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JØRGENSEN GROUPS OF PARABOLIC TYPE II (COUNTABLY INFINITE CASE)

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1. Introduction

1.1. It is one of the most important problems in the theory of Kleinian groups to decide whether or not a subgroup G of the Möbius transformation group is discrete. For this problem there are two important and useful theorems: One is Poincaré's polyhedron theorem, which gives a sufficient condition for G to be discrete. The other is Jørgensen's inequality, which gives a necessary condition for a two-generator Möbius transformation group $\langle A, B\rangle$ to be discrete. Here we will consider extreme discrete groups (Jørgensen groups) for Jørgensen's inequality. This paper is the second part of a series of studies on Jørgensen groups (cf. Li-Oichi-Sato [4, 5]).

1.2. Let Möb denote the set of all linear fractional transformations (Möbius transformations)

$$A(z) = \frac{az+b}{cz+d}$$

of the extended complex plane $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, where a, b, c, d are complex numbers and the determinant $ad - bc = 1$. There is an isomorphism between Möb and $PSL(2, \mathbf{C})$. We always write elements of Möb as matrices with determinant 1 in this paper. We recall that Möb (= $PSL(2, \mathbf{C})$) acts on the upper half space H^3 of $\mathbf{R}^3$ as the group of conformal isometries of hyperbolic 3-space.

In this paper we use a Kleinian group in the same meaning as a discrete group. Namely, a Kleinian group is a discrete subgroup of Möb. A Kleinian group G is of *the first kind* if the limit set $\Lambda(G)$ of G is all of the extended complex plane $\hat{\mathbf{C}}$ and it is of *the second kind* otherwise. A subgroup G of Möb is said to be *elementary* if there exists a finite G-orbit in $\mathbf{R}^3$.

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1.3. The *trace* $\mathrm{tr}(A)$ of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (ad - bc = 1)$$

in $SL(2, \mathbf{C})$ is defined by $\mathrm{tr}(A) = a + d$. We remark that the trace of an element of Möb (= $PSL(2, \mathbf{C})$) is not well-defined, but Jørgensen number (defined later) is still well-defined after choosing matrix representatives.

1.4. Let A^* and B^* be matrices in $SL(2, \mathbf{C})$ representing the Möbius transformations A and B, respectively. As A^* and B^* are determined by A and B to within a factor of -1, we see that the commutator $A^*B^*(A^*)^{-1}(B^*)^{-1}$ (resp. $(A^*)^2$) are uniquely determined by A and B (resp. A). Thus we may write $\mathrm{tr}(ABA^{-1}B^{-1}) = \mathrm{tr}(A^*B^*(A^*)^{-1}(B^*)^{-1})$ and $\mathrm{tr}^2(A) = \mathrm{tr}^2(A^*)$.

In 1976 Jørgensen obtained the following important theorem, which gives a necessary condition for a non-elementary Möbius transformation group $G = \langle A, B\rangle$ to be discrete.

Theorem A (Jørgensen [1]). *Suppose that the Möbius transformations A and B generate a non-elementary discrete group. Then*

$$J(A, B) := |\,\mathrm{tr}^2(A) - 4| + |\,\mathrm{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$

The lower bound 1 *is best possible.*

1.5.

DEFINITION 1. Let A and B be Möbius transformations. The *Jørgensen number* $J(A, B)$ for the ordered pair (A, B) is defined by

$$J(A, B) := |\,\mathrm{tr}^2(A) - 4| + |\,\mathrm{tr}(ABA^{-1}B^{-1}) - 2|.$$

DEFINITION 2. A subgroup G of Möb is called a *Jørgensen group* if G satisfies the following four conditions:

(1) G is a two-generator group.
(2) G is a discrete group.
(3) G is a non-elementary group.
(4) There exist generators A and B of G such that $J(A, B) = 1$.

1.6. Jørgensen and Kiikka showed the following.

Theorem B (Jørgensen-Kiikka [2]). *Let $\langle A, B\rangle$ be a non-elementary discrete group with $J(A, B) = 1$. Then A is elliptic of order at least seven or A is parabolic.*

If $\langle A, B\rangle$ is a Jørgensen group such that A is parabolic and $J(A, B) = 1$, then we call it *a Jørgensen group of parabolic type*. There are infinitely many Jørgensen groups of parabolic type (Jørgensen-Lascurain-Pignataro [3], Sato [8]).

Now it gives rise to the following problem.

Problem 1. Find all Jørgensen groups of parabolic type.

1.7. Let $\langle A, B\rangle$ be a marked two-generator group such that A is parabolic. Then we can normalize A and B as follows:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{\sigma,\mu} = \begin{pmatrix} \mu\sigma & \mu^2\sigma - 1/\sigma \\ \sigma & \mu\sigma \end{pmatrix},$$

where $\sigma \in \mathbf{C}\backslash\{0\}$ and $\mu \in \mathbf{C}$. See [4] for this normalization. We can easily see that $J(A, B_{\sigma,\mu}) = |\sigma|^2$.

We denote by $G_{\sigma,\mu}$ the marked group generated by A and $B_{\sigma,\mu}$: $G_{\sigma,\mu} = \langle A, B_{\sigma,\mu}\rangle$. We say that $(\sigma, \mu) \in \mathbf{C}\backslash\{0\} \times \mathbf{C}$ *is the point representing a marked group* $G_{\sigma,\mu}$ *and that* $G_{\sigma,\mu}$ *is the marked group corresponding to a point* (σ, μ).

1.8. In [8], Sato considered the case of $\mu = ik$ ($k \in \mathbf{R}$). Namely, he considered marked two-generator group $G_{\sigma,ik} = \langle A, B_{\sigma,ik}\rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{\sigma,ik} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix},$$

where $\sigma \in \mathbf{C}\backslash\{0\}$ and $k \in \mathbf{R}$.

Now we have the following conjecture.

Conjecture. For any Jørgensen group G of parabolic type there exists a marked group $G_{\sigma,ik}(\sigma \in \mathbf{C}\backslash\{0\}, k \in \mathbf{R})$ such that $G_{\sigma,ik}$ is conjugate to G.

If this conjecture is true, then it is sufficient to consider the case of $\mu = ik$ in order to find all Jørgensen groups of parabolic type. In this paper we only consider the case of $\mu = ik$.

1.9. Let C be the following cylinder:

$$C = \{(\sigma, ik) \mid |\sigma| = 1, k \in \mathbf{R}\}.$$

Theorem C (Sato [8]). *If a marked two-generator group* $G_{\sigma,ik} = \langle A, B_{\sigma,ik}\rangle$ $(\sigma \in \mathbf{C}\backslash\{0\}, k \in \mathbf{R})$ *is a Jørgensen group, then the point* (σ, ik) *representing* $G_{\sigma,ik}$ *lies on the cylinder* C.

