in \overline{\mathbf{Q}}$ in the same situation as in Section 3, with special interest for the case $z = t_j$ $(j = 1,\ldots,n)$. If $D(z) \in \overline{\mathbf{Q}}$, then $T(k,z)$ is of CM type, namely it is isogenous to a product of simple abelian varieties with complex multiplication. By inspection of the monodromy group we obtain the following detailed result.

In the case that the monodromy group is finite, namely the solution is an algebraic function, we always have $D(z) \in \overline{\mathbf{Q}}$. Moreover $T(k,z)$ does not depend on z and it is a fixed abelian variety of CM type.

If the monodromy group is infinite, the family $\{T(k,z) : z \in U\}$ corresponds to a 1-dimensional variety in the moduli space of abelian varieties. In the case of a non-arithmetic monodromy group, the André-Oort conjecture predicts that we have only finite numbers of $T(k,z)$'s with complex multiplication in this family. Much progress on this prediction is given by Edixhoven and Yafeev [8]. Cohen and Wüstholz [6] showed how their result can be applied to various problems concerned with hypergeometric functions, filling a serious gap in the third author's paper [20]. It is therefore highly likely that we have only finitely many algebraic points z with $D(z) \in \overline{\mathbf{Q}}$.

In the case of an arithmetic monodromy group, we have infinitely many $T(k,z)$'s of CM type. For the Gauss hypergeometric function, the Schwarz image is algebraic if and only if the corresponding Prym variety $T(k, z)$ is of CM type provided the corresponding corresponding differential form is holomorphic satisfying a necessary additional condition, see [19, Corollary 5 (ii)]. This is however not the case for our differential equation $L(t)u = 0$. We have very rarely an algebraic point z with $D(z)$ in $\overline{\mathbf{Q}}$. In fact, for fixed $z = \tau \in U \cap \overline{\mathbf{Q}}$, arbitrary $n \in \mathbf{Z}_{>0}$ and arbitrary $t = (t_1, \cdots, t_n) \in \left(\overline{\mathbf{Q}}\right)^n$ we have infinetly many Schwarz images $\{D_{t,j}(\tau)\colon n \in \mathbf{Z}_{>0},\ t \in \left(\overline{\mathbf{Q}}\right)^n,\ j = 1, \ldots, 2^n\}$. There are at most two algebraic values among them. This is the main result in this section.

The full statement is given in Theorem 4.1 and Theorem 4.2. For the proofs of this section, we need applications of Wüstholz' Analytic Subgroup Theorem [24]. As an Appendix, we include a proof, kindly provided to us by Paula B. Cohen, of the relevant linear independence result, originally announced in [23], for period integrals of the second kind.

For a transcendental value $D(z)$, a method of Hirata-Kohno [10] gives even its transcendence measure. If we wish to extend our study to the case where we have regular singularities at more than three points (namely with non-triangle monodromy group), we encounter a problem stated in the work of D.V. Chudnovsky and G.V. Chudnovsky [2]: in which cases do we have a monodromy group definable over $\overline{\mathbf{Q}}$? In our triangle case the monodromy group is always realized in $GL\left(2, \overline{\mathbf{Q}}\right)$. This is a subject outside the immediate context of our present work.

1. Preliminaries about accessory parameters

We consider a Fuchsian differential equation

$$\left\{\frac{d^2}{dz^2} + p(z)\frac{d}{dz} + q(z)\right\} u(z) = 0 \quad (p(z), q(z) \in \mathbf{C}(z))$$

of order 2 with $n+3$ regular singularities $t_1, \ldots, t_{n+3}$. We have at most a simple pole for $p(z)$ and a double pole for $q(z)$ at $z = t_i$ $(i = 1, \ldots, n+3)$. The characteristic equation at $z = t_i$ is given by

$$f(t) = t(t-1) + p_i t + q_i = 0$$

with $p_i = \lim_{z\to t_i}(z - t_i)p(z)$, $q_i = \lim_{z\to t_i}(z - t_i)^2 q(z)$. In the case $t_i = \infty$, we use $p_i = 2 - \lim_{z\to 0} p(1/z)/z$, $q_i = \lim_{z\to 0} q(1/z)/z^2$ for the coefficients. Two roots of the characteristic equation give the exponents e_i, e_i' at $z = t_i$. The sum of all exponents satisfies Fuchs' relation

$$\sum_{i=1}^{n+3}(e_i + e_i' - 1) = -2. \tag{1.1}$$

The table of exponents is called the Riemann scheme. If $n = 0$ the Riemann scheme determines a Fuchsian differential equation in a unique way. However in the case where we have $n > 0$ there appear several *accessory parameters* in the corresponding differential equation, so the differential equation is not uniquely determined by the Riemann scheme. We shall restrict our study to the case of simple apparent singularities for $z = t_1, \dots, t_n$ and ordinary singularities at $z = 0, 1, \infty$. In other words, we assume that we have the Riemann scheme of the form

$$\begin{pmatrix} 0 & 1 & t_1 & \cdots & t_n & \infty \\ 0 & 0 & 0 & \cdots & 0 & \mu' \\ \nu_0 & \nu_1 & 2 & \cdots & 2 & \mu'' \end{pmatrix}. \tag{1.2}$$

Corresponding to this Riemann scheme we have the Fuchsian differential equation $Lu = 0$ with

$$L = \frac{d^2}{dz^2} + \left(\frac{1-\nu_0}{z} + \frac{1-\nu_1}{z-1} - \sum_{i=1}^{n} \frac{1}{z-t_i} \right) \frac{d}{dz} + \frac{1}{z(z-1)} \left(\mu'\mu'' + \sum_{i=1}^{n} \frac{A_i}{z-t_i} \right), \tag{1.3}$$

where $A_1, \dots, A_n$ are the accessory parameters.

In general we have a logarithmic singularity at $z = t_j$, so we have to check a non-logarithmicity condition for these accessory parameters.

For a Fuchsian differential equation of order 2, in a neighborhood of a regular singularity $z = t_j$, we can rewrite it in the form

$$\begin{cases} \left\{ (z-t_j)^2 \dfrac{d^2}{dz^2} + (z-t_j)p^*(z)\dfrac{d}{dz} + q^*(z) \right\} u(z) = 0 \\ p^*(z) = p_0 + p_1(z-t_j) + p_2(z-t_j)^2 + \cdots \\ q^*(z) = q_0 + q_1(z-t_j) + q_2(z-t_j)^2 + \cdots . \end{cases}$$

Consider a solution of the form

$$u(z) = (z-t_j)^s \left(c_0 + c_1(z-t_j) + c_2(z-t_j)^2 + \cdots \right)$$

with an exponent s at $z = t_j$. If s is not a positive integer, we obtain a formal solution $u(z) = (z-t_j)^s \left(1 + \sum_{m \geq 1} c_m (z-t_j)^m \right)$ at $z = t_j$ by the recurrence relation

$$f(m-s)c_m + R_m = 0$$

with

$$R_m = \sum_{i+l=m,\ i<m} (ic_i p_l + c_i q_l).$$

If we have the exponent $s = 2$ the above recurrence condition becomes

$$m(m-2)c_m + R_m = 0.$$

When we have $R_2 \neq 0$ our recurrence procedure stops at $m = 2$. In this case we have a logarithmic singularity. Therefore we have a non-logarithmic singularity, that is an apparent singularity, if and only if $R_2 = c_0 q_2 + c_1 p_1 + c_1 q_1 = 0$. We may assume $c_0 = 1$, therefore $c_1 = R_1 = c_0 q_1 = q_1$, and we have $R_2 = q_2 + q_1^2 + p_1 q_1 = 0$. For our equation $Lu = 0$ with L in (1.3) we have

$$\begin{aligned} p^*(z) &= -1 + \left(\frac{1-\nu_0}{t_j} + \frac{1-\nu_1}{t_j - 1} - \sum_{i \neq j} \frac{1}{t_j - t_i} \right) (z - t_j) + \cdots \\ q^*(z) &= \frac{A_j}{t_j(t_j-1)} (z - t_j) \\ &\quad + \frac{1}{t_j(t_j-1)} \left\{ -A_j \left(\frac{1}{t_j} + \frac{1}{t_j - 1} \right) + \left(\mu' \mu'' + \sum_{i \neq j} \frac{A_i}{t_j - t_i} \right) \right\} (z - t_j)^2 + \cdots , \end{aligned}$$

for every t_j $(j = 1, \ldots, n)$. We obtain the following result:

Lemma 1.1. *The non-logarithmicity condition at $z = t_j$ for $Lu = 0$ in* (1.3) *is given by*

(1.4)
$$A_j^2 - \left(\nu_0(t_j - 1) + \nu_1 t_j + \sum_{i \neq j} \frac{t_j(t_j-1)}{t_j - t_i} \right) A_j + t_j(t_j - 1) \left(\mu' \mu'' + \sum_{i \neq j} \frac{A_i}{t_j - t_i} \right) = 0.$$

We also have:

Lemma 1.2. *The system of quadratic equations*

$$\begin{cases} X_1^2 + c_{12} X_2 + c_{13} X_3 + \cdots + c_{1n} X_n = \alpha_1 \\ c_{21} X_1 + X_2^2 + c_{23} X_3 + \cdots + c_{2n} X_n = \alpha_2 \\ \qquad \cdots \\ c_{n1} X_1 + c_{n2} X_2 + \cdots + c_{n,n-1} X_{n-1} + X_n^2 = \alpha_n \end{cases}$$

(*with all* $c_{ij} \neq 0$) *has* 2^n *solutions* $(X_1, \ldots, X_n) \in \mathbf{C}^n$ *counting multiplicities.*

Proof. Let V be the intersection of the above n quadratic hypersurfaces in the compactification $\mathbf{P}^n$ of the $(X_1, \ldots, X_n)$-space $\mathbf{C}^n$. It is apparent that V has no intersection with the hyperplane at infinity. As a consequence, V is a 0-dimensional algebraic set. Bezout's theorem therefore gives the required number of solutions as the degree of the intersection of n quadratic hypersurfaces. $\square$

According to this Lemma we obtain the following:

Proposition 1.1. *For any pairwise different apparent singularities* $t_1, \dots, t_n \in \mathbf{P}^1 - \{0, 1, \infty\}$ *we have* 2^n *possibilities for the* n*-tuples of accessory parameters* $(A_1, \dots, A_n)$.

REMARK 1.1. We can treat the case of general exponent differences at t_i possibly exceeding 2 as some confluent case of our equation. If we have exponents 0 and l at $z = t_i$, we get the differential equation putting $(l-1)/(z-t_i)$ instead of $1/(z-t_i)$ in (1.3). For the case $t = 0, 1$ we get the equation by putting $\nu_0 + 1$, $\nu_1 + 1$ instead of ν_0, ν_1, respectively, with the A_i's being given as the limit values determined by (1.4). The case $t = \infty$ can be reduced to the case $t = 0$ by the inversion $t \to 1/t$.

2. Isomonodromy properties and reduction procedure

We consider first the differential equation $L(u(z)) = 0$ with the differential operator (1.3) in the special case $n = 1$, i.e.

$$L(t, A) = L = \frac{d^2}{dz^2} + \left(\frac{1-\nu_0}{z} + \frac{1-\nu_1}{z-1} - \frac{1}{z-t}\right)\frac{d}{dz} + \left(\mu'\mu'' + \frac{A}{z-t}\right)\frac{1}{z(z-1)}, \tag{2.1}$$

satisfying the non-logarithmicity condition of Lemma 1.1 for $A = A_1$, $t = t_1$. It corresponds to the Riemann scheme

$$\begin{pmatrix} 0 & \infty & 1 & t \\ 0 & \mu' & 0 & 0 \\ \nu_0 & \mu'' & \nu_1 & 2 \end{pmatrix}. \tag{2.2}$$

By Lemma 1.1, the accessory parameter A runs over a 2 sheeted Riemann surface

$$R\colon\ A^2 - (\nu_0(t-1) + \nu_1 t)A + \mu'\mu'' t(t-1) = 0$$

over the space of t with two ramification points in general. We may therefore consider R as $\mathbf{P}^1$ with a natural projection π from R to the t space $\mathbf{P}^1$.

We sometimes write $L = L(t)$ as well, keeping in mind that for most t there are two choices for L.

