GEVREY REGULARIZING EFFECT FOR THE INITIAL VALUE PROBLEM FOR A DISPERSIVE OPERATOR

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Introduction

Gevrey regularizing effect or analytic regularizing effect for (nonlinear) Schrödinger equations is studied in [1], [2], [7], [10], and Gevrey regularizing effect for the equations of Schrödinger type is studied in [3], [4], [5], [6], [9]. Especially, concerning Gevrey regularizing effect for nonlinear Schrödinger equations

$$
(1) \qquad \begin{cases} i\partial_t u + \Delta u = f(u), \\ \quad u(0, x) = u_0(x), \end{cases}
$$

the following result is obtained: Assume that the initial data $u_0(x)$ belongs to a Gevrey class of order s. Then, the solution $u(t, x)$ belongs to a Gevrey class of order $\max(s/2, 1)$ in $t \neq 0$ (in case $f(u)$ is a polynomial, for simplicity). A simple extension to a higher order of Δ is

$$
(2) \qquad \begin{cases} i\partial_t u + \Delta^l u = f(u), \\ \quad u(0, x) = u_0(x). \end{cases}
$$

In this case, we easily obtain that the solution $u(t, x)$ of (2) belongs to a Gevrey class of order $\max(s/(2l), 1)$ in $t \neq 0$ if the initial data $u_0(x)$ belongs to a Gevrey class of order s. In the present paper, we replace Δ^l in (2) by a semi-elliptic operator A and consider a dispersive equation

$$
(3) \qquad \begin{cases} Lu \equiv i\partial_t u + Au = f(u), \\ \quad u(0, x) = u_0(x). \end{cases}
$$

Let $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ be a vector of even numbers m_k and set $|\alpha : \mathbf{m}| = \sum_{k=1}^{n} \alpha_k/m_k$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then, the operator A is defined by

$$
A = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha D_x^\alpha,
$$

where a_α are real constants and $D_x = (D_{x_1}, \ldots, D_{x_n})$, $D_{x_k} = -i\partial/\partial x_k$. We assume

(A.0) A is semi-elliptic,

that is, $a(\xi) = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha \xi^\alpha$ is not zero for $\xi \neq 0$.

Now, we introduce some notations. For $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, we set $m = \max\{m_1, \ldots, m_n\}$, $\rho = (\rho_1, \ldots, \rho_n)$ with $\rho_k = m/m_k$ and $\rho'_k = 1/m_k$. Write $P = t\partial_t + \sum_{k=1}^n \rho'_k x_k \partial_{x_k}$. Then, we have $[L, P] = L$. Denote $\lambda(\xi) = \left(1 + \sum_{k=1}^n \xi_k^{m_k}\right)^{1/m}$, a basic weight function, and denote by $H_{\lambda,r} = \{u \in \mathcal{S}; \lambda(\xi)^r \hat{u}(\xi) \in L_2\}$ a Sobolev space with respect to $\lambda(\xi)$, whose norm $\|u\|_{\lambda,r}$ is $\|\lambda(\xi)^r \hat{u}(\xi)\|_{L_2}$. Let S_λ^κ be a class of symbols $q(x, \xi)$ satisfying

$$|\partial_\xi^\alpha \partial_x^\beta q(x, \xi)| \leq C_{\alpha,\beta} \lambda(\xi)^{\kappa-|\alpha|}$$

and we also use same notations S_λ^κ to a class of pseudo-differential operators $Q = q(X, D_x)$ with a symbol $q(x, \xi)$ in S_λ^κ. Then, $Q \in S_\lambda^\kappa$ maps $H_{\lambda,r}$ to $H_{\lambda,r-\kappa}$ (see [8]).

Now, we state our main results. First, we consider the case the non-linear term $f(u)$ is a polynomial of u and $\bar{u}$. In this case we take n_0 such that n_0 satisfies $n_0 > \mu_0/2$ for $\mu_0 = \sum_{k=1}^n \rho_k \equiv \sum_{k=1}^n m/m_k$. In the following, we denote $P_0 = \sum_{k=1}^n \rho'_k x_k \partial_{x_k}$.

Theorem 1. *Assume* (A.0) *and that* $f(u)$ *is a polynomial of* u *and* $\bar{u}$ *with* $f(0) = 0$. *Then, for the initial data* $u_0(x)$ *satisfying*

$$\|P_0^l u_0\|_{\lambda,n_0} \leq C M_1^l l!^s, \tag{4}$$

there exists a positive constant T *such that the equation* (3) *has a unique solution* $u(t, x)$ *in* $C^0([0, T]; H_{\lambda,n_0}) \cap C^1([0, T]; H_{\lambda,n_0-m})$ *and it satisfies* $P^l u(t, x) \in C^0([0, T]; H_{\lambda,n_0})$ *with an estimate*

$$\sup_t \|P^l u(t, x)\|_{\lambda,n_0} \leq C M^l l!^s \tag{5}$$

for any l.

For the regularizing effect with respect to the space variables we have

Theorem 2. *Let* $\sigma = \max(s/m, 1)$ *and assume that the assumption in* Theorem 1 *are valid. Then, under the condition* (4) *the solution* $u(t, x)$ *of* (3) *satisfies the following property: For any* C^∞-*function* $\varphi(x)$ *there exist constants* $C = C_\varphi$ *and* $M = M_\varphi$ *such that*

$$\|\varphi(x)\partial_x^\alpha u(t, x)\|_{\lambda,n_0} \leq C M^{|\alpha|} t^{-\nu} \alpha!^{\rho\sigma} \tag{6}$$

holds with $\nu = [\rho \cdot \alpha]_*$.

Here, $\alpha!^{\rho\sigma} = \alpha_1!^{\rho_1\sigma} \alpha_2!^{\rho_2\sigma} \cdots \alpha_n!^{\rho_n\sigma}$, $\rho \cdot \alpha = \rho_1\alpha_1 + \cdots + \rho_n\alpha_n$ and $[r]_*$ is the smallest integer ν such that $\nu \geq r$.

REMARK 1.   When $s \leq m$ and $m_k = m$, then $u(t, x)$ is locally analytic with respect to x_k.

In case that the nonlinear term $f(u)$ is a general function of u and $\bar{u}$, we introduce the following assumption (depending on s).

(A.1)$_s$ For every positive number K, there exist constants $C = C_K$ and $M_2 = M_{2,K}$ such that

$$|\partial_u^k \partial_{\bar{u}}^{k'} f(u)| \leq C M_2^{k+k'} k!^s k'!^s \qquad \text{for } |u| \leq K$$

holds, where $\partial_{\bar{u}}$ is the differentiation with respect to the complex conjugate of u.

In this case we assume that the constant n_0 satisfies $n_0 > \mu_0/2$ and $n_0 m_k/m$ are integers for any k.