If (σ, ik) is a point on the cylinder C, then we can set $\sigma = -ie^{i\theta}$ $(0 \leq \theta \leq 2\pi)$. If a point $(-ie^{i\theta}, ik)$ on the cylinder C represents a Jørgensen group, then we say that the group is a *Jørgensen group of parabolic type* (θ, k).

Now it gives rise to the following problem.

Problem 2. Find all Jørgensen groups of parabolic type (θ, k).

We divide Jørgensen groups of this type into three parts as follows:
Part 1. $|k| \leq \sqrt{3}/2$, $0 \leq \theta \leq 2\pi$ (finite case).
Part 2. $\sqrt{3}/2 < |k| \leq 1$, $0 \leq \theta \leq 2\pi$ (countably infinite case).
Part 3. $1 < |k|$, $0 \leq \theta \leq 2\pi$ (uncountably infinite case).

By some lemmas in [8], it suffices to consider the case of $0 \leq \theta \leq \pi/2$ and $k \geq 0$ in order to find Jørgensen groups of parabolic type (θ, k).

In the previous paper [4] we found all Jørgensen groups in the case where $0 \leq \theta \leq \pi/2$ and $0 \leq k \leq \sqrt{3}/2$, that is, we obtain the following theorem.

Theorem D (finite case) (Li-Oichi-Sato [4]). (i) *There are sixteen Jørgensen groups in* $D = \{(\theta, k) \in \mathbf{R} \mid 0 \leq \theta \leq \pi/2, 0 \leq k \leq \sqrt{3}/2\}$.
(ii) *Nine of them are Kleinian groups of the first kind and seven groups are of the second kind.*

Furthermore, in [5] we found all Jørgensen groups in the case where $0 \leq \theta \leq \pi/2$ and $k > 1$ (uncountably infinite case). Therefore we found all Jørgensen groups of parabolic type (θ, k), that is, Problem 2 is completely solved.

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2. Main theorem

In this section we will state that we find all Jørgensen groups in Part 2.

For simplicity we write $B_{\theta,k}$ for $B_{-ie^{i\theta},ik}$. Let A and $B_{\theta,k}$ $(k \in \mathbf{R}, 0 \leq \theta \leq \pi/2)$ be the following matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{\theta,k} = \begin{pmatrix} ke^{i\theta} & ie^{-i\theta}(k^2e^{2i\theta} - 1) \\ -ie^{i\theta} & ke^{i\theta} \end{pmatrix}.$$

We obtain the following theorem.

Main Theorem. *The group* $G_{\theta,k}$ *with* $0 \leq \theta \leq \pi/2$ *and* $\sqrt{3}/2 < k \leq 1$ *is a Jørgensen group* (*discrete*) *if and only if one of the following conditions holds.*
(a) $\theta = 0$ *and* $k = 1$. *In this case,* $G_{\theta,k}$ *is a Kleinian group of the second kind, and* $\Omega(G_{\theta,k})/G_{\theta,k}$ *is a union of two Riemann surfaces with signature* $(0; 2, 3, \infty)$.
(b) $\theta = 0$ *and* $k = \cos(\pi/n)$ $(n = 7, 8, \ldots)$. *In this case,* $G_{\theta,k}$ *is a Kleinian group of*

the second kind, and $\Omega(G_{\theta,k})/G_{\theta,k}$ *is a union of two Riemann surfaces with signatures* $(0; 2, 3, n)$ *and* $(0; 2, 3, \infty)$.
(c) $\theta = \pi/4$ *and* $k = 1$. *In this case,* $G_{\theta,k}$ *is a Kleinian group of the first kind, and the volume* $V(G_{\pi/4,1})$ *of the* 3*-orbifold for* $G_{\pi/4,1}$ *is*

$$V(G_{\pi/4,1}) = 8\left\{2L\left(\frac{\pi}{4}\right) - L\left(\frac{\pi}{12}\right) - L\left(\frac{5\pi}{12}\right)\right\},$$

where $L(\theta)$ *is the Lobachevskiĭ function*:

$$L(\theta) = -\int_0^\theta \log|2\sin u|\,du.$$

(d) (Sato-Yamada [9]) $\theta = \pi/2$ *and* $k = 1$. *In this case,* $G_{\theta,k}$ *is a Kleinian group of the second kind, and* $\Omega(G_{\theta,k})/G_{\theta,k}$ *is a Riemann surface with signature* $(0; 3, 3, \infty)$.
(e) (Sato-Yamada [9]) $\theta = \pi/2$ *and* $k = \cos(\pi/n)$ $(n = 7, 8, \ldots)$. *In this case,* $G_{\theta,k}$ *is a Kleinian group of the second kind, and* $\Omega(G_{\theta,k})/G_{\theta,k}$ *is a Riemann surface with signature* $(0; 3, 3, n)$.

The proof of the main theorem is given in Sections 4–9.

Corollary. *There are countably infinite Jørgensen groups on the region* $\{(\theta, k) \mid 0 \le \theta \le \pi/2, \sqrt{3}/2 < k \le 1\}$.

3. Poincaré's polyhedron theorem

In this section we will state Poincaré's polyhedron theorem following Maskit [6, p. 73]. The theorem gives a sufficient condition for a subgroup of the Möbius transformation group to be discrete.

Let $\mathbf{P}$ be a polyhedron in the upper half space H^3. We assume that for each side s of $\mathbf{P}$, there is a side s', not necessarily distinct from s, and there is an element $g_s \in$ Möb, satisfying the following conditions:
(i) $g_s(s) = s'$
(ii) $g_{s'} = g_s^{-1}$.
Then the isometries g_s are called the *side pairing transformations*.

Let G be the group generated by the side pairing transformations. If there is a side s, with $s' = s$, then condition (i) implies that $g_s^2 = 1$. If this occurs, the relation $g_s^2 = 1$, is called a *reflection relation*.

The side pairing transformations induce an equivalence relation on $\overline{\mathbf{P}}$ (the closure of $\mathbf{P}$). Let $\mathbf{P}^*$ be the space of equivalence classes so that projection $p\colon \overline{\mathbf{P}} \to \mathbf{P}^*$ is continuous and open. We assume that for every point $z \in \mathbf{P}^*$, $p^{-1}(z)$ is a finite set.