We have the following classical well-known fact:

Lemma 2.1 (Local lemma). *Take* $(z_0, A_0) \in \mathbf{P}^1 - \{0, 1, \infty\} \times R$ *and suppose* $\pi(A_0) \neq z_0$. *We consider the solutions* $f_i(z, A)$ $(i = 0, 1)$ *of* $Lu = 0$ *defined in some neighbourhood of* (z_0, A_0) *with the initial conditions*

$$\left(f_0(z_0, A), \frac{\partial f_0}{\partial z}(z_0, A)\right) = (1, 0),$$
$$\left(f_1(z_0, A), \frac{\partial f_1}{\partial z}(z_0, A)\right) = (0, 1).$$

Then we can find a neighbourhood $U \times W$ *of* (z_0, A_0) *where the solutions* $f_i(z, A)$ $(i = 0, 1)$ *are holomorphic in* 2 *variables.*

We have:

Lemma 2.2 (Semi-global lemma). *Take any point* $z_0 \in \mathbf{P}^1 - \{0, 1, \infty\}$. *Then we can find a neighbourhood* U *of* z_0 *such that the solution* $f_i(z, A)$ *in the previous lemma at* z_0 *is holomorphic in* 2 *variables in* $U \times (R - \pi^{-1}(z_0))$.

Proof. Let $\varepsilon(t)$ be a neighbourhood of z_0 and let $f_i(z, A)$ be the solutions given by convergent power series in z with $t = \pi(A) \neq z_0$. They give a holomorphic function in some neighbourhood of $\{z_0\} \times R - \pi^{-1}(z_0)$. Take a simply connected open neighbourhood U of z_0 with $U \cap \{0, 1, \infty\} = \emptyset$. For every A with $t = \pi(A) \neq z_0$ we can make an analytic continuation of this series solution $f_i(z, A)$ to U as a holomorphic function of one variable. According to Hartogs' continuity theorem we have an extension of $f_i(z, A)$ holomorphic in $U \times R - \pi^{-1}(z_0)$. $\square$

DEFINITION. The above system $\{f_0(z, A; z_0), f_1(z, A; z_0)\}$ is said to be *initially conditioned* at z_0.

REMARK 2.1. As we see in the proof of the following proposition, $f_i(z, A)$ $(i = 0, 1)$ is meromorphic on $U \times R$ and has a polar divisor along $U \times \pi^{-1}(z_0)$.

In the following we will write sometimes $f_i(z, t)$ and $f_i(z, t; z_0)$ instead of $f_i(z, A)$ and $f_i(z, A; z_0)$ for $t = \pi(A)$. As functions of t these functions are multivalued but can be assumed to be locally single valued meromorphic at least outside the critical values of π.

Proposition 2.1 (Isomonodromy property). *Let* α *be an arc connecting* t_0 *and* t_1 *in the* t *space and avoiding the critical values of* π. *Let* $\overline{\alpha}$ *be one of the liftings of* α *to* R. *Let* z_0 *be a point different from* α. *Let* γ *be an arbitrary loop in the* z*-domain* $\mathbf{P}^1 - \{0, 1, \infty\}$ *with the terminal point* z_0. *Then we can find a basis* $\{\varphi(z, A), \psi(z, A)\}$ $(A \in \overline{\alpha})$ *of* $L(A)u = 0$ *in* (2.1) *so that the circuit matrix of the differential equation induced from* γ *with respect to this basis does not depend on* $A \in \overline{\alpha}$.

Proof. Let $f_i(z, A) = f_i(z, A; z_0)$ $(i = 0, 1)$ be the initially conditioned system at z_0. Let $\tilde{f}_0(z, A)$ and $\tilde{f}_1(z, A)$ be the result of the continuation along γ. We have an expression

$$\begin{pmatrix} \tilde{f}_0(z, A) \\ \tilde{f}_1(z, A) \end{pmatrix} = \begin{pmatrix} p_{00}(A) & p_{01}(A) \\ p_{10}(A) & p_{11}(A) \end{pmatrix} \begin{pmatrix} f_0(z, A) \\ f_1(z, A) \end{pmatrix}. \tag{2.3}$$

Here the matrix (p_{ij}) is a circuit matrix of γ with respect to the basis $\{f_0(z, A), f_1(z, A)\}$. Because $f_i(z, A)$ is holomorphic on $U \times (R - \pi^{-1}(z_0))$, $p_{ij}(A)$ is holomorphic on $R - \pi^{-1}(z_0)$. If $\pi(A) = z_0$, the function $f_i(z, A; z_0)$ is not defined. We must show that the matrix $(p_{ij}(A))$ in (2.3) is holomorphic at $A \in \pi^{-1}(z_0)$ also. The problem is local, so we consider a product neighbourhood $U \times W$ of $(z, t) = (z_0, z_0)$. We take another point z_1 in U and make $g_i(z, t) = f_i(z, A; z_1) = f_i(z, t; z_1)$ by the same procedure. It is holomorphic on $U \times (W - \{z_1\})$. We have a relation

$$\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} c_{00}(t) & c_{01}(t) \\ c_{10}(t) & c_{11}(t) \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \end{pmatrix},$$

where $c_{ij}(t)$ is holomorphic on $W - \{z_0, z_1\}$. Put

$$\begin{pmatrix} g_0'(z, t) \\ g_1'(z, t) \end{pmatrix} = \begin{pmatrix} (g_1)_z(z_0, t) & -(g_0)_z(z_0, t) \\ -g_1(z_0, t) & g_0(z_0, t) \end{pmatrix} \begin{pmatrix} g_0(z, t) \\ g_1(z, t) \end{pmatrix},$$

and

$$w(t) = \det \begin{bmatrix} g_0(z_0, t) & g_1(z_0, t) \\ (g_0)_z(z_0, t) & (g_1)_z(z_0, t) \end{bmatrix}.$$

Here we see that $w(t)$, the Wronskian at $z = z_0$, is holomorphic and non-zero on $0 < |t - z_0| < \delta$ for some small $\delta > 0$. But it has a zero at $t = z_0$, because $z = z_0$ $(= t)$ is a singularity of the equation $L(t)u = 0$. So $f_i(z, t) = (1/w(t))g_i'(z, t)$ has a polar divisor along $t = z_0$ as mentioned in the above remark.

Then we have again the same circuit matrix for γ with respect to the system $\{g_0', g_1'\}$:

$$\begin{pmatrix} \tilde{g}_0'(z, t) \\ \tilde{g}_1'(z, t) \end{pmatrix} = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} \begin{pmatrix} g_0'(z, t) \\ g_1'(z, t) \end{pmatrix}. \tag{2.4}$$

Because $g_i'(z, t)$ is holomorphic in some neighbourhood of (z_0, z_0), (2.4) means that $p_{ij}(t)$ is holomorphic at $t = z_0$. So it is holomorphic on the whole compact Riemann surface R, hence the circuit matrix (2.3) is a constant matrix. □

Theorem 2.1 (Ritter [14]). *We consider the differential equation $L(t)u = 0$ for the operator* (2.1). *Let α be an arc connecting $t = 0$ and $t = 1$. Let $z = z_0$ be a point different from α. Let $f_i(z, t)$ be the initially conditioned system at z_0.*

Take an arbitrary point $t \in \alpha$, and let $\varphi(z, t)$ be an arbitrary solution of $L(t)u = 0$. We put $\varphi(z, t) = c_0 f_0(z, t) + c_1 f_1(z, t)$. Set $\varphi_i(z) = \varphi(z, i) = c_0 f_0(z, i) + c_1 f_1(z, i)$ $(i = 0, 1)$. Then $\varphi(z, t)$ is expressed as a **C***-linear combination $x_0(t)\varphi_0 + x_1(t)\varphi_1$. Here, $f_j(z, 0)$ and $f_j(z, 1)$ $(j = 0, 1)$ are solutions of the Gauss hypergeomet-*

ric differential equations corresponding to the Riemann schemes

$$\begin{pmatrix} 0 & \infty & 1 \\ 0 & \mu' & 0 \\ \nu_0+1 & \mu'' & \nu_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \infty & 1 \\ 0 & \mu' & 0 \\ \nu_0 & \mu'' & \nu_1+1 \end{pmatrix},$$

respectively.

Proof. Note that all the functions above are defined on some neighbourhood $U \times W$ of $\{z_0\} \times \alpha$. The matrix

$$\begin{pmatrix} \varphi_0 & \varphi & \varphi_1 \\ f_{00} & f_0 & f_{01} \\ f_{10} & f_1 & f_{11} \end{pmatrix}$$

is of rank two for any point $(z,t) \in U \times W$, where we put $f_{ij} = f_i(z,j)$, $0 \le i, j \le 1$. It induces the relation

$$\varphi(z,t) = \frac{p_0(z,t)}{p(z,t)}\varphi_0(z,t) + \frac{p_1(z,t)}{p(z,t)}\varphi_1(z,t),$$

with

$$p(z,t) = \det\begin{bmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{bmatrix}, \quad p_0(z,t) = \det\begin{bmatrix} f_0 & f_{01} \\ f_1 & f_{11} \end{bmatrix}, \quad p_1(z,t) = \det\begin{bmatrix} f_{00} & f_0 \\ f_{10} & f_1 \end{bmatrix}.$$

Let us fix t for the moment. According to Proposition 2.1, these three minors of

$$\begin{pmatrix} f_{00} & f_0 & f_{01} \\ f_{10} & f_1 & f_{11} \end{pmatrix}$$

behave in the same manner as multivalued functions. The function $p_0(z,t)$ has the shape $z^{\nu_0}(z-1)^{\nu_1}H(z)$ with a holomorphic function $H(z)$ on $\mathbf{C}$. The situation is the same for $p(z,t)$ and $p_1(z,t)$. The coefficients $x_0 = p_0(z,t)/p(z,t)$ and $x_1 = p_1(z,t)/p(z,t)$ are single valued and holomorphic on the affine space of z. By Fuchs' relation $\nu_0 + \mu' + \mu'' + \nu_1 = 0$ they are holomorphic at $z = \infty$. Therefore these three minors are holomorphic on the whole plane $\mathbf{P}^1$. They must be constants in z, and hence $x_0 = x_0(t)$ and $x_1 = x_1(t)$ depend only on the variable t. □

Now we return to the Riemann scheme

$$\begin{pmatrix} 0 & \infty & 1 & t_1 & \cdots & t_n \\ 0 & \mu' & 0 & 0 & \cdots & 0 \\ \nu_0 & \mu'' & \nu_1 & 2 & \cdots & 2 \end{pmatrix} \tag{2.5}$$

with the Fuchs relation

$$\nu_0 + \mu' + \mu'' + \nu_1 = 1 - n.$$

We consider the corresponding Fuchsian differential operator (1.3) and the differential equation $Lu = 0$ having no logarithmic singularity. When $t = (t_1, \ldots, t_n)$ move on $(\mathbf{P}^1)^n$, the accessory parameters $A = (A_1, \ldots, A_n)$ constitute a compact covering variety R over the compactified space $(\mathbf{P}^1)^n$.

For the moment we fix the exponents and $n - 1$ apparent singularities $t_2, \ldots, t_n$. A solution f therefore becomes a function of z and t_1. According to Remark 1.1 we may allow t_i to be equal to $0, 1, \infty$ and also allow them to coincide.

Let α be an arc in the t_1 space connecting 0 and 1 and passing through a point $t_1 = \rho$. Let $z = z_0$ be a point different from α. By the same procedure as in Lemma 2.2 and Proposition 2.1 we find a system of solutions $\{f_i(z, A_1; z_0)\}$ $(i = 0, 1)$ holomorphic on $U \times R(t_1)$ with some neighbourhood U of z_0 where $R(t_1)$ denotes the one dimensional covering variety over t_1 space obtained as a connected component of the restriction of R. In the same way as shown in Proposition 2.1 this system has a constant circuit matrix for a fixed loop γ in the z space $\mathbf{C} - \{0, 1, t_1, t_2, \ldots, t_n\}$. We call it the initially conditioned system at z_0 with respect to t_1.