Theorem 3.   *Assume* (A.0) *and* (A.1)$_s$ *with* $f(0) = 0$. *Then, for the initial data* $u_0(x)$ *satisfying* (4), *there exists a positive constant* T *such that the equation* (3) *has a unique solution* $u(t, x)$ *in* $C^0([0, T]; H_{\lambda, n_0}) \cap C^1([0, T]; H_{\lambda, n_0 - m})$, *and it satisfies* $P^l u(t, x) \in C^0([0, T]; H_{\lambda, n_0})$ *with the estimate* (5) *for any* l.

Theorem 4.   *Let* σ *satisfy* $\max(s/m, 1) \leq \sigma \leq s$ *and assume that* (A.0) *and* (A.1)$_\sigma$ *with* $f(0) = 0$. *Then, under the condition* (4) *the solution* $u(t, x)$ *of* (3) *satisfies the following property: for any* C^∞-*function* $\varphi(x)$ *there exist constants* $C = C_\varphi$ *and* $M = M_\varphi$ *such that* (6) *holds with* $\nu = [\rho \cdot \alpha]_*$.

REMARK 2.   In $m = 2$, the problem (3) coincides with the problem (1) and in this case Theorems 3 and 4 are already proved in [7]. So, in the following, we always assume $m \geq 4$.

REMARK 3.   In Theorem 3 and Theorem 4 we assume $n_0 m_k/m$ are integers for any k. We conjecture that this assumption can be removed if we treat the idea of Bessov spaces.

The outline of the present paper is as follows. In Section 1 we give preliminaries. In Section 2 we show the existence of the solution of (3) by using the idea in [7]. Finally, in Section 3, we show the regularizing effect for the solution of (3). Since the operator A is semi-elliptic, we use not homogeneous factorials with respect to the space variables. So, we emphasize that we have to pay attention to very carefully the power of constants corresponding to radius of convergence, the power of factorial and so on, in order to proceed the proof of Theorem 2 and Theorem 4.

1. Preliminary

As in Introduction, we always denote $\mu_0 = \sum_{j=1}^{n} \rho_j \equiv \sum_{j=1}^{n} m/m_j$.

Lemma 1.1. *Let r satisfy $r > \mu_0/2$. Then, we have $\mathcal{B}^0 \subset H_{\lambda,r}$.*

Proof.   Using polar coordinates, we know that $\lambda(\xi)^{-2r}$ is integrable in $\mathbb{R}^n$. Hence, we obtain for $u \in \mathcal{S}$

$$(1.1) \qquad |u(x)| = \left| \int \lambda(\xi)^{-r} \{\lambda(\xi)^r \hat{u}(\xi)\} \, d\!\!\!/\xi \right| \leq \left\{ \int \lambda(\xi)^{-2r} \, d\!\!\!/\xi \right\}^{1/2} \|u\|_{\lambda,r}$$

$$\leq C_1 \|u\|_{\lambda,r}.$$

This implies $\mathcal{B}^0 \subset H_{\lambda,r}$. □

In (1.1) and in what follows, we use $d\!\!\!/\xi = (2\pi)^{-n} \, d\xi$.

Proposition 1.2. *Let n_0 satisfy $n_0 > \mu_0/2$. Then, we have for $u, v \in H_{\lambda,n_0}$*

$$(1.2) \qquad\qquad\qquad \|uv\|_{\lambda,n_0} \leq C_2 \|u\|_{\lambda,n_0} \|v\|_{\lambda,n_0}.$$

Proof.   We may only prove (1.2) for $u, v \in \mathcal{S}$. Denote $|\eta|_{\mathbf{m}} = \sum_{j=1}^{n} |\eta_j|^{m_j/m}$. Then, we can find a constant c_0 such that

$$|\lambda(\xi - \eta) - \lambda(\xi)| \leq \frac{1}{2}\lambda(\xi) \qquad \text{for} \quad |\eta|_{\mathbf{m}} \leq c_0 \lambda(\xi).$$

This implies

$$\int \frac{1}{\lambda(\xi - \eta)^{2n_0} \lambda(\eta)^{2n_0}} \, d\!\!\!/\eta$$

$$= \int_{|\eta|_{\mathbf{m}} \leq c_0 \lambda(\xi)} \frac{1}{\lambda(\xi - \eta)^{2n_0} \lambda(\eta)^{2n_0}} \, d\!\!\!/\eta + \int_{|\eta|_{\mathbf{m}} \geq c_0 \lambda(\xi)} \frac{1}{\lambda(\xi - \eta)^{2n_0} \lambda(\eta)^{2n_0}} \, d\!\!\!/\eta$$

$$\leq C_2' \lambda(\xi)^{-2n_0} \int_{|\eta|_{\mathbf{m}} \leq c_0 \lambda(\xi)} \frac{1}{\lambda(\eta)^{2n_0}} \, d\!\!\!/\eta + C_2' \lambda(\xi)^{-2n_0} \int_{|\eta|_{\mathbf{m}} \geq c_0 \lambda(\xi)} \frac{1}{\lambda(\xi - \eta)^{2n_0}} \, d\!\!\!/\eta$$

$$\leq C_2'' \lambda(\xi)^{-2n_0}.$$

Now, set $\hat{u}_{n_0}(\xi) = \lambda(\xi)^{n_0} \hat{u}(\xi)$ and $\hat{v}_{n_0}(\xi) = \lambda(\xi)^{n_0} \hat{v}(\xi)$ for $u, v \in \mathcal{S}$. Then, we have

$$|\widehat{uv}(\xi)|^2 = \left| \int \lambda(\xi - \eta)^{-n_0} \lambda(\eta)^{-n_0} \hat{u}_{n_0}(\xi - \eta) \hat{v}_{n_0}(\eta) \, d\!\!\!/\eta \right|^2$$

$$\leq \int \lambda(\xi - \eta)^{-2n_0} \lambda(\eta)^{-2n_0} \, d\!\!\!/\eta \int |\hat{u}_{n_0}(\xi - \eta)|^2 |\hat{v}_{n_0}(\eta)|^2 \, d\!\!\!/\eta$$

$$\leq C_2'' \lambda(\xi)^{-2n_0} \int |\hat{u}_{n_0}(\xi - \eta)|^2 |\hat{v}_{n_0}(\eta)|^2 \, d\!\!\!/\eta.$$

Multiplying both sides by $\lambda(\xi)^{2n_0}$ and integrating with respect to ξ, we get (1.2).
□

Lemma 1.3. *Let n_0 satisfy $n_0 > \mu_0/2$ and $n_{0,k} \equiv n_0 m_k/m$ be integers for any k. Then, if $u \in L^2$ satisfies $D_{x_k}^{n_{0,k}} u \in L^2$ for $k = 1, \ldots, n$, we have $u \in H_{\lambda,n_0}$ and*

$$(1.3) \qquad \|u\|_{\lambda,n_0} \leq C_3 \left\{ \|u\| + \sum_{k=1}^n \|D_{x_k}^{n_{0,k}} u\| \right\}.$$

We note that, for $u \in H_{\lambda,n_0}$, we have $D_x^\alpha u \in L_2$ with α satisfying $\rho \cdot \alpha \leq n_0$, since $D_x^\alpha \in S_\lambda^{\rho \cdot \alpha}$. Hence, we get