Next we will define a cycle of edges of $\mathbf{P}$ and a cycle transformation. Let e_1 be an edge. It lies on the boundary of two sides of $\mathbf{P}$. Let s_1 be one side of them. Then there is a side s_1' and there is a side pairing transformation g_1 with $g_1(s_1) = s_1'$. Set

$e_2 = g_1(e_1)$. The edge e_2 lies on the boundary of two sides, one of them is s_1'. Let s_2 be the other side. Then there are a side s_2' and a side pairing transformation g_2 with $g_2(s_2) = s_2'$. Continuing this manner, we generate a sequence $\{e_n\}$ of edges and a sequence $\{g_n\}$ of side pairing transformations.

Since each point of e_1 is equivalent to at most finitely many other points of $\overline{\mathbf{P}}$, the sequence of edges is periodic. Let m be the least period. The cyclically ordered sequence of edges $\{e_1, e_2, \ldots, e_m\}$ is called *a cycle of edges* and m is the *period* of the cycle. Two cycles are *equivalent* if they both contain the same set of edges. Each edge lies in exactly one equivalence class of cycle. We have $g_m \cdots g_1(e_1) = e_1$, and call $h = g_m \cdots g_1$ the *cycle transformation* at e_1. The relation of the form $h^t = 1$ is called the *cycle relation*. The following theorem is well-known.

Theorem E (Poincaré's Polyhedron Theorem (Maskit [6, p. 73])). *Let* $\mathbf{P}$ *be a polyhedron with side pairing transformations satisfying the following conditions* (1) *through* (6). *Then,* G*, the group generated by the side pairing transformations, is discrete and* $\mathbf{P}$ *is a fundamental polyhedron for* G*, and the reflection relations and cycle relations form a complete set of relations for* G:

(1) *For each side* s *of* $\mathbf{P}$*, there is a side* s' *and there is an element* $g_s \in G$ *satisfying* $g_s(s) = s'$ *and* $g_{s'} = g_s^{-1}$.

(2) $g_s(\mathbf{P}) \cap \mathbf{P} = \emptyset$.

(3) *For every point* $z \in \mathbf{P}^*$, $p^{-1}(z)$ *is a finite set.*

(4) *Let* e *be an edge and let* h *be the cycle transformation at* e. *Then for each edge* e*, there is a positive integer* t *such that* $h^t = 1$.

(5) *Let* $\{e_1, e_2, \ldots, e_m\}$ *be any cycle of edges of* $\mathbf{P}$ *and let* $\alpha(e_k)$ $(k = 1, 2, \ldots, m)$ *be the angle measured from inside* $\mathbf{P}$ *at the edge* e_k. *Let* q *be the smallest positive integer such that* $h^q = 1$, *where* h *is the cycle transformation at* e_1. *Then the equality*

$$\sum_{k=1}^{m} \alpha(e_k) = \frac{2\pi}{q}$$

holds.

(6) $\mathbf{P}^*$ *is complete.*

4. Proof of the case $\theta = 0$

In this section we will prove the main theorem in the case of $\theta = 0$. For $\theta = 0$ we have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{0,k} = \begin{pmatrix} k & i(k^2-1) \\ -i & k \end{pmatrix} \quad (k \in \mathbf{R}).$$

We set $G_k = \langle A, B_k \rangle$.

4.1. We set $C_k = AB_kA^{-1}B_k^{-1}$ and $D_k = B_k^{-1}C_k$. Then we have

$$C_k = \begin{pmatrix} -ik & -(k^2-1)+ik \\ -1 & 1+ik \end{pmatrix} \text{ and } D_k = \begin{pmatrix} -i & i \\ 0 & i \end{pmatrix}.$$

Furthermore we have

$$C_kA = \begin{pmatrix} -ik & -(k^2-1) \\ -1 & ik \end{pmatrix} \text{ and } D_kA = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

We can easily see the following lemmas and so omit the proofs.

Lemma 4.1. *Let A, B_k, C_k, D_k, C_kA and D_kA be the Möbius transformations represented by the above matrices. Then the following hold.*
(i) *A is a parabolic transformation with a fixed point ∞.*
(ii) (1) *B_k is a parabolic transformation with a fixed point 0 if $k = 1$.*
(2) *B_k is an elliptic transformation of order n (integer $n \geq 7$) if $0 < k < 1$, where $k = \cos(\pi/n)$.*
(iii) *C_k is an elliptic transformation of order three with fixed points $(1+i(2k\pm\sqrt{3}))/2$.*
(iv) *D_k is an elliptic transformation of order two with fixed points $1/2$ and ∞.*
(v) *C_kA is an elliptic transformation of order two with fixed points $(k\pm 1)i$.*
(vi) *D_kA is an elliptic transformation of order two with fixed points 0 and ∞.*

Lemma 4.2. *Let A, B_k, C_k, D_k, C_kA and D_kA be the Möbius transformations represented by the above matrices. Let G_k be the group generated by A and B_k, and G_k^* be the group generated by A, C_k and D_k : $G_k = \langle A, B_k\rangle$ and $G_k^* = \langle A, C_k, D_k\rangle$. Then $G_k = G_k^*$.*

4.2. We will prove (a) and (b) of the main theorem. The part is proved parallel to 6.2 in [4]. Let $\theta = 0$ and $k = 1$ or $k = \cos\pi/n$ ($n = 7, 8, \ldots$). Let G_k^* be the group generated by A, C_k and D_k, that is, $G_k^* = \langle A, C_k, D_k\rangle$. We denote by H^3 and $\bar{H}^3$ the upper half space and its closure, respectively: $H^3 = \{(z,t) \in \mathbf{C}\times\mathbf{R} \mid t > 0\}$ and $\bar{H}^3 = \{(z,t) \in \mathbf{C}\times\mathbf{R} \mid t \geq 0\}$. We will define some sides F_j ($j = 1, 2, \ldots, 6$) in $\bar{H}^3$ as follows:

$F_1 = \{(z,t) \in \bar{H}^3 \mid x = 0,\ y \geq 0,\ t \geq 0,\ |z - ik|^2 + t^2 \geq 1\}$,
$F_2 = \{(z,t) \in \bar{H}^3 \mid x = 1,\ y \geq 0,\ t \geq 0,\ |z - (1+ik)|^2 + t^2 \geq 1\}$,
$F_3 = \{(z,t) \in \bar{H}^3 \mid 0 \leq x \leq 1/2, y = 0,\ t \geq 0,\ |z - ik|^2 + t^2 \geq 1\}$,
$F_4 = \{(z,t) \in \bar{H}^3 \mid 1/2 \leq x \leq 1, y = 0,\ t \geq 0,\ |z - (1+ik)|^2 + t^2 \geq 1\}$,
$F_5 = \{(z,t) \in \bar{H}^3 \mid 0 \leq x \leq 1/2, y \geq 0,\ |z - ik|^2 + t^2 = 1,\ t \geq 0\}$,
$F_6 = \{(z,t) \in \bar{H}^3 \mid 1/2 \leq x \leq 1, y \geq 0,\ |z - (1+ik)|^2 + t^2 = 1,\ t \geq 0\}$.