Proposition 2.2 (Reduction procedure). *We consider the situation stated above. Let $\overline{\alpha}$ be an arbitrary lifting of α to $R(t_1)$ and let $A_\rho = \pi^{-1}(\rho) \cap \overline{\alpha}$. Let $\{f_i(z, t_1)\} = \{f_i(z, A_1; z_0)\}$ $(i = 0, 1)$, $A_1 \in \overline{\alpha}$ be the initially conditioned system. Let $\varphi(z, \rho) = c_0 f_0(z, \rho) + c_1 f_1(z, \rho)$ be an arbitrary solution for $L(\rho)u = L(\rho, t_2, \ldots, t_n)u = 0$. Put $\varphi_i(z) = c_0 f_0(z, i) + c_1 f_1(z, i)$ $(i = 0, 1)$. Then $\varphi(z, \rho)$ is expressed as a linear combination*

$$\varphi(z, \rho) = x_0(\rho)\varphi_0(z) + x_1(\rho)\varphi_1(z).$$

Where φ_0 and φ_1 are solutions of the differential equations for the Riemann schemes

$$\begin{pmatrix} 0 & \infty & 1 & t_2 & \cdots & t_n \\ 0 & \mu' & 0 & 0 & \cdots & 0 \\ \nu_0 + 1 & \mu'' & \nu_1 & 2 & \cdots & 2 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & \infty & 1 & t_2 & \cdots & t_n \\ 0 & \mu' & 0 & 0 & \cdots & 0 \\ \nu_0 & \mu'' & \nu_1 + 1 & 2 & \cdots & 2 \end{pmatrix},$$

respectively.

Successive application of this reduction procedure gives

Theorem 2.2 (Linear dependence). *Let $f(z, t_1, \ldots, t_n)$ be a solution of $Lu = 0$ with L in* (1.3) *satisfying the nonlogarithmicity condition in* Lemma 1.1. *Suppose it is holomorphic at $z = 0$. Then we have an expression*

$$f(z,t) = x_0(t)F(a,b,c;z) + x_1(t)F(a,b,c+1;z) + \cdots + x_n(t)F(a,b,c+n;z) \tag{2.6}$$

in terms of Gauss hypergeometric functions where $(a,b,c) = (\mu', \mu'', 1-n-\nu_0)$.

3. Integral representation and algebro-geometric aspects

3.1. Associate hypergeometric functions. Throughout Sections 3 and 4 we will assume that the angular parameters in the non-apparent singularities satisfy the condition

$$\nu_0, \nu_1, \nu_\infty := \mu'' - \mu' \in \mathbf{Q} - \mathbf{Z}, \;\; \nu_0 \pm \nu_1 \pm \nu_\infty \notin \mathbf{Z}. \tag{3.1}$$

First we concentrate our attention on the equation $Lu = 0$ having *one* apparent singularity with L of (2.1). According to the Fuchs relation we may put the Riemann scheme in the form

$$\begin{pmatrix} 0 & 1 & \infty & t \\ 0 & 0 & \mu' & 0 \\ \nu_0 & \nu_1 & \mu'' & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \infty & t \\ 0 & 0 & -\dfrac{1}{2}(\nu_0+\nu_1+\nu_\infty) & 0 \\ \nu_0 & \nu_1 & -\dfrac{1}{2}(\nu_0+\nu_1-\nu_\infty) & 2 \end{pmatrix}. \tag{3.2}$$

Now let us consider the integral representation of the solution for our differential equation. Recall that two Gauss' hypergeometric functions $F(a,b,c;z)$, $F(a',b',c';z)$ are said to be *associate* if

$$a \equiv a', \;\; b \equiv b', \;\; c \equiv c' \mod \mathbf{Z}$$

or equivalently, if the respective angular parameters satisfy

$$\nu_0 \equiv \nu_0', \;\; \nu_1 \equiv \nu_1', \;\; \nu_\infty \equiv \nu_\infty' \mod \mathbf{Z} \text{ and } \nu_0+\nu_1+\nu_\infty \equiv \nu_0'+\nu_1'+\nu_\infty' \mod 2\mathbf{Z}.$$

For a fixed hypergeometric function $F(a,b,c;z)$ the functions in the set of all associate functions generate a 2-dimensional vector space over the field of rational functions $\mathbf{C}(z)$, and the operator d/dz acts on it as a linear transformation. In our case all the parameters a, b, c are in $\mathbf{Q}$. Therefore the vector space can be considered over the field $\mathbf{Q}(z)$ as well.

Recall the integral representation of the hypergeometric function:

$$F(a,b,c;z) = \frac{1}{B(b,c-b)} \int_1^\infty u^{a-c}(u-1)^{c-b-1}(u-z)^{-a}\,du$$

$$= \frac{\rho}{B(b, c-b)} \int_\gamma u^{a-c}(u-1)^{c-b-1}(u-z)^{-a}\, du$$

for the Pochhammer cycle γ going around 1, ∞ and some constant ρ in a certain cyclotomic field. Take two associate hypergeometric Riemann schemes

$$R_1 : \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & -\frac{1}{2}(\nu_0+\nu_1+\nu_\infty) \\ \nu_0+1 & \nu_1 & -\frac{1}{2}(\nu_0+\nu_1-\nu_\infty) \end{pmatrix},$$

$$R_2 : \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & -\frac{1}{2}(\nu_0+\nu_1+\nu_\infty) \\ \nu_0 & \nu_1+1 & -\frac{1}{2}(\nu_0+\nu_1-\nu_\infty) \end{pmatrix},$$

and define

$$\mu_0 = \frac{1}{2}(1-(\nu_0+1)+\nu_1-\nu_\infty),$$
$$\mu_1 = \frac{1}{2}(1+(\nu_0+1)-\nu_1-\nu_\infty),$$
$$\mu_z = \frac{1}{2}(1-(\nu_0+1)-\nu_1+\nu_\infty),$$
$$\mu_\infty = \frac{1}{2}(1+(\nu_0+1)+\nu_1+\nu_\infty).$$

For the hypergeometric differential equations corresponding to these Riemann schemes we have the integral representation of the solutions

$$\int_\gamma \eta, \quad \int_\gamma \eta'$$

with

$$\eta = \eta(z) = \frac{du}{y}, \quad \eta' = \eta'(z) = \frac{u-1}{u}\eta,$$

respectively. Here η and η' are differentials of the first or the second kind on the hypergeometric curve $X(k, z)$, a projective nonsingular model of

$$y^k = u^{k\mu_0}(u-1)^{k\mu_1}(u-z)^{k\mu_z},$$

where k denotes the least common denominator of μ_0, μ_1, μ_z. The multiplicative group $\langle\zeta_k\rangle$ acts on $X(k, z)$ by

$$(y, u) \mapsto (\zeta_k^{-1} y, u).$$

The homology group $H_1(X(k,z),\mathbf{Z})$ is a $\mathbf{Z}[\zeta_k]$ module. Let $H^1_{\mathrm{DR}}(X(k,z))$ be the $\mathbf{C}$-vector space of all differentials of the second kind on $X(k,z)$ modulo exact differentials. It splits into eigenspaces via the action of ζ_k. For each eigenvalue of primitive k-th roots of unity, the corresponding eigen space is always 2-dimensional, see e.g. [21]. For every $z \notin \mathbf{C} - \{0,1\}$, set $V_{\mathbf{C}}(z) = \mathbf{C}\eta \oplus \mathbf{C}\eta'$. It is the eigenspace for ζ_k, and it contains all differentials belonging to the hypergeometric functions associate to $\int_\gamma \eta(z)$. (Often we will omit the parameter z when it is clear that we consider the equations depending on z.)

Let γ_1, γ_2 be $\mathbf{Z}[\zeta_k]$-linearly independent in $H_1(X(k,z),\mathbf{Z})$. According to Theorem 2.2 we obtain a basis of solutions of the differential equation corresponding to the Riemann scheme (3.2) by the periods $\int_{\gamma_1}\eta_1$, $\int_{\gamma_2}\eta_1$ with $\eta_1 = c_1'\eta(z) + c_2'\eta'(z)$. We define the developing map by

$$D(z) = \frac{\int_{\gamma_1}\eta_1}{\int_{\gamma_2}\eta_1}. \tag{3.3}$$

As an arithmetic side-remark for use in the next section we point out that under this normalization the values $D(0)$, $D(1)$, $D(\infty)$ are algebraic or ∞. This can be seen by a study of the monodromy group: arguments already used by Felix Klein [9] show that it acts on the homology group of the curve hence gives an action on the values $D(z)$ by fractional linear transformations with coefficients in the cyclotomic field $\mathbf{Q}(\zeta_k)$. Since $D(0)$, $D(1)$, $D(\infty)$ are fixed points under this group, the claim follows. Another important point is the interpretation of the functions $\int_{\gamma_i}\eta_1$ as periods on certain abelian varieties. For all proper divisors d of k there is an obvious morphism of the curve $X(k,z)$ onto $X(d,z)$ inducing an epimorphism

$$\operatorname{Jac} X(k,z) \to \operatorname{Jac} X(d,z).$$

Let $T(k,z)$ be the connected component of 0 in the intersection of all kernels of these epimorphisms, namely it is the Prym variety for the covering Riemann surface $X(k,z) \to \mathbf{P}^1$ with ramifications. Then it is known by [20], [21] that $T(k,z)$ is an abelian variety of dimension $\varphi(k)$ where φ denotes Euler's function. The abelian variety $T(k,z)$ has generalized complex multiplication by the cyclotomic field, so

$$\mathbf{Q}(\zeta_k) \subseteq \operatorname{End}_0 T(k,z) := \mathbf{Q} \otimes_{\mathbf{Z}} \operatorname{End} T(k,z).$$

We consder the action of ζ_k on the vector space $H^0(T(k,z),\Omega)$ of the differentials of the first kind. Let W_n be the eigenspace for the eigenvalue ζ_k^n of this action. We have

$$r_n = \dim W_n = -1 + \sum_j \langle \mu_j n \rangle,$$

where $\langle s\rangle$ denotes the fractional part $s-[s]$ of s, see e.g. [19] (on p. 23 use formula (4) with $N=2$). In particular, W_1 can be identified with the subspace of holomorphic differentials in $V_{\mathbf{C}}$. In the same way we tacitly identify also the differentials of the second kined in $V_{\mathbf{C}}$ with differentials in $H^1_{\mathrm{DR}}(T(k,z))$.

3.2. Description of the apparent singularity. As a first application of the integral representation, we obtain the explicit relation between the coefficients x_i and the apparent singularities t_i in Theorem 2.2. At first we consider the case with one apparent singularity t. Recall the classical relations between a, b, c, the angular parameters and the exponents

$$\begin{cases} \nu_0+1=1-c=1-\mu_0-\mu_z=\mu_1+\mu_\infty-1 \\ \nu_1=c-a-b=1-\mu_1-\mu_z=\mu_0+\mu_\infty-1 \\ \nu_\infty=b-a=\mu_z+\mu_\infty-1=1-\mu_0-\mu_1 \end{cases} . \tag{3.4}$$