Corollary 1.4. *Suppose that n_0 satisfies $n_0 > \mu_0/2$ and that $n_{0,k} \equiv n_0 m_k/m$ be integers for any k. Then, the norm $\|u\|_{\lambda,n_0}$ is equivalent to $\|u\| + \sum_{k=1}^n \|D_{x_k}^{n_{0,k}} u\|$.*

Proof of Lemma. Using Fourier transformation, we have

$$\|u\|_{\lambda,n_0}^2 = \|\lambda(\xi)^{n_0} \hat{u}(\xi)\|^2 = \int \left(1 + \sum_{k=1}^n \xi_k^{m_k} \right)^{2n_0/m} |\hat{u}(\xi)|^2 \, d\xi$$

$$\leq C_3' \int \left(1 + \sum_{k=1}^n \xi_k^{2n_{0,k}} \right) |\hat{u}(\xi)|^2 \, d\xi = C_3' \left\{ \|u\|^2 + \sum_{k=1}^n \|D_{x_k}^{n_{0,k}}\|^2 \right\}.$$

Hence, we get (1.3). □

Lemma 1.5. *Let r satisfy $r > \mu_0$ and $q_1, q_2, \ldots, q_{\nu+1}$ be nonnegative constants such that $q_1 + \cdots + q_{\nu+1} \leq r$. Then, we have*

$$(1.4) \qquad \iint \frac{1}{\lambda(\xi - \eta^1)^{r-q_1}} \prod_{j=2}^{\nu+1} \frac{1}{\lambda(\eta^{j-1} - \eta^j)^{r-q_j}} \, d\tilde{\eta}^\nu \leq C_4 \qquad (\eta^0 = \xi, \ \eta^{\nu+1} = 0),$$

where $\tilde{\eta}^\nu = (\eta^1, \ldots, \eta^\nu)$ and $d\tilde{\eta}^\nu = d\eta^1 \cdots d\eta^\nu$.

Proof. For $\nu = 2$ we have

$$\int \frac{1}{\lambda(\xi - \eta)^{r-q_1} \lambda(\eta)^{r-q_2}} \, d\eta$$

$$\leq C_4' \int_{|\eta|_{\mathbf{m}} \leq \lambda(\xi - \eta)} \frac{1}{\lambda(\eta)^{2r-q_1-q_2}} \, d\eta + C_4' \int_{|\eta|_{\mathbf{m}} \geq \lambda(\xi - \eta)} \frac{1}{\lambda(\xi - \eta)^{2r-q_1-q_2}} \, d\eta$$

$$\leq C_4''.$$

Hence, we get (1.4) for $\nu = 2$. For $\nu \geq 3$ we get (1.4) by induction on ν. □

Proposition 1.6. *Let n_0 satisfy $n_0 > \mu_0/2$ and $n_0 m_k/m$ be an integer for each k. Suppose that $g(u) \in C^\infty(\mathbb{C};\mathbb{C})$ satisfies $|\partial_u^\nu \partial_{\bar{u}}^{\nu'} g(u)| \leq M_K$ for $\nu + \nu' \leq n_0$ and $|u| \leq K$. Then, for $u,\ v \in H_{\lambda,n_0}$, we have $g(u)v \in H_{\lambda,n_0}$ and*

$$(1.5) \qquad \|g(u)v\|_{\lambda,n_0} \leq C_5 M_K G(\|u\|_{\lambda,n_0})\|v\|_{\lambda,n_0},$$

where $G(\cdot)$ is a polynomial of order n_0 and C_5 is a constant depending only on n and n_0.

Proof.   From the differentiation of composite function we have

$$
\begin{aligned}
D_{x_k}^{n_{0,k}}\{g(u)v\} &= g(u)D_{x_k}^{n_{0,k}}v \\
&+ \sum_{l=1}^{n_{0,k}} \sum_{\nu=1}^{l} \sum_{\nu'+\nu''=\nu} \frac{n_{0,k}!}{(n_{0,k}-l)!\nu'!\nu''!} \partial_u^{\nu'}\partial_{\bar{u}}^{\nu''} g(u) \\
&\qquad \times \sum_{\substack{l_1+\cdots+l_\nu=l \\ l_j \geq 1}} \prod_{j=1}^{\nu'} \frac{1}{l_j!} D_{x_k}^{l_j} u \prod_{j=\nu'+1}^{\nu} \frac{1}{l_j!} D_{x_k}^{l_j} \bar{u} \cdot D_{x_k}^{n_{0,k}-l}v.
\end{aligned}
$$

Take a constant K such that $|u(x)| \leq K$, which is assured by Lemma 1.1. Then, we have

$$(1.6) \qquad \|D_{x_k}^{n_{0,k}}\{g(u)v\}\| \leq C_5' M_K \|v\|_{\lambda,n_0}$$

$$+ C_5' M_K \sum_{l=1}^{n_{0,k}} \sum_{\nu=1}^{l} \sum_{\substack{l_1+\cdots+l_\nu=l \\ l_j \geq 1}} \Big\| \prod_{j=1}^{\nu} D_{x_k}^{l_j} u \cdot D_{x_k}^{n_{0,k}-l}v \Big\|.$$

In order to estimate $\|\prod_{j=1}^{\nu} D_{x_k}^{l_j} u \cdot D_{x_k}^{n_{0,k}-l} v\|$ we set for $l_1, \ldots, l_\nu$ satisfying $l_1 + \cdots + l_\nu = n_{0,k} - l$

$$\Lambda(\xi, \tilde{\eta}^\nu) = \prod_{j=1}^{\nu} \lambda(\eta^{j-1} - \eta^j)^{n_0 - \rho_k l_j} \cdot \lambda(\eta^\nu)^{n_0 - \rho_k(n_{0,k}-l)},$$

with $\tilde{\eta}^\nu = (\eta^1, \ldots, \eta^\nu)$, $\eta^0 = \xi$ and $\rho_k = m/n_{0,k}$. Then, we have

$$(1.7) \qquad \Big| \mathcal{F}\Big[\prod_{j=1}^{\nu} D_{x_k}^{l_j} u \cdot D_{x_k}^{n_{0,k}-l} v \Big](\xi) \Big|$$

$$= \Big| \iint \prod_{j=1}^{\nu} \Big\{ (\eta_k^{j-1} - \eta_k^j)^{l_j} \hat{u}(\eta^{j-1} - \eta^j) \Big\} \Big\{ (\eta_k^\nu)^{n_{0,k}-l} \hat{v}(\eta^\nu) \Big\} d\tilde{\eta}^\nu \Big|$$

$$\leq \Big| \iint \Lambda(\xi, \tilde{\eta}^\nu)^{-1} \prod_{j=1}^{\nu} \hat{u}_{n_0}(\eta^{j-1} - \eta^j) \cdot \hat{v}_{n_0}(\eta^\nu) d\tilde{\eta}^\nu \Big|$$

$$\leq \left\{ \iint \Lambda(\xi, \tilde{\eta}^\nu)^{-2} \, d\tilde{\eta}^\nu \right\}^{1/2}$$

$$\times \left\{ \iint \prod_{j=1}^{\nu} |\hat{u}_{n_0}(\eta^{j-1} - \eta^j)|^2 \cdot |\hat{v}_{n_0}(\eta^\nu)|^2 \, d\tilde{\eta}^\nu \right\}^{1/2}.$$