Let $\mathbf{P}$ be the polyhedron in H^3 bounded by the above six sides F_j ($j = 1, 2, \ldots, 6$) and the complex plane $\mathbf{C}$. Then there are three side pairing transformations A, D_k and C_k of $\mathbf{P}$ as follows: $F_2 = A(F_1)$, $F_4 = D_k(F_3)$, $F_5 = C_k(F_6)$ (see

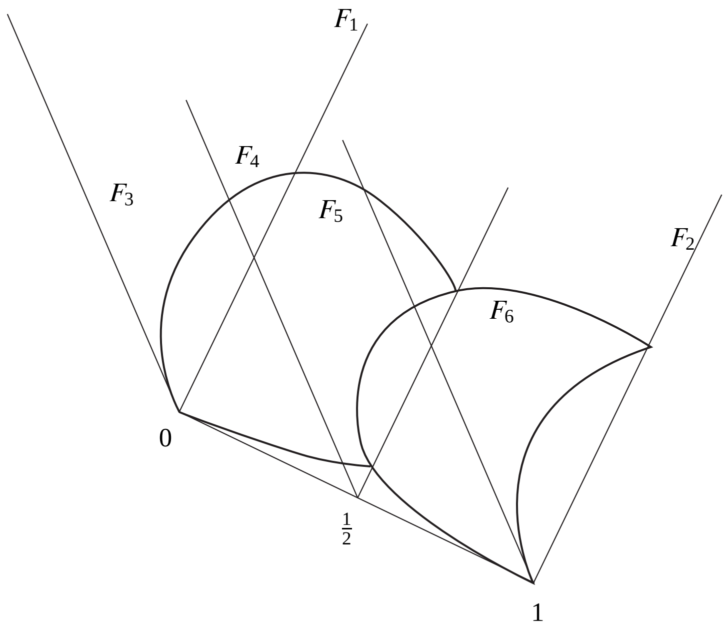

Fig. 4.1. $(k = 1)$

Fig. 4.1 and Fig. 4.2).

Next we denote the edges e_j $(j = 1, 2, \dots, 6)$ of $\mathbf{P}$ as follows:

$e_1 = F_1 \cap F_3$; $e_2 = F_2 \cap F_4$; $e_3 = F_3 \cap F_4$; $e_4 = F_1 \cap F_5$; $e_5 = F_2 \cap F_6$; $e_6 = F_5 \cap F_6$.

Let $\theta_j = \alpha(e_j)$ $(j = 1, 2, \dots, 6)$ be the angles measured from inside $\mathbf{P}$ at e_j. Then it is easily seen that $\theta_1 = \theta_2 = \pi/2$, $\theta_3 = \pi$, $\theta_4 = \theta_5 = \pi/2$, $\theta_6 = 2\pi/3$. Then $\theta_1 + \theta_2 = 2\pi/2$, $\theta_3 = 2\pi/2$, $\theta_4 + \theta_5 = 2\pi/2$, $\theta_6 = 2\pi/3$. Since $e_2 = A(e_1)$ and $e_1 = D_k(e_2)$ (resp. $e_5 = A(e_4)$ and $e_4 = C_k(e_5)$) we have that $D_k A$ (resp. $C_k A$) are the cycle transformations at e_1 (resp. e_4) and that both of the cycle transformations are elliptic of order two by Lemma 4.1. Since $e_3 = D_k(e_3)$ (resp. $e_6 = C_k(e_6)$), we have D_k (resp. C_k) are the cycle transformations at e_3 (resp. e_6) and that both of the cycle transformations are elliptic of order two (resp. of order three) by Lemma 4.1.

By the above we see that the conditions (1) through (5) in Theorem E are satisfied. The completeness condition (6) in Theore E is easily shown by using Ratcliffe [7, Theorem 11.1.2]. Thus G_k^* is a Kleinian group of the second kind. By Lemma 4.2, G_k is a Kleinian group of the second kind and so a Jørgensen group.

We define some half lines, segments and circular arcs s_j $(j = 1, 2, \dots, 8)$ on the complex plane $\mathbf{C}$ as follows:

$s_1 = \{z \in \mathbf{C} \mid x = 0,\ \ y \geq k + 1\}$,
$s_2 = \{z \in \mathbf{C} \mid x = 1,\ \ y \geq k + 1\}$,
$s_3 = \{z \in \mathbf{C} \cap F_5 \mid 0 \leq x \leq 1/2,\ \ k + \sqrt{3}/2 \leq y \leq k + 1\}$,
$s_4 = \{z \in \mathbf{C} \cap F_6 \mid 1/2 \leq x \leq 1,\ \ k + \sqrt{3}/2 \leq y \leq k + 1\}$,

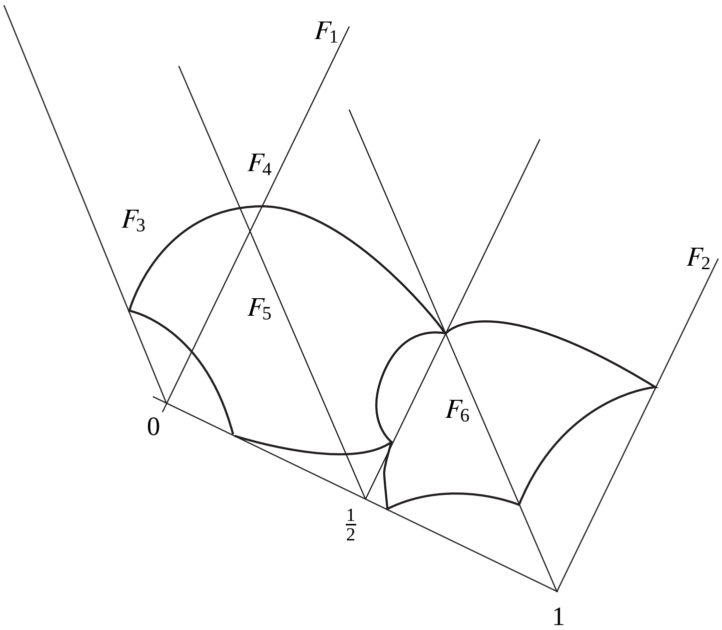

Fig. 4.2. $\left(\frac{\sqrt{3}}{2} < k < 1\right)$

$s_5 = \{z \in \mathbf{C} \cap F_5 \mid \sqrt{1-k^2} \le x \le 1/2,\ \ 0 \le y \le k - \sqrt{3}/2\}$,
$s_6 = \{z \in \mathbf{C} \cap F_6 \mid 1/2 \le x \le 1 - \sqrt{1-k^2},\ \ 0 \le y \le k - \sqrt{3}/2\}$,
$s_7 = \{z \in \mathbf{C} \mid \sqrt{1-k^2} \le x \le 1/2,\ \ y = 0\}$,
$s_8 = \{z \in \mathbf{C} \mid 1/2 \le x \le 1 - \sqrt{1-k^2},\ \ y = 0\}$.