Theorem 3.1. *We consider the equation* $Lu=0$ *with the Riemann scheme* (3.2). *Define*

$$g(x,z)=(c-a-b)z+c(z-1)x,$$
$$h(x,z)=\frac{(a-c)(c-b)}{c}z-c(z-1)x,$$

where x *is the ratio* x_1/x_0 *of the coefficients*

$$f(z,t)=x_0F(a,b,c;z)+x_1F(a,b,c+1;z)$$

in Theorem 2.2 *for one apparent singularity case. Then we have*

$$h(x,t)=xg(x,t). \tag{3.5}$$

By using $D(z)$ in (3.3) the apparent singularity property is expressed by the condition that $(d/dz)D$ vanishes in $z=t$, in other words that the Wronskian

$$\int_{\gamma_1}\eta_2\int_{\gamma_2}\eta_1-\int_{\gamma_1}\eta_1\int_{\gamma_2}\eta_2=0$$

where the differential

$$\eta_2(z)=\eta_2:=\frac{d}{dz}\eta_1(z)=\frac{d}{dz}(w\eta+v\eta')=\frac{\mu_z}{u-z}\eta_1(z)$$

is again a differential of the second kind lying in the same eigenspace $V_{\mathbf{C}}(z)$ as η_1. For generic z this 2-dimensional space $V_{\mathbf{C}}$ is generated by η_1 and η_2 because any two different associate hypergeometric functions generate the vector space of all associate hypergeometric functions having dimension 2 over the field $\mathbf{C}(z)$ of rational functions. This property remains true for the corresponding differentials in the $\mathbf{C}$-vector space $V_{\mathbf{C}}(z)$ if we replace z by any special number $t \in \mathbf{C} - \{0, 1\}$ with some possible exceptions corresponding to the fact that the $\mathbf{C}(z)$-coordinates in $V_{\mathbf{C}}(z)$ with respect to the basis $\eta_1(z)$, $\eta_2(z)$ may become singular for $z = t$. This happens precisely for apparent singularities: $D'(t) = 0$ as above implies that the same linear dependence of periods

$$\frac{\int_{\gamma_1} \eta_2}{\int_{\gamma_2} \eta_2} = \frac{\int_{\gamma_1} \eta_1}{\int_{\gamma_2} \eta_1}$$

is valid for η_2 as for η_1 at least if not all periods of η_1 vanish. That this is impossible may be seen by considering the global behaviour of $\int \eta_1$ as a function of z: Otherwise, 0 would be a fixed point of all monodromy substitutions. Since γ_1, γ_2 generate the homology $H_1(T(k, t), \mathbf{Q}) = \mathbf{Q} \otimes_{\mathbf{Z}} H_1(T(k, t), \mathbf{Z})$ as a vector space over $\mathbf{Q}(\zeta_k)$ and since η_1, η_2 belong to the same $\mathbf{Q}(\zeta_k)$-eigenspace, there is a constant C such that

$$\int_\gamma \eta_1 = C \int_\gamma \eta_2 \text{ for all } \gamma \in H_1(T(k, t), \mathbf{Z}).$$

Then $\eta_1 - C\eta_2$ would be a second kind differential with vanishing periods on the entire homology, hence $\eta_1 - C\eta_2 = 0 \in H_{\mathrm{DR}}(T(k, t))$. So we may characterize the apparent singularity by the fact that in this point the differentials η_1 and η_2 become linearly dependent. If we express both as $\mathbf{C}(z)$-linear combinations of a given basis, e.g. η and η', by means of Gauss' *relationes inter functiones contiguas*, this gives an algebraic relation between $z = t$ and v/w with coefficients in $\mathbf{Q}$.

Proposition 3.1. *For any $t \in \mathbf{C} - \{0, 1\}$, in the family of 2-dimensional $\mathbf{Q}(\zeta_k)$-eigenspaces $V_{\mathbf{C}}(z) \subset H^1_{\mathrm{DR}}(T(k, z))$ there are at most two 1-dimensional families of subspaces $\mathbf{C}\eta_1(z) = \mathbf{C}(w\eta(z) + v\eta'(z))$ containing for the special point $z = t$ both*

$$\eta_1 \text{ and } \eta_2 = \frac{\mu_z}{u - t}\eta_1$$

hence giving a developing map D with $D'(t) = 0$. These 1-dimensional eigenspaces have generating differentials $\eta_{11}(z)$, $\eta_{12}(z)$ which are for $z = t$ defined over a quadratic extension of $\mathbf{Q}(t)$.

We give a proof in the spirit of classical function theory. Consider a linear combination

$$f_x(z) = F(a, b, c; z) + xF(a, b, c + 1; z)$$

and its first derivative with respect to z which can be written ([9], p. 10) as

$$\begin{aligned} f_x'(z) &= \frac{ab}{c}F(a+1,b+1,c+1;z) + \frac{xab}{c+1}F(a+1,b+1,c+2;z) \\ &= \frac{b(c-1)}{(c-1-b)z}\left(F(a,b+1,c;z) - F(a,b,c-1;z)\right) \\ &\quad + \frac{bcx}{(c-b)z}\left(F(a,b+1,c+1;z) - F(a,b,c;z)\right) \end{aligned}$$

observing that $c - b \notin \mathbf{Z}$ by our assumptions on the angular parameters: recall the formulas (3.4). Rewrite the expression for $f_x'(z)$ as a $\mathbf{Q}(x,z)$-linear combination

$$\frac{1}{z(z-1)}\left(g(x,z)F(a,b,c;z) + h(x,z)F(a,b,c+1;z)\right)$$

of $F(a,b,c;z)$ and $F(a,b,c+1;z)$ using Gauss' relations ([7], p. 103, formulas (30), (41), (42)). The resulting coefficients g, h are in $\mathbf{Q}[x,z]$, linear in x and z. A straightforward but lengthy calculation gives them in explicit form as

$$\begin{aligned} g(x,z) &= (c-a-b)z + c(z-1)x, \\ h(x,z) &= \frac{(a-c)(c-b)}{c}z - c(z-1)x \end{aligned}$$

(recall that by our assumptions on the angular parameters $c = -\nu_0 \neq 0$).

Now consider analytic continuations of all functions involved here along some nontrivial loop starting and ending at z, avoiding the singularities 0, 1, ∞, and denote the resulting new branches by adding a tilde. Since 1, x, $g(x,z)$, $h(x,z)$ remain unchanged, we obtain the matrix equation

$$\begin{pmatrix} f_x(z) & \tilde{f}_x(z) \\ f_x'(z) & \tilde{f}_x'(z) \end{pmatrix} = \frac{1}{z(z-1)}\begin{pmatrix} 1 & x \\ g(x,z) & h(x,z) \end{pmatrix}\begin{pmatrix} F(a,b,c;z) & \tilde{F}(a,b,c;z) \\ F(a,b,c+1;z) & \tilde{F}(a,b,c+1;z) \end{pmatrix}.$$

For almost all loops, the first row on the left side forms the numerator and the denominator of a corresponding developing map D. As in our arguments concerning the integral representation, $D'(t) = 0$ is equivalent to the determinant condition

$$f_x'(t)\tilde{f}_x(t) - \tilde{f}_x'(t)f_x(t) = 0.$$

Since the matrix on the right side is nonsingular for almost all loops, the matrix in the middle has to be singular, hence

$$h(x,t) = xg(x,t)$$

gives an algebraic relation between x and t with coefficients in $\mathbf{Q}$, quadratic in x and linear in t, proving both Theorem and Proposition. □

3.3. More apparent singularities. Recall that by Proposition 1.1 of Section 1 there are generically 2^n different Fuchsian differential equations of second order with angular parameters

$$\nu_0, \nu_1, \nu_\infty \in \mathbf{Q} - \mathbf{Z}, \ \ \nu_0 \pm \nu_1 \pm \nu_\infty \notin \mathbf{Z}$$

in the regular singularities $0, 1, \infty$ and apparent singularities in $t_1, \ldots, t_n$ (which may be supposed to be simple if they are pairwise different). As one may expect, to these correspond 2^n basis functions given by period integrals of families of differentials in $V_{\mathbf{C}}(z) \subset H^1_{\mathrm{DR}}(X(k, z))$.

Proposition 3.2. *For a Zariski dense subset of n-tuples $(t_1, \ldots, t_n) \in \mathbf{C}^n$ the 2^n second order Fuchsian differential equations $L = 0$ with regular singularities 0, 1, ∞, angular parameters*

$$\nu_0, \nu_1, \nu_\infty \in \mathbf{Q} - \mathbf{Z}, \ \ \nu_0 \pm \nu_1 \pm \nu_\infty \notin \mathbf{Z}$$

and apparent singularities $t_1, \ldots, t_n$ are solved by linear combinations of associate hypergeometric functions

$$f(x_1, \ldots, x_n; z) := F(a, b, c; z) + \sum_{j=1}^{n} x_j F(a, b, c + j; z)$$

where $F(a, b, c; z)$ belongs to the angular parameters $\nu_0 + n$, ν_1, ν_∞. The corresponding integral representations are given by period integrals of 2^n families of differentials $\eta_{1j}(z)$, $j = 1, \ldots, 2^n$, for each z generating 2^n 1-dimensional subspaces of $V_{\mathbf{C}}$, and the apparent singularities t are characterized by the property that in the points t the differentials

$$\eta_{1j}(z) \ \textit{and} \ \frac{d}{dz}\eta_{1j}(z)$$

are multiples of each other.

The main part of this statement is only a reformulation of Proposition 1.1 and Theorem 2.2 in the language of differentials of the second kind used for the proof of Proposition 3.1. One may try to give a direct proof based on the same idea as the proof of Proposition 3.1, with the only difference that the linear combination $f(x_1, \ldots, x_n; z)$ has to be rewritten first as a linear combination of the first two contiguous functions by successive application of some Gauss relation ([7, p. 103, (30)]). The result is

$$z^{1-n} r(x_2, \ldots, x_n, z) F(a, b, c; z) + z^{1-n} s(x_1, \ldots, x_n, z) F(a, b, c + 1; z)$$

where the coefficients r, s are polynomials defined over $\mathbf{Q}$ linear in $x_1, \dots, x_n$ and of degree $\leq n-1$ in z. Differentiation with respect to z leads as before to n determinant conditions for $t = t_1, \dots, t_n$

$$f'(x_1, \dots, x_n, t)\tilde{f}(x_1, \dots, x_n, t) - \tilde{f}'(x_1, \dots, x_n, t) f(x_1, \dots, x_n, t) = 0,$$

where the derivative—with a similar application of Gauss' contiguity relations as before—can be written as

$$z^{-n}(1-z)^{-1} g(x_1, \dots, x_n, z) F(a, b, c; z) + z^{-n}(1-z)^{-1} h(x_1, \dots, x_n, z) F(a, b, c+1; z)$$

with polynomials g, h defined over $\mathbf{Q}$ linear in $x_1, \dots, x_n$ and of degree $\leq n$ in z. Finally one obtains n algebraic equations

$$s(x_1, \dots, x_n, t_j) h(x_1, \dots, x_n, t_j) = r(x_2, \dots, x_n, t_j) g(x_1, \dots, x_n, t_j), \quad j = 1, \dots, n,$$

all quadratic in $x_1, \dots, x_n$. Using the above argument we obtain the relation for the case $n = 2$ in explicit form.

Proposition 3.3. *For the solution*

$$f(x, y; z) := F(a, b, c; z) + x F(a, b, c+1; z) + y F(a, b, c+2; z),$$

of $Lf = 0$ in the case of two apparent singularities t_1, t_2, we have

$$s(x, y; t_i) h(x, y; t_i) - r(y; t_i) g(x, y; t_i) = 0 \tag{3.6}$$

with

$$\begin{aligned}
r(y; z) &= -\left(c\,(1+c)\,y\,(-1+z)\right) + (1-a+c)\,(1-b+c)\,z \\
s(x, y; z) &= (1-a+c)\,(1-b+c)\,xz - (1+c)\,y\,(c - (1-a-b+2c)\,z) \\
g(x, y; z) &= -\left(z\,(c\,(1+x\,(-1+z)) - (a+b)\,z)\right) \\
&\qquad - \frac{c\,(-1+z)\left((1+c)^2 y\,(-1+z) + (-1+a-c)\,(-1+b-c)\,z\right)}{(1-a+c)\,(1-b+c)} \\
h(x, y; z) &= -\left(z\,((1+c)\,y\,(-1+z) - (-1+a+b-c)\,z)\right) \\
&\qquad + \frac{z\left(c^2\,(1+x\,(-1+z)) + abz - (a+b)\,cz\right)}{c} \\
&\quad - \frac{(c + (-1+a+b)\,z - 2cz)\left(-\left((1+c)^2 y\,(-1+z)\right) + (-1+a-c)\,(1-b+c)\,z\right)}{(-1+a-c)\,(1-b+c)}
\end{aligned}$$

leading to the following relation between x, y, $t = t_1$, t_2:

$$
\begin{aligned}
0 = {} & \big(c^2(1+x)\,y + 2c^3(1+x)\,y + c^4(1+x)\,y\big) \\
& + \big[c^3(1+x)\,((-2+a+b)\,x - 4y) + c^4(1+x)\,(-x-2y) + abcy \\
& \quad + c^2\left((-1-a\,(-1+b)+b)\,x + (1-a)\,(-1+b)\,x^2 + (-2+ab-2x)\,y\right)\big]\,t \\
& + \big[(-1+a)\,a\,(-1+b)\,b + c^4(1+x)\,(1+x+y) \\
& \quad + c^3(1+x)\,(-2\,(-1+a+b) + (2-a-b)\,x + 2y) \\
& \quad + c^2\{1 + a^2 - 3b + b^2 + a\,(-3+4b) + \left(2 + a^2 + 4a\,(-1+b) - 4b + b^2\right)x \\
& \qquad + (-1+a)\,(-1+b)\,x^2 + (1-ab+x)\,y\} \\
& \quad + c\big(-((1-b)\,b\,(1+x)) - a^2(-1-x+b\,(2+x)) \\
& \qquad - a\left(1 + x + b^2(2+x) + b\,(-2\,(2+x)+y)\right)\big)\big]\,t^2
\end{aligned}
$$

As a consequence we know that the equation (3.6) is of degree 2 in t and gives four solutions (x, y) for any fixed pair (t_1, t_2) as predicted by Proposition 1.1 and Theorem 2.2.