Since an inequality

$$\int \Lambda(\xi, \tilde{\eta}^\nu)^{-2} \, d\tilde{\eta}^\nu \leq C_4$$

holds from Lemma 1.5, we get

$$\left\| \prod_{j=1}^{\nu} D_{x_k}^{l_j} u \cdot D_{x_k}^{n_{0,k}-l} v \right\|^2 \leq C_4 \|u\|_{\lambda,n_0}^{2\nu} \|v\|_{\lambda,n_0}^2$$

by squaring both sides of (1.7) and integrating with respect to ξ. This yields

$$\| D_{x_k}^{n_{0,k}} \{g(u)v\} \| \leq C_6 M_K G(\|u\|_{\lambda,n_0}) \|v\|_{\lambda,n_0}$$

with (1.6). Now, we use Corollary 1.4. Then, we have

$$\|g(u)v\|_{\lambda,n_0} \leq C_7 \left\{ \|g(u)v\| + \sum_{k=1}^{n} \| D_{x_k}^{n_{0,k}} \{g(u)v\} \| \right\}$$

$$\leq (n+1)C_7 C_6 M_K G(\|u\|_{\lambda,n_0}) \|v\|_{\lambda,n_0}.$$

This proves Lemma 1.6. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Finally, we quote two lemmas from [7].

Lemma 1.7. *There exists a constant C_8 without depending on l such that*

$$(1.8) \qquad\qquad \sum_{l'+l''=l} \frac{1}{(l'+1)^2(l''+1)^2} \leq C_8 \frac{1}{(l+1)^2}.$$

Lemma 1.8. *For a multi-index β and an integer l we assume that the integers $\mu_j \ (\geq 1) \ (j = 1, \ldots, k)$ satisfy $\mu_1 + \cdots + \mu_k = |\beta| + l$. Then, we have*

$$\beta! l! \sum_{\substack{\beta^1+\cdots+\beta^k=\beta \\ l_1+\cdots+l_k=l \\ |\beta^j|+l_j=\mu_j}} \prod_{j=1}^{k} \frac{(|\beta^j|+l_j)!}{\beta^j! l_j!} = (|\beta|+l)!.$$

2.   Existence of the solution

In this section we prove Theorem 3 by using Proposition 1.2 and Proposition 1.6. Then, we can prove Theorem 1 by the same method only by using Proposition 1.2. So, throughout this section we always assume that n_0 satisfies $n_0 > \mu_0/2$ and that $n_0 m_k/m$ are integers for any k. Since we can prove Theorem 3 by the same method of proving Theorem 1.1 in [7], we only give outline of the proof.

First, for a vector field Q with analytic coefficients in $\mathbb{R}^n$ or $[0, T] \times \mathbb{R}^n$ we define a function space $G_R^s(Q; H_{\lambda, n_0})$ by

$$G_R^s(Q; H_{\lambda, n_0}) = \{u \in H_{\lambda, n_0} \,;\, \|u\|_{G_R^s(Q; H_{\lambda, n_0})} < \infty\}$$

or

$$G_R^s(Q; H_{\lambda, n_0}) = \{u = u(t) \in C^\infty([0, T] \,;\, H_{\lambda, n_0}) \,;$$
$$\|u(t)\|_{G_R^s(Q; H_{\lambda, n_0})} < \infty \text{ for any } t\}$$

with

$$\|u\|_{G_R^s(Q; H_{\lambda, n_0})} = \|u\|_{\lambda, n_0} + \sum_{l=1}^\infty \frac{R^{l-1} \|Q^l u\|_{\lambda, n_0}}{l!(l-1)^{s-1}}.$$

Especially, for $P = t\partial_t + \sum_{k=1}^n \rho_k' x_k \partial_{x_k}$ we denote $G_R^s(P; H_{\lambda, n_0})$ by $\mathcal{G}^s(R)$, $\|u\|_{G_R^s(P; H_{\lambda, n_0})}$ by $\|u\|_{\mathcal{G}^s(R)}$ and set

$$|||u|||_{\mathcal{G}^s(R)} = \sup_{t \in [0, T]} \|u\|_{\mathcal{G}^s(R)}.$$

Next, we write $\|u\|_{\mathcal{G}^s(R)}' = \|u\|_{\mathcal{G}^s(R)} - \|u\|_{\lambda, n_0}$. Then, as in Lemma 2.3 in [7], we have

Lemma 2.1.   *Let $f(u)$ be a C^∞-function satisfying* $(A.1)_s$ *and suppose that $u \equiv u(t, x)$ belongs to $G_R^s(P; H_{\lambda, n_0})$ for a constant R. Then, there exists a constant R_1 depending on R such that an inequality*

$$\|f(u)\|_{\mathcal{G}^s(R_1)}' \leq \frac{C_9 \|u\|_{\mathcal{G}^s(R_1)}'}{1 - C_{10} R_1 \|u\|_{\mathcal{G}^s(R_1)}'}$$

holds with constants C_9 and C_{10} depending only f, $\|u\|_{\lambda, n_0}$, n_0 and n.

Now, we give the outline of the proof of Theorem 3.

Proof of Theorem 3.   Consider a linearized equation with respect to (3):

$$(2.1) \qquad \begin{cases} Lu \equiv i\partial_t u + Au = f(v), \\ \quad u(0, x) = u_0(x). \end{cases}$$

Then, the solution of (2.1) is written as

$$u = e^{itA}u_0 + i^{-1}\int_0^t e^{i(t-s)A}f(v(s))\,ds$$

with the evolution operator e^{itA} for $i\partial_t + A$. Since A is defined by $A = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha D_x^\alpha$ with real coeffients a_α, an operator e^{itA} maps $H_{\lambda,r}$ to itself for any r with the estimate $\|e^{itA}u\|_{\lambda,r} = \|u\|_{\lambda,r}$. This implies

$$(2.2)\qquad \|u\|_{\lambda,n_0} \le \|u_0\|_{\lambda,n_0} + T\|f(v)\|_{\lambda,n_0}.$$

Now, we use $[L, P] = L$. Then, from (2.1) we have with $P_0 = \sum_{k=1}^n \rho_k' x_k \partial_{x_k}$

$$\begin{cases} LP^l u = (P+1)^l f(v), \\ (P^l u)|_{t=0} = P_0^l u_0, \end{cases}$$

which yields

$$\|P^l u\|_{\lambda,n_0} \le \|P_0^l u_0\|_{\lambda,n_0} + T\|(P+1)^l\{f(v)\}\|_{\lambda,n_0}.$$