We denote by S_1 (resp. S_2) the polygons bounded by the curves s_1, s_2, s_3 and s_4 (resp. s_5, s_6, s_7 and s_8), and we denote by S_3 (resp. S_4) the mirror images of S_1 (resp. S_2) with respect to the real axis. Let p_j $(j = 1, 2, \ldots, 7)$ be the vertices of the polygons S_1 and S_2 as follows: $p_1 = s_1 \cap s_3$, $p_2 = s_2 \cap s_4$, $p_3 = s_3 \cap s_4$, $p_4 = s_5 \cap s_6$, $p_5 = s_5 \cap s_7$, $p_6 = s_6 \cap s_8$ and $p_7 = s_7 \cap s_8$. Since $p_2 = A(p_1)$, $p_1 = C_k(p_2)$, $p_3 = C_k(p_3)$ (resp. $p_4 = C_k(p_4)$, $p_5 = C_k(p_6)$, $p_6 = D_k(p_5)$ and $p_7 = D_k(p_7)$; $s_2 = A(s_1)$, $s_4 = C_k(s_3)$, $s_5 = C_k(s_6)$, $s_8 = D_k(s_9)$ and $S_3 = D_k(S_1)$ (resp. $S_4 = D_k(S_2)$) we have by Lemma 4.1 that $\Omega(G_k)/G_k$ is a union of two Riemann surfaces with signature $(0; 2, 3, \infty)$ for $k = 1$ and that $\Omega(G_k)/G_k$ is a union of two Riemann surfaces with signatures $(0; 2, 3, n)$ and $(0; 2, 3, \infty)$ for $k = \cos(\pi/n)$ $(n = 7, 8, \ldots)$. □

4.3. We will prove that $G_{\theta,k}$ is not discrete in the case of $\theta = 0$ and k with $\cos(\pi/2m) < k < \cos(\pi/(2m+2))$ and $k \neq \cos(\pi/(2m+1))$ $(m = 3, 4, \ldots)$. We set

$$T_k = \left(\frac{1}{2\sqrt{1-k^2}}\right)^{1/2} \begin{pmatrix} 1 & -\sqrt{1-k^2} \\ 1 & \sqrt{1-k^2} \end{pmatrix} \ \ (0 < k < 1).$$

Then we have

$$A^* := T_k A T_k^{-1} = \frac{1}{2\sqrt{1-k^2}} \begin{pmatrix} 2\sqrt{1-k^2}-1 & 1 \\ -1 & 2\sqrt{1-k^2}+1 \end{pmatrix} \ (0<k<1),$$

$$B_k^* := T_k B_k T_k^{-1} = \begin{pmatrix} k+i\sqrt{1-k^2} & 0 \\ 0 & k-i\sqrt{1-k^2} \end{pmatrix} \ (0<k<1).$$

Thus

$$(B_k^*)^n = \begin{pmatrix} \left(k+i\sqrt{1-k^2}\right)^n & 0 \\ 0 & \left(k-i\sqrt{1-k^2}\right)^n \end{pmatrix} \ (0<k<1).$$

We set $\cos\theta = k$. Then $e^{i\theta} = k+i\sqrt{1-k^2}$ and $e^{-i\theta} = k-i\sqrt{1-k^2}$.

Then we have

$$J(A, B_k^n) = J(A^*, (B_k^*)^n) = \left| \frac{-2+2\cos 2n\theta}{4(\sin\theta)^2} \right|^2.$$

Thus $0 \leq J(A, B_k^n) < 1$ if and only if $\cos(\pi/2m) < k < \cos(\pi/(2m+2))$ ($n = 2m+1, m = 3, 4, \ldots$). We note that if $k = \cos(\pi/(2m+1))$ ($m = 3, 4, \ldots$), then $G_k = \langle A, B_k^{2m+1} \rangle$ is an elementary group. For the other cases G_k is a non-elementary group. By Theorem A, we can see that G_k is not a Kleinian group and not a Jørgensen group for k with $\cos(\pi/2m) < k < \cos(\pi/(2m+2))$ and $k \neq \cos(\pi/(2m+1))$ ($m = 3, 4, \ldots$). □

5. Proof of the case $\theta = \pi/4$

In this section we will prove the main theorem in the case of $\theta = \pi/4$. For $\theta = \pi/4$, we have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B := B_{\pi/4,1} = \begin{pmatrix} e^{\pi i/4} & ie^{\pi i/4}(-1+i) \\ -ie^{\pi i/4} & e^{\pi i/4} \end{pmatrix}.$$

We set $G = \langle A, B \rangle$.

5.1. We set $S := ABAB^{-1}A^{-1}BA^{-1}B^{-1}ABAB^{-1}$, $T := ABAB^{-1}A^{-1}BA^{-1}B^{-1}AB^{-1}A^{-1}B$ and $U := ABAB^{-1}A^{-1}BA^{-1}B^{-1}A$.

Then we have

$$S = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}, \quad T = \begin{pmatrix} -i & -2 \\ 0 & i \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad SB = \begin{pmatrix} e^{-\pi i/4} & e^{-\pi i/4}(1+i) \\ -ie^{-\pi i/4} & e^{-\pi i/4} \end{pmatrix}.$$

We easily see the following lemma and so omit the proof.

Lemma 5.1. *Let A, B, S, T and U be the Möbius transformations represented by the above matrices. Then the following hold*:
(i) A *is a parabolic transformation with a fixed point* ∞.
(ii) S *is an elliptic transformation of order two with fixed points* i *and* ∞.
(iii) T *is an elliptic transformation of order two with fixed points* $-i$ *and* ∞.
(iv) U *is an elliptic transformation of order two with fixed points* i *and* $-i$.
(v) SA *is an elliptic transformation of order two with fixed points* $(-1+2i)/2$ *and* ∞.
(vi) TA *is an elliptic transformation of order two with fixed points* $(-1-2i)/2$ *and* ∞.
(vii) UA *is an elliptic transformation of order three with fixed points* $(-1+\sqrt{3}\,i)/2$ *and* $(-1-\sqrt{3}\,i)/2$.
(viii) $U^{-1}\cdot(SB)^{-1}\cdot AB = I$, *where* I *is the identity mapping.*
(ix) $T^{-1}\cdot(SB)^{-1}\cdot B = I$.
(x) $S^{-1}\cdot SB\cdot B^{-1} = I$.
(xi) $U\cdot SB\cdot A^{-1}\cdot B^{-1} = I$.