4. Apparent singularities and transcendence

4.1. Schwarz maps with algebraic values at algebraic arguments. Now we describe the arithmetic implications of the other sections on Fuchsian differential equations having three singularities 0, 1, ∞ with angular parameters $\nu_0, \nu_1, \nu_\infty \in \mathbf{Q} - \mathbf{Z}$ and n apparent singularities $t_j \neq 0, 1, \infty$, $j = 1, \ldots, n$. If z is algebraic, the curve $X(k, z)$, the Jacobian and its Prym part $T(k, z)$, all differentials of the second kind η, η', used in the previous section become defined over $\overline{\mathbf{Q}}$, and then these differentials generate a 2-dimensional $\overline{\mathbf{Q}}$-vector space $V_{\overline{\mathbf{Q}}}$ with $V_{\mathbf{C}} = \mathbf{C} \otimes_{\overline{\mathbf{Q}}} V_{\overline{\mathbf{Q}}}$. If we consider apparent singularities $t_1, \ldots, t_n$ lying in $\overline{\mathbf{Q}}$, also all differentials $\eta_{1j}(z)$, $j = 1, \ldots, 2^n$ constructed in Proposition 3.2 and Theorem 2.2 lie in $V_{\overline{\mathbf{Q}}}$ for all algebraic z, in particular for the t_i themselves. Recall further that for all $z \neq 0, 1$ the cycles γ_1, γ_2 become cycles on $T(k, z)$ generating the homology $H_1(T(k, z), \mathbf{Q}) = \mathbf{Q} \otimes_{\mathbf{Z}} H_1(T(k, z), \mathbf{Z})$ as a vector space over $\mathbf{Q}(\zeta_k)$.

Proposition 4.1. *Suppose* $\tau \in \overline{\mathbf{Q}}$, $\neq 0, 1$ *and that* $T(k, \tau)$ *is a simple abelian variety with* $\mathbf{Q}(\zeta_k) = \mathrm{End}_0\, T(k, \tau)$. *Then all periods*

$$\int_\gamma \eta_1, \quad \gamma \in H_1(T(k, \tau), \mathbf{Z})$$

of a fixed nonzero $\eta_1 \in V_{\overline{\mathbf{Q}}} \subset H^1_{\mathrm{DR}}(T(k, \tau))$ *generate a* $\overline{\mathbf{Q}}$*-vector space* Π *of dimension* 2.

The action of the endomorphisms on $H_1(T(k, \tau), \mathbf{Z})$ and η_1 shows $\dim_{\overline{\mathbf{Q}}} \Pi \leq 2$.

The proof that we have in fact = 2 relies on a result going back to Wüstholz [23]. A proof worked out by Paula B. Cohen, see Theorem 6.1, is provided in the Appendix. To apply Theorem 6.1, take $N = 1$, $A = A_1$, $n_1 = \varphi(k)$, complete η_1 by other $\mathbf{Q}(\zeta_k)$-eigendifferentials $\eta_2, \ldots, \eta_{2\varphi(k)}$ to a basis of $H^1_{\mathrm{DR}}(T(k,\tau))$. Then Theorem 6.1 gives

$$\dim_{\overline{\mathbf{Q}}} \widehat{V}_A = 2 + 4\varphi(k).$$

Since the periods of every η_j generate an at most 2-dimensional $\overline{\mathbf{Q}}$-vector space, this upper bound 2 has to be attained for every η_j, in particular $\dim_{\overline{\mathbf{Q}}} \Pi = 2$. □

On the other hand, this vector space is generated by $\int_{\gamma_1} \eta_1$, $\int_{\gamma_2} \eta_1$. If their quotient $D(\tau)$ is an algebraic number, the $\overline{\mathbf{Q}}$-vector space Π generated by all periods $\int_\gamma \eta_1$ has dimension 1, therefore $T(k,\tau)$ cannot be a simple abelian variety with $\mathbf{Q}(\zeta_k) = \mathrm{End}_0\, T(k,z)$. More precisely, we can show

Proposition 4.2. *If τ and $D(\tau) = \delta \in \overline{\mathbf{Q}}$ are algebraic, the abelian variety $T(k,\tau)$ is of CM type, i.e. isogenous to a product of simple abelian varieties with complex multiplication. More precisely,*

1. *$T(k,\tau)$ is isogenous to the product of two abelian varieties A, A', both of complex dimension $(1/2)\varphi(k)$ and with endomorphism algebra $\mathbf{Q}(\zeta_k)$,*
2. *or $T(k,\tau)$ has complex multiplication by a quadratic extension K of $\mathbf{Q}(\zeta_k)$ and is isogenous to a pure power B^m of a simple abelian variety B with complex multiplication.*

Proof. An argument going back to Bertrand ([1], Section 1, Example 3) gives two possibilities for $T(k,\tau)$ (see also [19], Proposition 5 and its proof):

If there are zero-divisors of $\mathrm{End}_0(T(k,\tau))$ commuting with $\mathbf{Q}(\zeta_k)$, their kernels give proper abelian subvarieties A stable under the action of $\mathbf{Q}(\zeta_k)$. Such an A has complex dimension $< [\mathbf{Q}(\zeta_k) : \mathbf{Q}]$, hence $= (1/2)[\mathbf{Q}(\zeta_k) : \mathbf{Q}]$ by well known divisibility relations between dimensions of abelian varieties and their endomorphism algebras. Then A has complex multiplication by $\mathbf{Q}(\zeta_k)$, and its cofactor in $T(k,\tau)$ does as well.

Otherwise, the endomorphisms of $T(k,\tau)$ commuting with $\mathbf{Q}(\zeta_k)$ form a field K, either $\mathbf{Q}(\zeta_k)$ itself or a quadratic extension of it, and $T(k,\tau)$ is isogenous to a pure power B^m of a simple abelian variety. Under the isomorphism

$$H^1_{\mathrm{DR}}(T(k,\tau)) \cong (H^1_{\mathrm{DR}}(B))^m,$$

η_1 corresponds to an m-tuple $(\eta_1^1, \ldots, \eta_1^m)$ of differentials in $H^1_{\mathrm{DR}}(B)$, all defined over $\overline{\mathbf{Q}}$. The components η_1^j lie all in the same $(\mathrm{End}_0\, B)$-eigenspace, otherwise the periods of $\int_\gamma \eta_1$, $\gamma \in H_1(T(k,\tau), \mathbf{Z})$, could not lie in the 1-dimensional $\overline{\mathbf{Q}}$-vector space Π. Using this reasoning, the same Π contains all periods of all η_1^j, and by $\dim_{\overline{\mathbf{Q}}} \Pi = 1$ again, this is possible only for abelian varieties with complex multiplication. By Satz 4 of [20], this implies $[K : \mathbf{Q}(\zeta_k)] = 2$. $\square$

4.2. Implications of complex multiplication. Our hypergeometric functions are algebraic if and only if all associate functions are algebraic if and only if their monodromy groups are finite—some degenerate cases like polynomials (occurring for certain integer parameters a, b) are excluded by our assumptions on the angular parameters. Then clearly D is an algebraic function as well and the algebraic apparent singularity $t = \tau$ will have an algebraic image $D(\tau) = \delta$. We will see that this corresponds precisely to a special situation of the first case discussed in Proposition 4.2.

Proposition 4.3. *Let τ be algebraic $\neq 0, 1$. The following two conditions are equivalent.*
1. *The abelian variety $T(k, \tau)$ is isogenous to $A \oplus A'$ where both abelian varieties A, A' have complex multiplication by $\mathbf{Q}(\zeta_k)$ with the same CM type.*
2. *The monodromy group of $L = 0$ is finite.*

Proof. The first condition implies that the CM type of $T(k, \tau)$ (see the end of Subsection 3.1) satisfies

$$r_n = 0 \text{ or } 2 \text{ for all } n \in (\mathbf{Z}/k\mathbf{Z})^*,$$

and this property is valid for all $T(k, z)$, not only for $T(k, \tau)$. But then it is well known by results of Shimura (see [20], Section 7) that all $T(k, z)$ are isogenous to a square A^2 of a fixed abelian variety with complex multiplication. Therefore the monodromy group is finite. The converse direction follows in the same way by the fact that in the case of a finite monodromy group, the family of all $T(k, z)$ has complex dimension zero, see again [20], Section 7, or [5] for a more general version. $\square$

For the other cases we obtain the following result.

Proposition 4.4. *Suppose that the monodromy group of $Lu = 0$ is infinite, and let $\tau \neq 0, 1, \infty$ be an algebraic point with $D(\tau) \in \overline{\mathbf{Q}}$ such that $T(k, \tau)$ is of CM type. Then the 2-dimensional $\mathbf{Q}(\zeta_k)$-eigenspace $V_{\overline{\mathbf{Q}}} \subset H^1_{\mathrm{DR}}(T(k, \tau))$ has only two 1-dimensional subspaces whose generating differentials η_{3i}, $i = 1, 2$, satisfy*

$$\frac{\int_{\gamma_1} \eta_{3i}}{\int_{\gamma_2} \eta_{3i}} \in \overline{\mathbf{Q}}, \quad i = 1, 2.$$

Proof. 1. Suppose the claim of the Proposition is not true. Then we show first that the periods of *all* $\eta_3 \in V_{\overline{\mathbf{Q}}}$ lie in a 1-dimensional $\overline{\mathbf{Q}}$-vector space Π.

Let $\eta_{31}, \eta_{32} \in V_{\overline{\mathbf{Q}}} \subset H^1_{\mathrm{DR}}(T(k,\tau))$ be linearly independent and let $\eta_{31} + \sigma\eta_{32}$, $\sigma \in \overline{\mathbf{Q}} - \{0\}$ be a third differential all with algebraic period quotients. Writing

$$\pi_{ij} := \int_{\gamma_i} \eta_{3j}, \quad i, j = 1, 2$$

we can suppose

$$\frac{\pi_{11}}{\pi_{21}} = \delta_1, \quad \frac{\pi_{12}}{\pi_{22}} = \delta_2, \quad \frac{\pi_{11} + \sigma\pi_{12}}{\pi_{21} + \sigma\pi_{22}} = \delta_3$$

all to be algebraic. Then

$$\frac{\delta_1\pi_{21} + \sigma\delta_2\pi_{22}}{\pi_{21} + \sigma\pi_{22}} = \delta_3$$

follows and the algebraicity of π_{21}/π_{22}, hence $\dim_{\overline{\mathbf{Q}}} \Pi = 1$.

2. By Wüstholz' analytic subgroup theorem [24], applied via Theorem 6.1 in a similar way as in the proof of Proposition 4.1, and by standard facts about complex multiplication ([12], Chapter I), two differentials of the second kind η_{31}, η_{32} defined over $\overline{\mathbf{Q}}$ on two simple abelian varieties B_1, B_2 defined over $\overline{\mathbf{Q}}$ lead to the same 1-dimensional $\overline{\mathbf{Q}}$-vector space Π generated by their respective periods if and only if

- B_1 and B_2 are isogenous,
- both have complex multiplication by the same field L and with isomorphic CM types,
- both η_{31} and η_{32} are eigendifferentials for the complex representation of L on the respective spaces of differentials,
- for some isogeny $\iota\colon B_1 \to B_2$, the pullback $\eta_{32} \circ \iota$ is a $\overline{\mathbf{Q}}$-multiple of η_{31}.