Combine this with (2.2) and Lemma 2.1. Then, for $u_0 \in G_R^s(P_0; H_{\lambda,n_0})$, there exists a constant R_1 $(< R)$ such that

$$(2.3)\qquad \begin{aligned} \|u\|_{\mathcal{G}^s(R_1)} &\le \|u_0\|_{G_{R_1}^s(P_0;H_{\lambda,n_0})} + T\|f(v)\|_{G_{R_1}^s(P+1;H_{\lambda,n_0})} \\ &\le \|u_0\|_{G_R^s(P_0;H_{\lambda,n_0})} + e^{R_1}T\|f(v)\|_{\mathcal{G}^s(R_1)} \\ &\le \|u_0\|_{G_R^s(P_0;H_{\lambda,n_0})} + e^{R_1}T\{\|f(v)\|_{\lambda,n_0} + \|f(v)\|_{\mathcal{G}^s(R_1)}'\} \\ &\le \|u_0\|_{G_R^s(P_0;H_{\lambda,n_0})} + e^{R_1}T\left\{C_{11}\|v\|_{\lambda,n_0} + \frac{C_9\|v\|_{\mathcal{G}^s(R_1)}'}{1 - C_{10}R_1\|v\|_{\mathcal{G}^s(R_1)}'}\right\}. \end{aligned}$$

Set $K = 2\|u_0\|_{G_R^s(P_0;H_{\lambda,n_0})}$ and write $\mathcal{G}^s(R, K) = \{u \in C^\infty([0, T]; \mathcal{G}^s(R)); \||u|\|_{\mathcal{G}^s(R)} \le K\}$. Then, denoting by S the mapping which corresponds v to the solution u of (2.1), the inequality (2.3) shows that S maps $\mathcal{G}^s(R_1, K)$ into $\mathcal{G}^s(R_1, K)$, if we take T small enough. Moreover, using Lemma 2.1 again, we can prove that

$$\||Su - Sv|\|_{\mathcal{G}^s(R_1)} \le C_{12}T\||u - v|\|_{\mathcal{G}^s(R_1)} \qquad \text{for } u, v \in \mathcal{G}^s(R, K)$$

with a constant C_{12} independent of T, u and v. Hence, if we retake a constant T such that $C_{12}T < 1$, S is a contraction mapping from $\mathcal{G}^s(R_1, K)$ to $\mathcal{G}^s(R_1, K)$, and we get a solution of (3). This proves Theorem 3. $\qquad\square$

3. Local Gevrey regularizing property

As in Section 2 we only prove Theorem 4. Then, we can prove Theorem 2 by the same method. So, throughout this section we always assume that n_0 satisfies $n_0 >$

$\mu_0/2$ and that $n_0 m_j/m$ are integers for any j. Now, let $u(t, x)$ be a solution constructed in Theorem 3. Then, since $[L, P] = L$, we have

$$LP^l u = (P + 1)^l \{f(u)\},$$

which implies

$$(3.1) \qquad AP^l u = -i\partial_t P^l u + (P + 1)^l \{f(u)\}$$

$$= -i\frac{1}{t} P^{l+1} u + i\frac{1}{t} \sum_{k=1}^{n} \rho'_k x_k \partial_{x_k} P^l u + (P + 1)^l \{f(u)\},$$

since $\partial_t = (1/t)P - (1/t) \sum_{k=1}^{n} \rho'_k x_k \partial_{x_k}$.

Proposition 3.1. *Let $u(t, x)$ be a solution of* (3). *Then, for an integer N_0 and a C_0^∞-function $\varphi(x)$, there exist constants C_{13} and C_{14} such that*

$$(3.2) \qquad \|\varphi(x) P^l u\|_{\lambda, n_0+1} \le C_{13} M^l t^{-1} (l+1)! \, l!^{s-1} \cdot \frac{1}{(l+1)^2},$$

$$(3.3) \qquad \|\varphi(x) P^l u\|_{\lambda, n_0+\nu} \le C_{14} M^{l+\nu-2} t^{1-\nu} (l+\nu-1)! \, (l+\nu-2)!^{s-1} \cdot \frac{1}{(l+1)^2}$$

hold for $2 \le \nu \le N_0 m$ in (3.3) *and for all integer l in* (3.2) *and* (3.3).

For the proof we prepare two lemmas.

Lemma 3.2. *Assume that* (3.2) *and* (3.3) *hold for $2 \le \nu \le Nm$. Then, we have for a C_0^∞-function $\chi(x)$ with $\chi \Subset \varphi$*

$$(3.4) \qquad \|\chi(x) P^l \{f(u)\}\|_{\lambda, n_0+Nm} \le C_{15} M^{l+Nm-1} t^{-Nm}$$

$$\times (l+Nm)! \, (l+Nm-1)!^{s-1} \cdot \frac{1}{(l+1)^2}$$

with a constant C_{15} depending on $\varphi(x)$, $\chi(x)$ and N and independent on l.

Here, for two C_0^∞-functions $\varphi(x)$ and $\chi(x)$, $\chi \Subset \varphi$ means that $\varphi(x) = 1$ on $\operatorname{supp} \chi$.

Lemma 3.3. *Assume that the following hold for C_0^∞-functions $\psi(x)$ and $\varphi(x)$ with $\varphi \Subset \psi$*

$$(3.5) \qquad \|\psi(x) P^l u\|_{\lambda, \mu} \le C_{16} M^{l+m'-1} t^{-j_1} (l+m')! \, (l+m'-1)!^{s-1} \cdot \frac{1}{(l+1)^2},$$

$$(3.6) \qquad \|\varphi(x) P^l \{f(u)\}\|_{\lambda, \mu} \le C_{17} M^{l+m''-1} t^{-j_2} (l+m'')! \, (l+m''-1)!^{s-1} \cdot \frac{1}{(l+1)^2}.$$

Then, we have

$$(3.7) \qquad \|\varphi(x)^\nu P^l u\|_{\lambda,\mu+\nu} \leq C_{18,\nu} M^{l+\tilde{m}-1} t^{-j'}(l+\tilde{m})!\,(l+\tilde{m}-1)!^{s-1} \cdot \frac{1}{(l+1)^2}$$

$$for \quad \nu \leq m,$$

with $j' = (j_1+1) \vee j_2$ *(for* $\nu \leq m/2$*),* $j' = \{(j_1+1) \vee j_2\} + 1$ *(for* $\nu > m/2$*) and* $\tilde{m} = (m'+1) \vee m''$*, where* $j_1 \vee j_2 = \max(j_1, j_2)$*.*

In order to fix the constant M in (3.2)–(3.7), we prepare

$$(3.8) \qquad \|P^l u\|_{\lambda,n_0} \leq C_{19} M^{l-1} l!\,(l-1)!^{s-1} \cdot \frac{1}{(l+1)^2},$$

$$(3.9) \qquad \|P^l\{f(u)\}\|_{\lambda,n_0} \leq C_{20} M^{l-1} l!\,(l-1)!^{s-1} \cdot \frac{1}{(l+1)^2},$$

$$(3.10) \qquad |\partial_u^{k'} \partial_{\tilde{u}}^{k''} f(u)| \leq C_{21} M_1^{k'+k''} k'!\,k''!\,(k'+k''-1)!^{s-1},$$

which is assured by the assumption and the result in the previous section. Then, the constant M in (3.2)–(3.7) is always taken such that (3.8), (3.9) and

$$(3.11) \qquad\qquad M_1 C_2 C_8 C_{19} < M$$

hold with the constants C_2 and C_8 in (1.2) and (1.8). Now, we start to prove Proposition 3.1 and Lemmas 3.2–3.3. First, admitting Lemmas 3.2 and 3.3, we prove Proposition 3.1.