We can easily see the following lemma.

Lemma 5.2. *Let A, B, S, T and U be the Möbius transformations represented by the above matrices. Let G be the group generated by A and B, and G^* be the group generated by A, S, T, U, B and SB : $G = \langle A, B\rangle$ and $G^* = \langle A, S, T, U, B, SB\rangle$. Then $G = G^*$.*

5.2. We will prove (c) of the main theorem. The part is similar to 6.2 in [4]. Let $\theta = \pi/4$ and $k = 1$. We denote by H^3 and $\bar{H}^3$ the upper half space and its closure, respectively: $H^3 = \{(z,t) \in \mathbf{C}\times\mathbf{R} \mid t > 0\}$ and $\bar{H}^3 = \{(z,t) \in \mathbf{C}\times\mathbf{R} \mid t \geq 0\}$.

We will define some sides F_j ($j = 1, 2, \dots, 12$) in $\bar{H}^3$ as follows:
$F_1 := \{(z,t) \in \bar{H}^3 \mid x = -1/2, -1 \leq y \leq 1,\ t \geq 0,\ |z|^2 + t^2 \geq 1,\ |z-i|^2 + t^2 \geq 1,\ |z+i|^2 + t^2 \geq 1\}$,
$F_2 := \{(z,t) \in \bar{H}^3 \mid x = 1/2, -1 \leq y \leq 1,\ t \geq 0,\ |z|^2 + t^2 \geq 1,\ |z-i|^2 + t^2 \geq 1,\ |z+i|^2 + t^2 \geq 1\}$,
$F_3 := \{(z,t) \in \bar{H}^3 \mid -1/2 \leq x \leq 0,\ y = 1,\ t \geq 0,\ |z-i|^2 + t^2 \geq 1\}$,
$F_4 := \{(z,t) \in \bar{H}^3 \mid 0 \leq x \leq 1/2,\ y = 1,\ t \geq 0,\ |z-i|^2 + t^2 \geq 1\}$,
$F_5 := \{(z,t) \in \bar{H}^3 \mid -1/2 \leq x \leq 0, y = -1,\ t \geq 0,\ |z+i|^2 + t^2 \geq 1\}$,
$F_6 := \{(z,t) \in \bar{H}^3 \mid 0 \leq x \leq 1/2,\ y = -1,\ t \geq 0,\ |z+i|^2 + t^2 \geq 1\}$,
$F_7 := \{(z,t) \in \bar{H}^3 \mid -1/2 \leq x \leq 0,\ -1/2 \leq y \leq 1/2,\ t \geq 0,\ |z|^2 + t^2 = 1\}$,
$F_8 := \{(z,t) \in \bar{H}^3 \mid 0 \leq x \leq 1/2,\ -1/2 \leq y \leq 1/2,\ t \geq 0,\ |z|^2 + t^2 = 1\}$,
$F_9 := \{(z,t) \in \bar{H}^3 \mid 0 \leq x \leq 1/2,\ -1 \leq y \leq -1/2,\ t \geq 0,\ |z+i|^2 + t^2 = 1\}$,
$F_{10} := \{(z,t) \in \bar{H}^3 \mid -1/2 \leq x \leq 0,\ 1/2 \leq y \leq 1,\ t \geq 0,\ |z-i|^2 + t^2 = 1\}$,

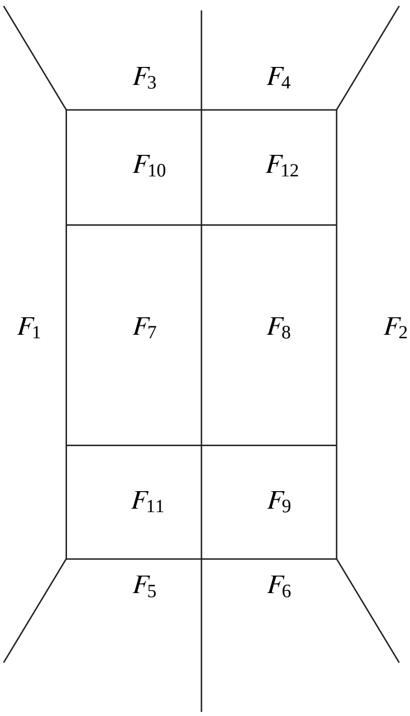

Fig. 5.1.

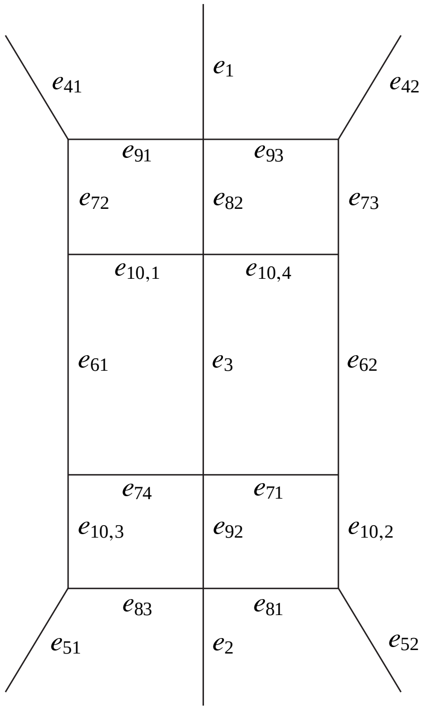

Fig. 5.2.

$F_{11} := \{(z,t) \in \bar{H}^3 \mid -1/2 \le x \le 0,\ \ -1 \le y \le -1/2,\ \ t \ge 0,\ \ |z+i|^2 + t^2 = 1\}$,
$F_{12} := \{(z,t) \in \bar{H}^3 \mid 0 \le x \le 1/2,\ \ 1/2 \le y \le 1,\ \ t \ge 0,\ \ |z-i|^2 + t^2 = 1\}$,
where $z = x + iy$, that is, $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

Let **P** be the polyhedron in H^3 bounded by the above twelve sides F_j $(j = 1, 2, \ldots, 12)$. Then there are six side pairing transformations A, S, T, U, B and SB of **P** as follows: $F_2 = A(F_1)$, $F_3 = S(F_4)$, $F_5 = T(F_6)$, $F_7 = U(F_8)$, $F_{10} = B(F_9)$, $F_{12} = SB(F_{11})$ (see Fig. 5.1 and Fig. 5.3).