Since Π does not change under isogenies, we may even assume without loss of generality that ι is an isomorphism, that B_1 and B_2 have the same CM type and that $\eta_{31} = \eta_{32} \circ \iota$.

3. Now consider the decomposition of $T(k,\tau)$ given in Proposition 4.2. In the first case considered there, the intersections of $V_{\overline{\mathbf{Q}}}$ with $H^1_{\mathrm{DR}}(A)$ and $H^1_{\mathrm{DR}}(A')$ are both 1-dimensional $\mathbf{Q}(\zeta_k)$-eigenspaces for the same eigenvalues, and by our assumption, their periods generate the same 1-dimensional $\overline{\mathbf{Q}}$-vector space Π. If A and A' are simple, we see by the above that they are isogenous of the same CM type. Therefore, Proposition 4.3 applies and gives a contradiction to the assumption about the monodromy group. If A or A' are not simple, they are isogenous to powers of simple abelian varieties B, B' with complex multiplication (see [12] or the details given below for the second case), and we can extend the same argument to see that B and B' are isogenous and Proposition 4.3 applies again.

In the second case of Proposition 4.2 recall that the simple factor B of $T(k,\tau)$ is determined as follows. Let Φ be the CM type of $T(k,\tau)$, i.e. a system of $\varphi(k) = (1/2)[K : \mathbf{Q}]$ representatives σ of the embeddings $K \to \mathbf{C}$ modulo complex conjugation, and let H be the maximal subgroup of $\mathrm{Gal}\, K/\mathbf{Q}$ leaving Φ invariant. Then B

has complex multiplication by the fixed field L of H ($[K : L] = |H| = m$), and its CM type $H\backslash\Phi$ consists of the different restrictions of the $\sigma \in \Phi$ to the subfield L. On B^m our 2-dimensional eigenspace $V_{\overline{\mathbf{Q}}}$ for the action of $\mathbf{Q}(\zeta_k)$ has a 1-dimensional $\overline{\mathbf{Q}}$-vector space Π of periods only if $\mathrm{Gal}\, K/\mathbf{Q}(\zeta_k) \subseteq H$, i.e. if it fixes the CM type of $T(k, \tau)$. But then again $T(k, \tau)$ is isogenous to some A^2 where A has complex multiplication by $\mathbf{Q}(\zeta_k)$ with CM type $(\mathrm{Gal}\, K/\mathbf{Q}(\zeta_k))\backslash\Phi$, and Proposition 4.3 gives a contradiction to our assumptions. ☐

Now we specialize τ to be an apparent singularity. Putting together the last three Propositions we can conclude

Theorem 4.1. *For a second order Fuchsian differential equation $Lu = 0$ with regular singularities* 0, 1, ∞, *angular parameters*

$$\nu_0, \nu_1, \nu_\infty \in \mathbf{Q} - \mathbf{Z}, \;\; \nu_0 \pm \nu_1 \pm \nu_\infty \notin \mathbf{Z},$$

one algebraic apparent singularity $\tau \neq 0, 1$ and with infinite monodromy group, the developing map D has algebraic values in the singularities if and only if

- *the abelian variety $T(k, \tau)$ is of CM type and*
- *on $T(k, \tau)$, a differential $\eta_{11}(\tau)$ or $\eta_{12}(\tau)$ constructed in* Proposition 3.1 *is a multiple of η_{31} or η_{32} found in* Proposition 4.4.

If the number n of apparent singularities t is > 1, we have generically 2^n different families of 1-dimensional spaces $\mathbf{C}\eta_{1j}(z)$ leading to developing maps. Together with the Proposition 4.4 we obtain

Theorem 4.2. *Let n be an integer > 1. For all second order Fuchsian differential equations $Lu = 0$ with regular singularities* 0, 1, ∞, *angular parameters*

$$\nu_0, \nu_1, \nu_\infty \in \mathbf{Q} - \mathbf{Z}, \;\; \nu_0 \pm \nu_1 \pm \nu_\infty \notin \mathbf{Z}$$

and algebraic apparent singularities $\tau_1, \ldots, \tau_n$ the developing maps D_i, $i = 1, \ldots, 2^n$, have the following properties.

- *For all τ_j the value $D_i(\tau_j)$ is algebraic only if the Prym variety $T(k, \tau_j)$ is of CM type.*
- *If moreover the monodromy group is infinite, for all components τ_j in a Zariski dense subset of algebraic n-tuples $(\tau_1, \ldots, \tau_n) \in \overline{\mathbf{Q}}^n$ the value $D_i(\tau_j)$ can be algebraic for at most two of the D_i, $i = 1, \ldots, 2^n$.*

5. Some graphics of Schwarz maps

We give some graphics of the Schwarz maps for our differential equations with one apparent singularity given by (2.1). We consider only the case with rational parameters a, b, c and a real apparent singularity t. According to [11] we determine the Schwarz map of a Gauss hypergeometric differential equation for $F(a,b,c;z)$. Set

$$F_{pq}(a,b,c;z) = F_{pq}(z) = \int_p^q \varphi(u)\,dt \quad p,q \in \{0,1,\infty,z\},\ \ p \neq q \tag{5.1}$$

$$\varphi(u) = u^{a-c}(1-u)^{c-b-1}(z-u)^{-a} \tag{5.2}$$

for a variable z with $\Im(z) > 0$. The path of the integral is a side or an extended side of the triangle $01z$ on the complex plane. So $F_{pq}(z)$ can be defined as a single valued function on the upper half plane $\mathbf{H}$.

The arguments of $\varphi(u)$ along these paths are determined according to the following table

$\cdot$	$\arg u$	$\arg(1-u)$	$\arg(z-u)$
$\overline{1\infty}$	0	$-\pi$	$\eta \to \pi$
$\overline{0z}$	ξ	$0 \to \eta - \pi$	ξ
$\overline{\infty 0}$	π	0	$0 \to \xi$
$\overline{1z}$	$0 \to \xi$	$\eta - \pi$	η
$\overline{01}$	0	0	$\xi \to \eta$
$\overline{z\infty}$	ξ	$\eta - \pi \to \xi - \pi$	$\xi + \pi$

with $0 < \xi = \arg z < \pi$, $0 < \eta = \arg(z-1) < \pi$.

Theorem 5.1 (Linear relations).

$$\begin{aligned}
&F_{01} - F_{0z} + F_{1z} = 0\\
&F_{1\infty} - F_{1z} - F_{z\infty} = 0\\
&F_{\infty 0} + F_{0z} + e^{2\pi i a} F_{z\infty} = 0\\
&F_{01} + e^{2\pi i(c-b)} F_{1\infty} + e^{2\pi i(c-a)} F_{\infty 0} = 0.
\end{aligned}$$

Theorem 5.2 (Connection formula).

$$(F_{1\infty}, F_{0z}) \begin{pmatrix} e^{2\pi i(c-b)} & e^{2\pi i a} \\ 1 & 1 \end{pmatrix} = (F_{\infty 0}, F_{1z}) \begin{pmatrix} -e^{2\pi i(c-a)} & -1 \\ 1 & e^{2\pi i a} \end{pmatrix},$$

$$(F_{1\infty}, F_{0z}) \begin{pmatrix} 1 & -e^{2\pi i(c-b)} \\ -1 & e^{2\pi i(c-a)} \end{pmatrix} = (F_{01}, F_{z\infty}) \begin{pmatrix} -1 & 1 \\ 1 & -e^{2\pi i c} \end{pmatrix}.$$

Theorem 5.3 (Expression via the Kummer solutions).

$$F_{1\infty}(z) = -e^{\pi i(-c+b-a)} \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b,c;z)$$

$$F_{0z}(z) = \frac{\Gamma(a-c+1)\Gamma(1-a)}{\Gamma(2-c)} z^{1-c} F(a-c+1, b-c+1, 2-c; z)$$
$$F_{\infty 0}(z) = e^{\pi i(a-c)} \frac{\Gamma(b)\Gamma(a-c+1)}{\Gamma(a+b-c+1)} F(a, b, a+b-c+1; 1-z)$$
$$F_{1z}(z) = -e^{-\pi i a} \frac{\Gamma(c-b)\Gamma(1-a)}{\Gamma(c-a-b+1)} (1-z)^{c-a-b} F(c-a, c-b, c-a-b+1; 1-z)$$
$$F_{01}(z) = \frac{\Gamma(a-c+1)\Gamma(c-b)}{\Gamma(a-b+1)} z^{-a} F\left(a, a-c+1, a-b+1; \frac{1}{z}\right)$$
$$F_{z\infty}(z) = -e^{\pi i(-c+b-a)} \frac{\Gamma(b)\Gamma(1-a)}{\Gamma(b-a+1)} z^{-b} F\left(b, b-c+1, b-a+1; \frac{1}{z}\right).$$

REMARK 5.1. These three theorems are stated in [11] as Theorem 4.4.1, Theorem 4.4.2 and formulas (4.4.10), (4.4.11) (there—as we believe—with typing errors; these are corrected above since we need them for the drawings below).

The Schwarz map for $F(a, b, c; z)$ is defined by

$$\Phi_0(z) = \frac{F_{0z}(a, b, c; z)}{F_{1\infty}(a, b, c; z)}$$

on $\{|z| < 1\} \cap \mathbf{H}$ and it has an analytic continuation on $\mathbf{H}$.

By using the connection formula we can get the image of the real line by the Schwarz map $\Phi_0(z)$.

In the following, the letter k has no longer the same meaning as on Sections 3 and 4. According to Theorem 2.2 the integrals

$$kF_{0z}(a, b, c; z) + (1-k)F_{0z}(a, b, c+1; z), \quad kF_{1\infty}(a, b, c; z) + (1-k)F_{1\infty}(a, b, c+1; z) \tag{5.3}$$

are the basis solutions of some differential equation for (2.1). So we define our Q-Schwarz map (Ritter's terminology [14]) by

$$\Phi(a, b, c; k, z) = \frac{kF_{0z}(a, b, c; z) + (1-k)F_{0z}(a, b, c+1; z)}{kF_{1\infty}(a, b, c; z) + (1-k)F_{1\infty}(a, b, c+1; z)}.$$

We obtain the following as a direct consequence of Theorem 3.1.

Proposition 5.1. *The apparent singularity t for* (2.1) *is given as a function of k*:

$$t = \frac{(k-1)(b-c+(2c-b)k)}{(2k-1)(b-c+(2c-a-b)k)}.$$

We show the images of the real line of $\Phi(1/8, 3/8, 3/4; k, z)$ and $\Phi(1/8, 3/8, -1/4; k, z)$ by plotting the discrete values of z for several k's in the interval $0 \leq k \leq 1$. In many cases the coordinates frame is pressed down, the horizontal and the vertical line segments are always the intervals $[-1, 1]$ and $[-1-i, -1]$. In these cases we have the angular parameters $(1/4, 1/4, 1/4)$. We used "Mathematica" to generate the figures.