    Proof of Proposition 3.1.   From (3.8) and (3.9), we have (3.5) and (3.6) with $j_1 = j_2 = 0$, $m' = m'' = 0$ and $\mu = n_0$. In this case we can take $\psi(x) = 1$ instead of the function with compact support. Hence, applying Lemma 3.3, we get for any C_0^∞-function $\varphi_1(x)$ and for any $\nu \leq m$

$$\|\varphi_1(x)^\nu P^l u\|_{\lambda,n_0+\nu} \leq C_{22,\nu} M^l t^{-j'}(l+1)!\,l!^{s-1} \cdot \frac{1}{(l+1)^2}$$

with $j' = 1$ $(\nu \leq m/2)$ and $j' = 2$ $(\nu > m/2)$. This implies (3.2) and (3.3) for $2 \leq \nu \leq m$, since $m \geq 4$. In order to prove (3.3) for $m < \nu \leq N_0 m$, we take C_0^∞-functions $\varphi_N(x)$, $N = 2, \ldots, N_0$ satisfying

$$\varphi_1 \Supset \varphi_2 \Supset \cdots \Supset \varphi_{N_0}$$

and prove

$$(3.12) \qquad \|\varphi_N(x) P^l u\|_{\lambda,n_0+\nu} \leq C_{23,N} M^{l+\nu-2} t^{1-\nu}(l+\nu-1)!\,(l+\nu-2)!^{s-1} \cdot \frac{1}{(l+1)^2}$$

$$for \quad 2 \leq \nu \leq Nm$$

by induction on N. Now, we assume (3.12). Then, from (3.2) and (3.12) we get (3.4) with $\chi(x) = \varphi_{N+1}(x)$ from Lemma 3.2. This implies (3.6) with $\varphi(x) = \varphi_{N+1}(x)$, $\mu = n_0 + Nm$, $j_2 = Nm$ and $m'' = Nm$. Moreover, from (3.12) with $\nu = Nm$, an inequality (3.5) holds with $\psi(x) = \varphi_N(x)$, $\mu = n_0 + Nm$, $j_1 = Nm - 1$ and $m' = Nm - 1$. Hence, from Lemma 3.3 we get

$$\|\varphi_{N+1}(x)^\nu P^l u\|_{\lambda, n_0 + Nm + \nu} \leq C_{24,\nu} M^{l+Nm-1} t^{-j'} (l + Nm)! \, (l + Nm - 1)!^{s-1} \cdot \frac{1}{(l+1)^2}$$

$$\text{for} \quad \nu \leq m$$

with $j' = Nm$ ($\nu \leq m/2$), $j' = Nm + 1$ ($\nu > m/2$). This proves that the inequality (3.12) holds for $Nm + 1 \leq \nu \leq (N+1)m$ with φ_N replaced by φ_{N+1}, and hence, we get (3.12) with N replaced by $N+1$. Summing up, by the induction on N we get (3.3) for $2 \leq \nu \leq N_0 m$ from (3.12). □

Proof of Lemma 3.2. Using

$$\lambda(\xi)^{Nm} = \sum_{|\beta'| \leq N} \frac{N!}{(N - |\beta'|)! \, \beta'!} \xi_1^{\beta_1' m_1} \cdots \xi_n^{\beta_n' m_n},$$

we write

$$\|\chi(x) P^l \{f(u)\}\|_{\lambda, n_0 + Nm} \leq C_{25} \sum_{|\beta : \mathbf{m}| \leq N} \|D_x^\beta \chi P^l \{f(u)\}\|_{\lambda, n_0}$$

$$\leq C_{25} \sum_{|\beta : \mathbf{m}| \leq N} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \|(D^{\beta - \gamma} \chi) D_x^\gamma P^l \{f(u)\}\|_{\lambda, n_0}.$$

For convenience of the notation below, we set $\tilde{\varphi}_0(x) = 1$ and $\tilde{\varphi}_\gamma(x) = \varphi(x)$ for $\gamma \neq 0$. Then, we have from (3.8), (3.2) and (3.3) for $2 \leq \nu \leq Nm$

$$\begin{cases} \|\tilde{\varphi}_0(x) P^l u\|_{\lambda, n_0} \leq C_{19} M^{l-1} l! \, (l-1)!^{s-1} \cdot \frac{1}{(l+1)^2}, \\[2mm] \|\tilde{\varphi}_\gamma(x) P^l u\|_{\lambda, n_0 + [\rho \cdot \gamma]_*} \leq C_{26} M^{l+[\rho \cdot \gamma]-1} t^{-[\rho \cdot \gamma]} \\[2mm] \qquad\qquad \times (l + [\rho \cdot \gamma])! \, (l + [\rho \cdot \gamma] - 1)!^{s-1} \cdot \frac{1}{(l+1)^2} \\[2mm] \qquad\qquad\qquad \text{for} \quad 0 < |\gamma| \leq Nm. \end{cases}$$

Here, we used $[\rho \cdot \gamma]_* \leq [\rho \cdot \gamma] + 1$ and $\rho \cdot \gamma = 1$ if $[\rho \cdot \gamma]_* = 1$. Hence, for $l + |\gamma| \geq 1$ we get from $\chi \Subset \tilde{\varphi}_\gamma$, Proposition 1.6 and (3.10)

$$\|(D^{\beta - \gamma} \chi) D_x^\gamma P^l \{f(u)\}\|_{\lambda, n_0}$$

$$\leq \sum_{k=1}^{l+|\gamma|} \sum_{k' + k'' = k} \frac{\gamma! \, l!}{k'! \, k''!} \left\| (D^{\beta - \gamma} \chi) \partial_u^{k'} \partial_{\bar{u}}^{k''} f(u) \right\|$$

$$\times \sum_{\substack{l_1+\cdots+l_k=l \\ \gamma^1+\cdots+\gamma^k=\gamma \\ l_j+|\gamma^j|\geq 1}} \prod_{j=1}^{k'} \frac{1}{\gamma^j!\,l_j!} D_x^{\gamma^j} P^{l_j} u \prod_{j=k'+1}^{k} \frac{1}{\gamma^j!\,l_j!} D_x^{\gamma^j} P^{l_j} \bar{u} \Big\|_{\lambda,n_0}$$

$$= \sum_{k=1}^{l+|\gamma|} \sum_{k'+k''=k} \frac{\gamma!\,l!}{k'!\,k''!} \Big\| (D^{\beta-\gamma}\chi) \partial_u^{k'} \partial_{\bar{u}}^{k''} f(u)$$