Next we denote the edges e_j and e_{jk} $(j = 1, 2, \ldots, 10;\ k = 1, 2, 3, 4)$ of **P** as follows:

$e_1 = F_3 \cap F_4$; $e_2 = F_5 \cap F_6$; $e_3 = F_7 \cap F_8$; $e_{41} = F_1 \cap F_3$, $e_{42} = F_2 \cap F_4$; $e_{51} = F_1 \cap F_5$, $e_{52} = F_2 \cap F_6$; $e_{61} = F_1 \cap F_7$, $e_{62} = F_2 \cap F_8$; $e_{71} = F_8 \cap F_9$, $e_{72} = F_1 \cap F_{10}$, $e_{73} = F_2 \cap F_{12}$, $e_{74} = F_7 \cap F_{11}$; $e_{81} = F_6 \cap F_9$, $e_{82} = F_{10} \cap F_{12}$, $e_{83} = F_5 \cap F_{11}$; $e_{91} = F_3 \cap F_{10}$, $e_{92} = F_9 \cap F_{11}$, $e_{93} = F_4 \cap F_{12}$; $e_{10,1} = F_7 \cap F_{10}$, $e_{10,2} = F_2 \cap F_9$, $e_{10,3} = F_1 \cap F_{11}$, $e_{10,4} = F_8 \cap F_{12}$ (see Fig. 5.2).

Let $\theta_j = \alpha(e_j)$ $(j = 1, 2, \ldots, 8)$ be the angles measured from inside **P** at e_j. Then it is easily seen that $\theta_1 = \pi$, $\theta_2 = \pi$, $\theta_3 = \pi$, $\theta_{41} = \theta_{42} = \pi/2$, $\theta_{51} = \theta_{52} = \pi/2$, $\theta_{61} = \theta_{62} = \pi/3$, $\theta_{71} = \theta_{74} = 2\pi/3$, $\theta_{72} = \theta_{73} = \pi/3$, $\theta_{81} = \theta_{83} = \pi/2$, $\theta_{82} = \pi$,

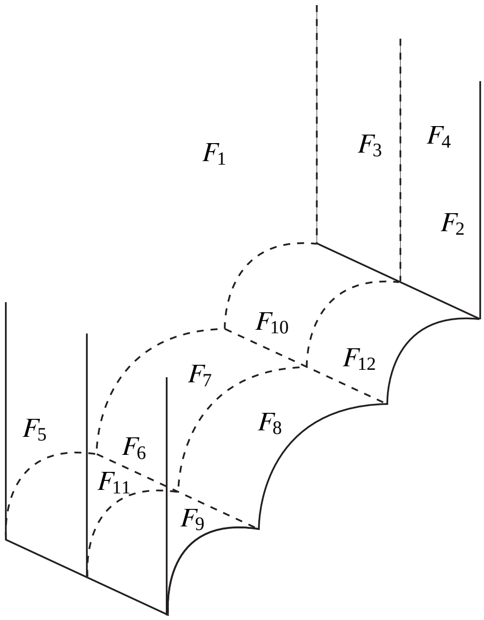

Fig. 5.3.

$\theta_{91} = \theta_{93} = \pi/2$, $\theta_{92} = \pi$, $\theta_{10,1} = \theta_{10,4} = 2\pi/3$, $\theta_{10,2} = \theta_{10,3} = \pi/3$. Thus we have the following: $\theta_1 = 2\pi/2$, $\theta_2 = 2\pi/2$, $\theta_3 = 2\pi/2$, $\theta_{41} + \theta_{42} = 2\pi/2$, $\theta_{51} + \theta_{52} = 2\pi/2$, $\theta_{61} + \theta_{62} = 2\pi/3$, $\theta_{71} + \theta_{72} + \theta_{73} + \theta_{74} = 2\pi/1$, $\theta_{81} + \theta_{82} + \theta_{83} = 2\pi/1$, $\theta_{91} + \theta_{92} + \theta_{93} = 2\pi/1$ and $\theta_{10,1} + \theta_{10,2} + \theta_{10,3} + \theta_{10,4} = 2\pi/1$.

By Lemma 5.1 we can see the following. Since $e_1 = S(e_1)$ (resp. $e_2 = T(e_2)$ and $e_3 = U(e_3)$), we have that S (resp. T and U) are the cycle transformations at e_1 (resp. e_2 and e_3) and that all of the cycle transformations are elliptic of order two. Since $e_{42} = A(e_{41})$ and $e_{41} = S(e_{42})$ (resp. $e_{52} = A(e_{51})$ and $e_{51} = T(e_{52})$; $e_{62} = A(e_{61})$ and $e_{61} = U(e_{62})$) we have that SA (resp. TA and UA) are the cycle transformations at e_{41} (resp. e_{51} and e_{61}) and that the cycle transformations are elliptic of order two (resp. two and three). Since $e_{72} = B(e_{71})$, $e_{73} = A(e_{72})$, $e_{74} = (SB)^{-1}(e_{73})$ and $e_{71} = U^{-1}(e_{74})$ (resp. $e_{82} = B(e_{81})$, $e_{83} = (SB)^{-1}(e_{82})$ and $e_{81} = T^{-1}(e_{83})$; $e_{92} = B^{-1}(e_{91})$, $e_{93} = (SB)(e_{92})$ and $e_{91} = S^{-1}(e_{93})$; $e_{10,2} = B^{-1}(e_{10,1})$, $e_{10,3} = A^{-1}(e_{10,2})$, $e_{10,4} = (SB)(e_{10,3})$ and $e_{10,1} = U(e_{10,4})$) we have that $U^{-1}\cdot(SB)^{-1}\cdot A\cdot B$ (resp. $T^{-1}\cdot(SB)^{-1}\cdot B$, $S^{-1}\cdot SB\cdot B^{-1}$, $U\cdot SB\cdot A^{-1}\cdot B^{-1}$) are the cycle transformations at e_{71} (resp. e_{81}, e_{91} and $e_{10,1}$) and that all cycle transformations are the identity mapping.

By the above we see that the conditions (1) through (5) in Theorem E are satisfied. The completeness condition (6) in Theorem E is easily shown by using Ratcliffe [7, Theorem 11.1.2]. Thus G^* is a Kleinian group of the first kind. By Lemma 5.2, G is a Kleinian group of the first kind and so a Jørgensen group.

We can easily calculate the volume of the polyhedron **P** constructed above, and so we omit the calculations. □

5.3. We will prove that $G_{\theta,k}$ is not discrete in the case of $\theta = \pi/4$ and $\sqrt{3}/2 < k < 1$. We have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/4,k} = \begin{pmatrix} ke^{\pi i/4} & i(k^2e^{\pi i/4} - e^{-\pi i/4}) \\ -ie^{\pi i/4} & ke^{\pi i/4} \end{pmatrix} \ (k \in \mathbf{R}).$$

We set $G_k = \langle A, B_k \rangle$.