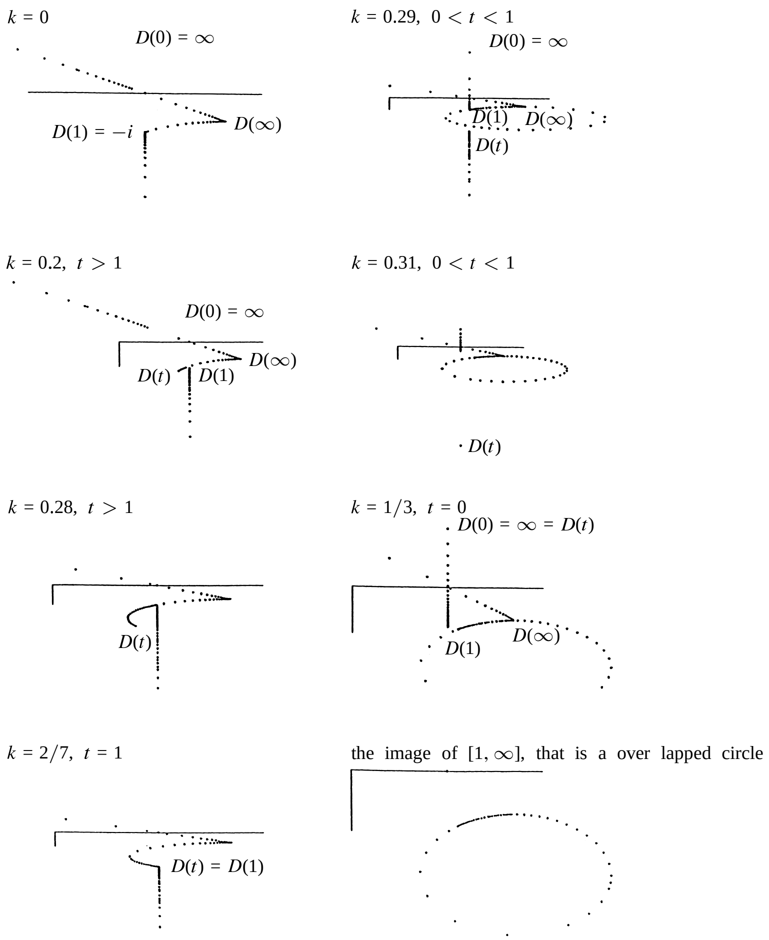
Animation of the Q-Schwarz map Φ for the combination of hypergeometric functions

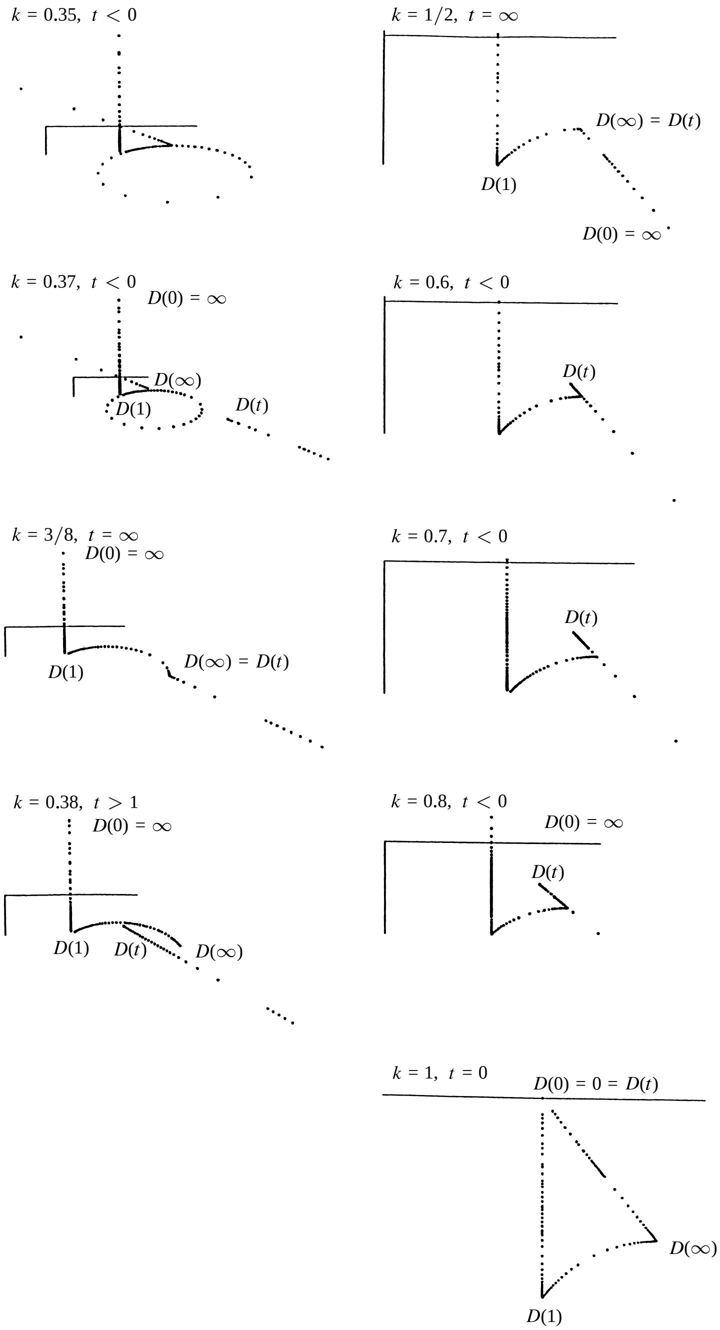
$$kF\left(\frac{1}{8}, \frac{3}{8}, \frac{3}{4}\right) + (1-k)F\left(\frac{1}{8}, \frac{3}{8}, 1+\frac{3}{4}\right).$$

The apparent singularity t is given by the function of k:

$$\frac{3(k-1)(3k-1)}{(2k-1)(8k-3)}.$$

The graphics are the images of the real line corresponding to several values of k indicated, and the images of singular points are occasionally indicated by hand writings.




$k = 0.35,\ t < 0$
$k = 1/2,\ t = \infty$
$D(\infty) = D(t)$
$D(1)$
$D(0) = \infty$
$k = 0.37,\ t < 0$
$D(0) = \infty$
$D(\infty)$
$D(1)$
$D(t)$
$k = 0.6,\ t < 0$
$D(t)$
$k = 3/8,\ t = \infty$
$D(0) = \infty$
$D(1)$
$D(\infty) = D(t)$
$k = 0.7,\ t < 0$
$D(t)$
$k = 0.38,\ t > 1$
$D(0) = \infty$
$D(1)$
$D(t)$
$D(\infty)$
$k = 0.8,\ t < 0$
$D(0) = \infty$
$D(t)$
$k = 1,\ t = 0$
$D(0) = 0 = D(t)$
$D(\infty)$
$D(1)$

Animation of the Q-Schwarz map for

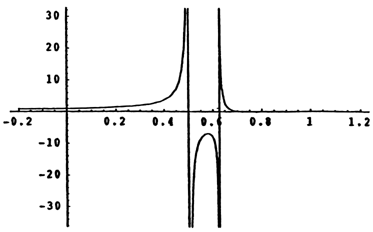
$$kF\left(\frac{1}{8}, \frac{3}{8}, \frac{-1}{4}\right) + (1-k)F\left(\frac{1}{8}, \frac{3}{8}, 1 + \frac{-1}{4}\right).$$

The apparent singularity t is given by the function of k:

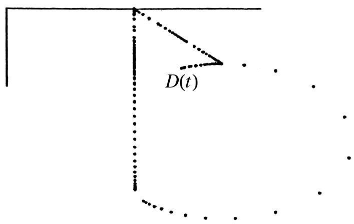
$$\frac{(k-1)(7k-5)}{(2k-1)(8k-5)},$$

its graphic is given below. These graphics are the continuation of the previous ones connected at $F(1/8, 3/8, 3/4)$.

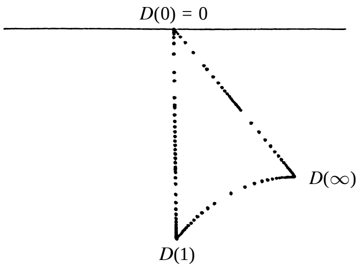
Graphic of the t as a function of k:



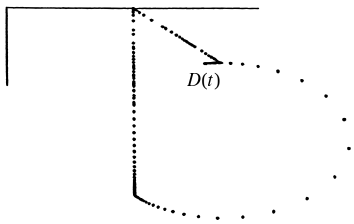
$k = 0.2$



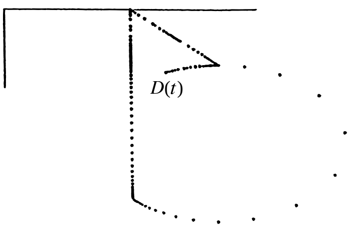
$k = 0$



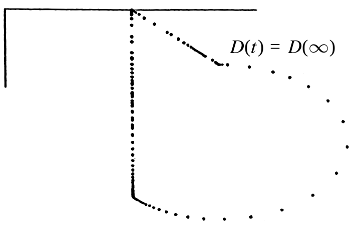
$k = 0.4$

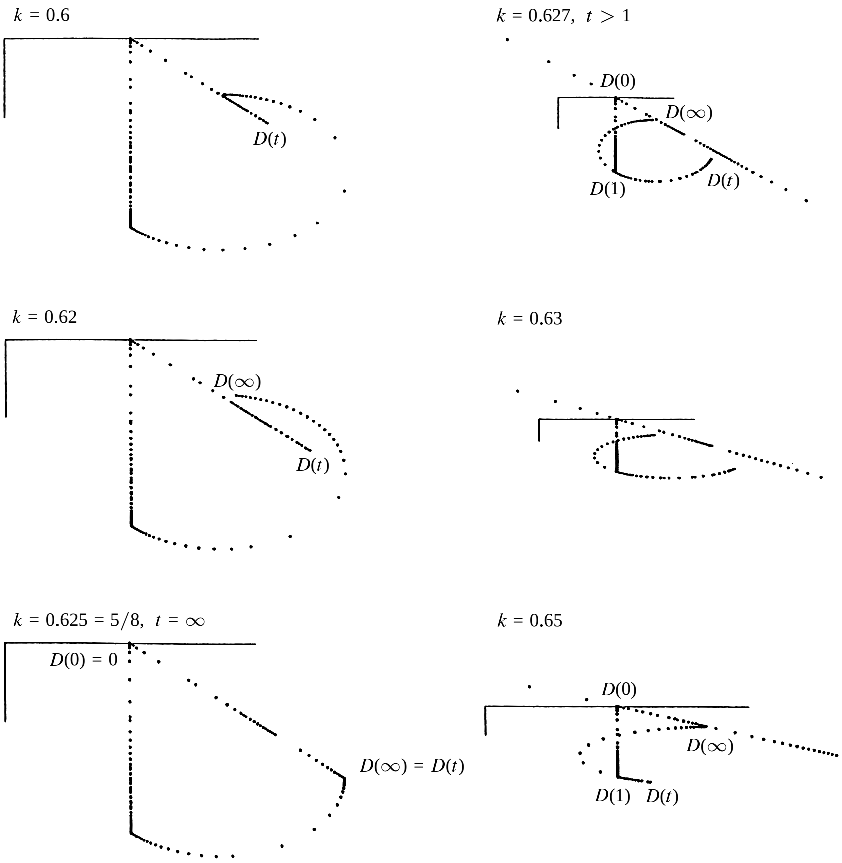


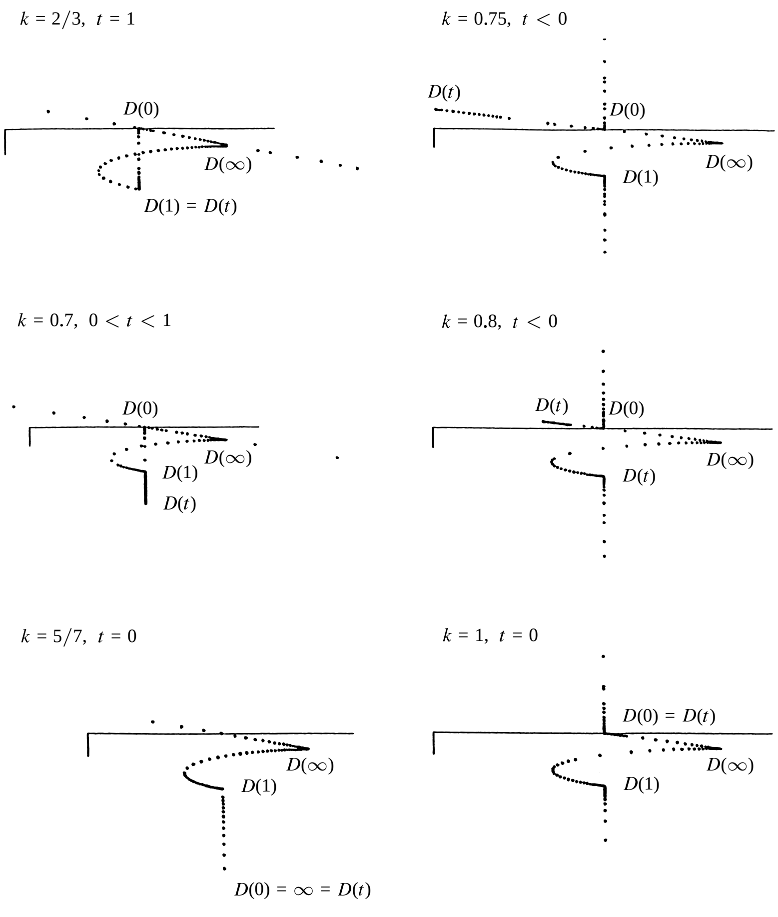
$k = 0.1$



$k = 1/2,\ t = \infty$




k = 0.6
$D(t)$
k = 0.627, $t > 1$
$D(0)$
$D(\infty)$
$D(1)$
$D(t)$
k = 0.62
$D(\infty)$
$D(t)$
k = 0.63
k = 0.625 = 5/8, $t = \infty$
$D(0) = 0$
$D(\infty) = D(t)$
k = 0.65
$D(0)$
$D(\infty)$
$D(1)$ $D(t)$