$$\times \sum_{\substack{l_1+\cdots+l_k=l \\ \gamma^1+\cdots+\gamma^k=\gamma \\ l_j+|\gamma^j|\geq 1}} \prod_{j=1}^{k'} \frac{1}{\gamma^j!\,l_j!} D_x^{\gamma^j} \tilde{\varphi}_{\gamma^j}(x) P^{l_j} u \prod_{j=k'+1}^{k} \frac{1}{\gamma^j!\,l_j!} D_x^{\gamma^j} \tilde{\varphi}_{\gamma^j}(x) P^{l_j} \bar{u} \Big\|_{\lambda,n_0}$$

$$\leq C_{27} \sum_{k=1}^{l+|\gamma|} \gamma!\,l! \cdot C_{21} M_1^k C_2^{k-1} (k-1)!^{s-1}$$

$$\times \sum_{\substack{l_1+\cdots+l_k=l \\ \gamma^1+\cdots+\gamma^k=\gamma \\ l_j+|\gamma^j|\geq 1}} \prod_{j=1}^{k} \Big\| \frac{1}{\gamma^j!\,l_j!} \tilde{\varphi}_{\gamma^j}(x) P^{l_j} u \Big\|_{\lambda,n_0+[\rho\cdot\gamma^j]_*}$$

$$\leq C_{27} C_{21} \sum_{k=1}^{l+|\gamma|} l!\,(k-1)!^{s-1} M_1^k C_2^{k-1} C_{26}^{|\gamma|} C_{19}^{(k-|\gamma|)_+} \Big(\sum_{\gamma^1+\cdots+\gamma^k=\gamma} \frac{\gamma!}{\gamma^1!\cdots\gamma^k!} \Big)$$

$$\times \max_{\gamma^j} \sum_{l_1+\cdots+l_k=l} \prod_{j=1}^{k} \frac{(l_j+[\rho\cdot\gamma^j])!}{l_j!} M^{l_j+[\rho\cdot\gamma^j]-1} t^{-[\rho\cdot\gamma^j]}$$

$$\times (l_j+[\rho\cdot\gamma^j]-1)!^{s-1} \cdot \frac{1}{(l_j+1)^2}$$

$$\leq C_{27}' C_{21} C_2^{-1} (C_{26}/C_{19})^{|\gamma|} l!\, M^{l+Nm} t^{-Nm}$$

$$\times \sum_{k=1}^{l+|\gamma|} \Big(\frac{M_1 C_2 C_{19}}{M} \Big)^k k^{|\gamma|} \cdot \frac{(l+Nm)!}{l!}$$

$$\times (l+Nm-1)!^{s-1} \sum_{l_1+\cdots+l_k=l} \frac{1}{(l_j+1)^2}$$

$$\leq C_{28} M^{l+Nm} t^{-Nm} (l+Nm)!\,(l+Nm-1)!^{s-1} \cdot \frac{1}{(l+1)^2}$$

$$\times \sum_{k=1}^{l+|\gamma|} \Big(\frac{M_1 C_2 C_8 C_{19}}{M} \Big)^k k^{|\gamma|}.$$

This proves (3.4), since we have from (3.11)

$$\sum_{k=1}^{l+|\gamma|} \left(\frac{M_1 C_2 C_8 C_{19}}{M} \right)^k k^{|\gamma|} \leq C_{29} M^{-1}. \qquad \square$$

Proof of Lemma 3.3.   We prove (3.7) by the induction on ν. For $\nu = 0$ the inequality (3.7) holds if we regard $\varphi(x)^0$ as $\psi(x)$. So, we assume that (3.7) holds for $\nu' < \nu$ and prove (3.7). Since $A = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha D_x^\alpha \in S_\lambda^m$ is semi-elliptic, we have

$$(3.13) \quad \|\varphi(x)^\nu P^l u\|_{\lambda,\mu+\nu} \leq C_{30}\{\|A\varphi(x)^\nu P^l u\|_{\lambda,\mu+\nu-m} + \|\varphi(x)^\nu P^l u\|_{\lambda,\mu+\nu-m}\}$$

$$\leq C_{30}\Big\{ \|\varphi(x)^\nu A P^l u\|_{\lambda,\mu+\nu-m} + \sum_{|\alpha:\mathbf{m}|=1} \|[a_\alpha D^\alpha, \varphi(x)^\nu] P^l u\|_{\lambda,\mu+\nu-m}$$

$$+ \|\varphi(x)^\nu P^l u\|_{\lambda,\mu+\nu-m} \Big\}.$$

Use $[a_\alpha D^\alpha, \varphi(x)^\nu] = -a_\alpha \sum_{0<\beta\leq\alpha} (-1)^{|\beta|} \binom{\alpha}{\beta} D_x^{\alpha-\beta}\{D_x^\beta \varphi(x)^\nu\}$ and

$$D_x^{\alpha-\beta} \in S_\lambda^{\rho\cdot(\alpha-\beta)} \subset S_\lambda^{m-[\rho\cdot\beta]}.$$

Then, we have

$$\|[a_\alpha D^\alpha, \varphi(x)^\nu] P^l u\|_{\lambda,\mu+\nu-m} \leq C_{31} \sum_{0<\beta\leq\alpha} \|\{D_x^\beta \varphi(x)^\nu\} P^l u\|_{\lambda,\mu+\nu-[\rho\cdot\beta]}$$

and, in case $\nu - [\rho\cdot\beta] \leq 0$, we have from $\varphi \Subset \psi$, $D_x^\beta \varphi(x)^\nu \in S_\lambda^0$ and (3.5)

$$\|\{D_x^\beta \varphi(x)^\nu\} P^l u\|_{\lambda,\mu+\nu-[\rho\cdot\beta]} = \|\{D_x^\beta \varphi(x)^\nu\} \psi(x) P^l u\|_{\lambda,\mu+\nu-[\rho\cdot\beta]}$$

$$\leq C_{32} \|\psi(x) P^l u\|_{\lambda,\mu}$$

$$\leq C_{32} C_{16} M^{l+m'-1} t^{-j_1} (l+m')! \, (l+m'-1)!^{s-1} \cdot \frac{1}{(l+1)^2}.$$

For the case $\nu - [\rho\cdot\beta] > 0$, we write $D_x^\beta \varphi(x)^\nu = \varphi_{\nu,\beta} \varphi^{\nu-|\beta|} = \varphi_{\nu,\beta} \varphi^{[\rho\cdot\beta]-|\beta|} \varphi^{\nu'}$ with a function $\varphi_{\nu,\beta}(x)$ and $\nu' = \nu - [\rho\cdot\beta]$. Then, noting $|\beta| < \nu$ and $\varphi_{\nu,\beta}\varphi^{[\rho\cdot\beta]-|\beta|} \in S_\lambda^0$, an inequality

$$\|\{D_x^\beta \varphi(x)^\nu\} P^l u\|_{\lambda,\mu+\nu-[\rho\cdot\beta]} \leq C_{33} \|\varphi(x)^{\nu'} P^l u\|_{\lambda,\mu+\nu'}$$

$$\leq C_{33} C_{18,\nu'} M^{l+\tilde{m}-1} t^{-j'} (l+\tilde{m})! \, (l+\tilde{m}-1)!^{s-1} \cdot \frac{1}{(l+1)^2}$$

holds by the assumption of the induction. In order to estimate the last term of (3.13), we use $\varphi \Subset \psi$, $\varphi(x)^\nu \in S_\lambda^0$ and (3.5) again. Then, we have

$$\|\varphi(x)^\nu P^l u\|_{\lambda,\mu+\nu-m} \leq C_{34} C_{16} M^{l+m'-1} t^{-j_1} (l+m')! \, (l+m'-1)!^{s-1} \cdot \frac{1}{(l+1)^2}.$$