We set $P_k := B_k^{-1}AB_kB_kAB_k^{-1}$. Then we have $J(A, P_k) = |2k-2|^2$. Thus $0 \leq J(A, P_k) < 1$ if and only if $1/2 < k < 3/2$. We easily see that the group G_k is an elementary group for k with $k = 1$ and G_k is a non-elementary group for k with $1/2 < k < 3/2$ ($k \neq 1$). By Theorem A we can see that G_k is a not discrete group for k with $\sqrt{3}/2 < k < 1$. □

6. Proof of the case of $\theta = \pi/6$

In this section we will prove that $G_{\theta,k}$ is not discrete in the case of $\theta = \pi/6$. We have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/6,k} = \begin{pmatrix} ke^{\pi i/6} & i(k^2e^{\pi i/6} - e^{-\pi i/6}) \\ -ie^{\pi i/6} & ke^{\pi i/6} \end{pmatrix} \ (k \in \mathbf{R}).$$

We set $G_k = \langle A, B_k \rangle$.

We set $E_k := AB_k^{-1}A^{-1}B_kA^{-1}B_k^{-1}AB_kA^{-1}B_kAB_k^{-1}$. Then we have $J(A, E_k) = |-1/2 - i(3\sqrt{3}/2 - 2k)|^2$. Thus $0 \leq J(A, E_k) < 1$ if and only if $\sqrt{3}/2 < k < \sqrt{3}$.

We note that G_k is a non-elementary group for k with $\sqrt{3}/2 < k \leq 1$. By Theorem A we can see that G_k is not a discrete group for k with $\sqrt{3}/2 < k \leq 1$. □

7. Proof of the case $\theta = \pi/3$

In this section we will prove that $G_{\theta,k}$ is not discrete in the case of $\theta = \pi/3$. We have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/3,k} = \begin{pmatrix} ke^{\pi i/3} & i(k^2e^{\pi i/3} - e^{-\pi i/3}) \\ -ie^{\pi i/3} & ke^{\pi i/6} \end{pmatrix} \ (k \in \mathbf{R}).$$

We set $G_k = \langle A, B_k \rangle$.

We set $F_k := AB_kA^{-1}B_k^{-1} \cdot AB_kAB_k^{-1} \cdot AB_k^{-1}A^{-1}B_k$.
Then we have $J(A, F_k) = |(\sqrt{3}k - 2) + i(k - \sqrt{3})|^2$. Thus $0 \leq J(A, F_k) < 1$ if and only if $\sqrt{3}/2 < k < \sqrt{3}$.

We note that G_k is a non-elementary group for k with $\sqrt{3}/2 < k \leq 1$. By Theorem A we can see that G_k is not a discrete group for k with $\sqrt{3}/2 < k \leq 1$. □

8. Proof of the case $\theta = \pi/2$

The main theorem (d) and (e) are proved by Sato-Yamada [9]. In this section we will prove that $G_{\theta,k}$ is not discrete in the case of $\theta = \pi/2$. For $\theta = \pi/2$, we have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/2,k} = \begin{pmatrix} ik & -(k^2+1) \\ 1 & ik \end{pmatrix} \quad (k \in \mathbf{R}).$$

We set $G_k = \langle A, B_k \rangle$.

We set $S_k := AB_k^{-1}AB_k$, $Q_k := AB_kAB_k^{-1}$, $V_k := A^{-1}S_k^{-1}A^{-1}Q_k^{-1}$. We can easily calculate V_k:

$$V_k = \begin{pmatrix} 2k^2-1 & 2ik-2ik^3 \\ 2ik & 2k^2-1 \end{pmatrix}.$$

We set

$$T_k = \left(\frac{1}{2\sqrt{1-k^2}}\right)^{1/2} \begin{pmatrix} 1 & -\sqrt{1-k^2} \\ 1 & \sqrt{1-k^2} \end{pmatrix} \quad (0 < k < 1).$$

Then for $0 < k < 1$ we have

$$A^* := T_kAT_k^{-1} = \frac{1}{2\sqrt{1-k^2}} \begin{pmatrix} 2\sqrt{1-k^2}-1 & 1 \\ -1 & 2\sqrt{1-k^2}+1 \end{pmatrix}$$

and

$$V_k^* := T_kV_kT_k^{-1} = \begin{pmatrix} (2k^2-1)-2ik\sqrt{1-k^2}-1 & 0 \\ 0 & (2k^2-1)+2ik\sqrt{1-k^2}-1 \end{pmatrix}.$$

We set $\cos\theta = k$. Then $e^{2i\theta} = (2k^2-1)+2ik\sqrt{1-k^2}$ and $e^{-2i\theta} = (2k^2-1)-2ik\sqrt{1-k^2}$.

Thus we have $J(A, V_k^n) = J(A^*, (V_k^*)^n) = |(-1+\cos 4n\theta)/(1-\cos 2\theta)|^2$. We have that $0 \le J(A, V_k^n) < 1$ if and only if $\cos(\pi/(2n-1)) < k < \cos(\pi/(2n+1))$ ($k \ne \cos\pi/2n$) ($n = 3, 4, \ldots$).

We note that if $k = \cos(\pi/2n)$ ($n = 3, 4, \ldots$), then $G_k = \langle A, V_k^n \rangle$ is an elementary group. G_k is a non-elementary group for k with $\cos(\pi/(2n-1)) < k < \cos(\pi/(2n+1))$ and $k \ne \cos(\pi/2n)$ ($n = 3, 4, \ldots$). By Theorem A, we can see that G_k is not a discrete group for k with $\cos(\pi/(2n-1)) < k < \cos(\pi/(2n+1))$ and $k \ne \cos(\pi/2n)$ ($n = 3, 4, \ldots$). □

9. Proof of the other cases

In this section we will see that $G_{\theta,k}$ is not discrete in the case where $0 < \theta < \pi/6$, $\pi/6 < \theta < \pi/4$, $\pi/4 < \theta < \pi/3$, $\pi/3 < \theta < \pi/2$, and $\sqrt{3}/2 < k \le 1$. The part follows from the following proposition.

Proposition (Li-Oichi-Sato [4]). *Let* $G_{\theta,k} = \langle A, B_{\theta,k} \rangle$ *be the group generated by* A *and* $B_{\theta,k}$. *If* $0 < \theta < \pi/6$, $\pi/6 < \theta < \pi/4$, $\pi/4 < \theta < \pi/3$, $\pi/3 < \theta < \pi/2$, *then* $G_{\theta,k} = \langle A, B_{\theta,k} \rangle$ *is not a discrete group for* $k \in \mathbf{R}$. □

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