$k = 2/3,\ t = 1$
$D(0)$
$D(\infty)$
$D(1) = D(t)$
$k = 0.75,\ t < 0$
$D(t)$
$D(0)$
$D(\infty)$
$D(1)$
$k = 0.7,\ 0 < t < 1$
$D(0)$
$D(\infty)$
$D(1)$
$D(t)$
$k = 0.8,\ t < 0$
$D(t)$
$D(0)$
$D(\infty)$
$D(t)$
$k = 5/7,\ t = 0$
$D(\infty)$
$D(1)$
$D(0) = \infty = D(t)$
$k = 1,\ t = 0$
$D(0) = D(t)$
$D(\infty)$
$D(1)$

6. Appendix. Linear independence of periods of the second kind

by Paula B. Cohen

In this appendix, we show that the arguments of [19], Section 2, together with those of Section 1 of [22] give rather directly a proof of Wüstholz' announcement, Theorem 5 of [23], in the case of abelian varieties defined over $\overline{\mathbf{Q}}$. Namely, we have:

Theorem 6.1. *Let A be an abelian variety isogenous over $\overline{\mathbf{Q}}$ to the direct product $A_1^{k_1} \times \cdots \times A_N^{k_N}$ of simple, pairwise non-isogenous abelian varieties A_ν defined over $\overline{\mathbf{Q}}$, with A_ν of dimension n_ν, $\nu = 1, \ldots, N$. Then the $\overline{\mathbf{Q}}$-vector space $\widehat{V}_A$ generated by $1, 2\pi i$ together with all periods of differentials, defined over $\overline{\mathbf{Q}}$, of the first and the second kind on A, has dimension*

$$\dim_{\overline{\mathbf{Q}}} \widehat{V}_A = 2 + 4 \sum_{\nu=1}^{N} \frac{n_\nu^2}{\dim_{\mathbf{Q}} \mathrm{End}_0 A_\nu}. \tag{6.1}$$

Proof. We use the notations of [19], Section 1, and assume the background necessary to understand Section 1 of [22] and its notations. For general information, the reader can consult [17], [18] and [15]—see also [13], [3] and [4] for the elliptic case. That $\widehat{V}_A$ has dimension over $\overline{\mathbf{Q}}$ bounded above by the right-hand-side of (6.1) is obvious once one observes that the induced action of $\mathrm{End}_0(A)$ on $\widehat{V}_A$ is by linear transformations defined over $\overline{\mathbf{Q}}$.

We now show that $\dim_{\overline{\mathbf{Q}}} \widehat{V}_A$ is bounded below by the right-hand-side of (6.1). Assume the contrary. As in [19], Section 2, let $\omega_1, \ldots, \omega_n$, $n = \dim A$, be a $\overline{\mathbf{Q}}$-basis of $H^0\left(A, \Omega_{\overline{\mathbf{Q}}}\right)$: the differentials, defined over $\overline{\mathbf{Q}}$, of the first kind on A. Choose $\gamma_1, \ldots, \gamma_m \in H_1(A, \mathbf{Z})$ such that the period vectors

$$\int_{\gamma_j} \underline{\omega} = \begin{pmatrix} \int_{\gamma_j} \omega_1 \\ \vdots \\ \int_{\gamma_j} \omega_n \end{pmatrix}, \quad j = 1, \ldots, m.$$

in the period lattice Λ of A form a basis of $\Lambda_{\mathbf{Q}} = \Lambda \otimes_{\mathbf{Z}} \mathbf{Q}$ over $\Phi(L)$, where Φ is the complex representation of $L = \mathrm{End}_0(A)$ induced by the isomorphism of A with $\mathbf{C}^n/\Lambda$. Let $\eta_1, \ldots, \eta_n$ be a $\overline{\mathbf{Q}}$-basis of $H^1_{\mathrm{DR}}(A)$ modulo $H^0\left(A, \Omega_{\overline{\mathbf{Q}}}\right)$. In particular, the η_i, $i = 1, \ldots, n$, are defined over $\overline{\mathbf{Q}}$. Our hypothesis is, then, that there exists a relation of the form

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \left\{ \alpha_{ij} \int_{\gamma_j} \omega_i + \beta_{ij} \int_{\gamma_j} \eta_i \right\} + \beta_0 \cdot 2\pi i + \alpha_0 = 0 \tag{6.2}$$

with $\alpha_0, \beta_0, \alpha_{ij}, \beta_{ij} \in \overline{\mathbf{Q}}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, not all zero. Let,

$$\eta^{(j)} = \sum_{i=1}^{n} \beta_{ij} \eta_j, \quad j = 1, \ldots, m.$$

By analogy with [22], the complex number,

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \beta_{ij} \int_{\gamma_j} \eta_i = \sum_{j=1}^{m} \int_{\gamma_j} \eta^{(j)},$$

can be written as $\int_\gamma \eta$ for a certain $\eta \in H^1_{\mathrm{DR}}(A^m)$ and $\gamma \in H_1(A^m, \mathbf{Z})$, where η corresponds to an element of $H^1(A^m, \mathcal{O}_{A^m}) \cong \mathrm{Ext}^1(A^m, \mathbf{G}_a)$. Namely, if $p_j \colon A^m \to A$ denotes the projection onto the j-th factor of A^m, $j = 1, \ldots, m$, then

$$\begin{aligned} \eta &= p_1^* \eta^{(1)} + \cdots + p_m^* \eta^{(m)} \\ \text{and} \quad \gamma &= p_1^* \gamma_1 + \cdots + p_m^* \gamma_m. \end{aligned}$$

Moreover η determines a commutative algebraic group variety G over $\overline{\mathbf{Q}}$ such that on the tangent space at the orgin $T(G) = T_e(G) \cong \mathbf{C} \times (\mathbf{C}^n)^m$ the vector,

$$v = \left(\int_\gamma \eta, \int_{\gamma_1} \underline{\omega}, \ldots, \int_{\gamma_m} \underline{\omega} \right)$$

is in $\mathrm{Ker}\exp_G$. The extension G corresponds to the sum of the extensions $\tilde{G}_j = p_j^* G_j \in \mathrm{Ext}^1(A^m, \mathbf{G}_a)$, where G_j is determined by $\eta^{(j)}$, $j = 1, \ldots, m$.

Assume for the moment that $\alpha_0 = \beta_0 = 0$. Now $T(G) = T(\mathbf{G}_a) \oplus T(A^m)$. We let z be the coordinate in $T(\mathbf{G}_a) \cong \mathbf{C}$ and $w_1^{(j)}, \ldots, w_n^{(j)}$ be the coordinates in $T(A)$ for the j-th factor of A^m, $j = 1, \ldots, m$. Let H be the hyperplane in $T(G) \cong \mathbf{C} \times (\mathbf{C}^n)^m$ given, with $w^{(j)} = {}^t\left(w_i^{(j)}\right)_{i=1,\ldots,n}$, $j = 1, \ldots, m$ by

$$F\left(z, w^{(1)}, \ldots, w^{(m)}\right) = F_1\left(w^{(1)}\right) + \cdots + F_m\left(w^{(m)}\right) + z = 0$$

where the $F_j\left(w^{(j)}\right)$, $j = 1, \ldots, m$, are the linear forms,

$$F_j\left(w^{(j)}\right) = \sum_{i=1}^{n} \alpha_{ij} w_i^{(j)}.$$

Then, from (6.2) we have $0 \neq v \in H$.

We may therefore apply the Wüstholz algebraic subgroup theorem as in Lemma 1 of [19] to deduce the existence of a proper connected algebraic subgroup W of G, defined over $\overline{\mathbf{Q}}$, with $v \in T(W) \subseteq H$.

Now, if $\mathbf{G}_a \cap W = \mathbf{G}_a$, then $\exp_G^{-1}(\mathbf{G}_a) \subseteq H$, so that for every $z \in \mathbf{C}$ the point $(z, 0, \dots, 0) \in \mathbf{C} \times (\mathbf{C}^n)^m$ is in H, which contradicts the defining equation for H. Hence $\dim(\mathbf{G}_a \cap W) = 0$ and so W is isogenous to an abelian subvariety W_0 of A^m, defined over $\overline{\mathbf{Q}}$ with $v_0 = \left(\int_{\gamma_1} \underline{\omega}, \dots, \int_{\gamma_m} \underline{\omega}\right) \in T(W_0)$. If W_0 were a proper abelian subvariety of A^m then, as $T(W_0)$ is defined over $\overline{\mathbf{Q}}$, there would exist a non-trivial $\overline{\mathbf{Q}}$-linear relation between the periods $\int_{\gamma_j} \omega_i$, $i = 1, \dots, n$, $j = 1, \dots, m$, and so by Proposition 1 in [19] there would be a non-trivial dependence relation over $\Phi(L)$ between the period vectors $\int_{\gamma_j} \underline{\omega}$, $j = 1, \dots, m$. Hence, $W_0 = A^m$ and so $F_1 \equiv \cdots \equiv F_m \equiv 0$.

These arguments show that W is isogenous to A^m. In particular, the element of $\mathrm{Ext}^1(A^m, \mathbf{G}_a)$ defined by G ist trivial as this same isogeny provides a splitting of the associated exact sequence. Therefore $\eta = 0$, so that we have finally $\alpha_{ij} = \beta_{ij} = 0$, $i = 1, \dots, n$, $j = 1, \dots, m$, which is the desired contradiction in the case $\alpha_0 = \beta_0 = 0$.

When we do not have $\alpha_0 = \beta_0 = 0$, we can use the same arguments as in [22] to conclude the proof. Let us briefly recall the line of those arguments. When $\alpha_0 = 0$, $\beta_0 \neq 0$, one argues with $G' = \mathbf{G}_m \times G$ instead of G to obtain an algebraic subgroup W' of G', defined over $\overline{\mathbf{Q}}$, with $\dim((\mathbf{G}_a \times \mathbf{G}_m) \cap W') = 0$. One then concludes as above.

When $\alpha_0 \neq 0$ one argues with $G'' = \mathbf{G}_a \times \mathbf{G}_m \times G$ as in the proof of [22]. One can now construct an algebraic subgroup W'' of G'', defined over $\overline{\mathbf{Q}}$, with $\dim(\mathbf{G}_m \cap W'') = 0$. However, one does not necessarily have $\dim((\mathbf{G}_a \times \mathbf{G}_a) \cap W'') = 0$ as $\dim((\mathbf{G}_a \times \mathbf{G}_a) \cap W'') = 1$ can also occur. In the latter case, one argues as in the preceding paragraphs (when we had $\alpha_0 = 0$) on the quotients $\overline{\mathbf{G}} = G''/\mathbf{G}_a$ and $\overline{W} = W''/\mathbf{G}_a$. One deduces then that $\overline{\mathbf{G}}$ is a trivial extension of A^m by $\mathbf{G}_a$, so that $\beta_{ij} = 0$, $i = 1, \dots, n$, $j = 1, \dots, m$, that $\overline{W}$ is isogenous to A^m, so that $\alpha_{ij} = 0$, $i = 1, \dots, n$, $j = 1, \dots, m$, and that $\beta_0 = 0$. It is then easy to show that $\alpha_0 = 0$ exactly as in [22]. $\square$

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Hironori Shiga
Institue of Mathematics and Physics
Chiba University
Yayoi-cho 1-33, Inage-ku
Chiba 263-8522, Japan

Toru Tsutsui
Faculty of Science
Chiba University
Yayoi-cho 1-33, Inage-ku
Chiba 263-8522, Japan

Jürgen Wolfart
Math. Seminar der Univ
Postfach 111932
D-60054 Frankfurt, Germany