Summing up, we get

$$(3.14) \quad \|\varphi(x)^\nu P^l u\|_{\lambda,\mu+\nu} \leq C_{30} \Big\{ \|\varphi(x)^\nu A P^l u\|_{\lambda,\mu+\nu-m}$$
$$+ C_{35} M^{l+\tilde{m}-1} t^{-j'} (l+\tilde{m})! \, (l+\tilde{m}-1)!^{s-1} \cdot \frac{1}{(l+1)^2} \Big\}.$$

Now, we use (3.1). Then, we have

(3.15)

$$\|\varphi(x)^\nu A P^l u\|_{\lambda,\mu+\nu-m} \leq \frac{1}{t} \|\varphi(x)^\nu P^{l+1} u\|_{\lambda,\mu+\nu-m}$$
$$+ \sum_{k=1}^n \frac{1}{t} \|\varphi(x)^\nu x_k \partial_{x_k} P^l u\|_{\lambda,\mu+\nu-m} + \|\varphi(x)^\nu (P+1)^l \{f(u)\}\|_{\lambda,\mu+\nu-m}.$$

We estimate each term of (3.15). For the first term we get

$$(3.16) \qquad \frac{1}{t} \|\varphi(x)^\nu P^{l+1} u\|_{\lambda,\mu+\nu-m} = \frac{1}{t} \|\varphi(x)^\nu \psi(x) P^{l+1} u\|_{\lambda,\mu+\nu-m}$$
$$\leq C_{36} \frac{1}{t} \|\psi(x) P^{l+1} u\|_{\lambda,\mu}$$
$$\leq C_{36} C_{16} M^{l+m'} t^{-j_1-1} (l+m'+1)! \, (l+m')!^{s-1} \cdot \frac{1}{(l+1)^2}$$

from (3.5). In order to estimate the second term of (3.15), we divide into two cases $\nu \leq m/2$ and $\nu > m/2$. Then, for the case $\nu \leq m/2$, we get from (3.5)

$$(3.17) \qquad \frac{1}{t} \|\varphi(x)^\nu x_k \partial_{x_k} P^l u\|_{\lambda,\mu+\nu-m} = \frac{1}{t} \|\varphi(x)^\nu x_k \partial_{x_k} \psi(x) P^l u\|_{\lambda,\mu+\nu-m}$$
$$\leq C_{37} t^{-1} \|\psi(x) P^l u\|_{\lambda,\mu}$$
$$\leq C_{37} C_{16} M^{l+m'-1} t^{-j_1-1} (l+m')! \, (l+m'-1)!^{s-1} \cdot \frac{1}{(l+1)^2},$$

since we have $\partial_{x_k} \in S_\lambda^{\rho_k}$ and $\nu - m + \rho_k = \nu - m + m/m_k \leq m/2 - m + m/2 = 0$. For the case of $\nu > m/2$, we use (3.7) with $\nu = m/2$, the assumption of the induction. Then, we have

$$(3.18) \quad \frac{1}{t} \|\varphi(x)^\nu x_k \partial_{x_k} P^l u\|_{\lambda,\mu+\nu-m}$$
$$\leq \frac{1}{t} \{ \|x_k \partial_{x_k} \varphi(x)^\nu P^l u\|_{\lambda,\mu} + \|x_k \{\partial_{x_k} \varphi(x)^\nu\} P^l u\|_{\lambda,\mu} \}$$
$$\leq C_{38} t^{-1} \|\varphi(x)^{m/2} P^l u\|_{\lambda,\mu+m/2}$$
$$\leq C_{38} C_{18,m/2} M^{l+\tilde{m}-1} t^{-\{(j_1+1)\vee j_2\}-1} (l+\tilde{m})! \, (l+\tilde{m}-1)!^{s-1} \cdot \frac{1}{(l+1)^2}.$$

Finally, we get from (3.6)

$$(3.19) \quad \|\varphi(x)^\nu (P+1)^l \{f(u)\}\|_{\lambda,\mu+\nu-m} \le \sum_{l'=0}^{l} \binom{l}{l'} \|\varphi(x)^\nu P^{l'} \{f(u)\}\|_{\lambda,\mu}$$

$$\le C_{39} C_{17} M^{l+m''-1} t^{-j_2} (l+m'')! \, (l+m''-1)!^{s-1} \cdot \frac{1}{(l+1)^2}.$$

Summing up, we have proved (3.7) from (3.14)–(3.19). $\qquad\qquad\square$

Now, we proceed to prove Theorem 4. From Proposition 3.1 we can estimate $\partial_x^\beta P^l u$ locally for β satisfying $|\rho \cdot \beta| \le mn$ if we take $N_0 = n$ in Proposition 3.1. Especially, taking a C_0^∞-function $\varphi(x)$ and a constant M_0 appropriately, the following inequality holds for β satisfying $m \le [\rho \cdot \beta]_* \le mn$

$$(3.20) \quad \|\varphi(x)^{\nu-1} \partial_x^\beta P^l u\|_{\lambda,n_0} \le M_0^{\nu+l+1-m} t^{1-\nu-l} (\nu+l-m)!^\sigma \, l!^{s-\sigma} \qquad (\nu = [\rho \cdot \beta]_*).$$

In fact, taking another C_0^∞-functions $\psi(x)$ with $\varphi \Subset \psi$ and using $\varphi(x)^{\nu-1} \partial_x^\beta \in S_\lambda^{\rho \cdot \beta}$ we get

$$\|\varphi(x)^{\nu-1} \partial_x^\beta P^l u\|_{\lambda,n_0} = \|\varphi(x)^{\nu-1} \partial_x^\beta \psi(x) P^l u\|_{\lambda,n_0} \le C_{40} \|\psi(x) P^l u\|_{\lambda,n_0+\nu}$$

$$\le C_{40} C_{14} M^{l+\nu-2} t^{1-\nu} (l+\nu-1)! \, (l+\nu-2)!^{s-1} \cdot \frac{1}{(l+1)^2}$$

$$\le C_{40} C_{14} T^l M^{\nu+l-2} t^{1-\nu-l} (\nu+l)!^\sigma \, (l+mn)!^{s-\sigma}$$

$$\le C_{40} C_{14} T^l M^{\nu+l-2} t^{1-\nu-l} M_3^{\nu+l} (\nu+l-m)!^\sigma \, l!^{s-\sigma}$$

$$= C_{40} C_{14} T^l M^{m-2} M_3^m (MM_3)^{\nu+l-m} t^{1-\nu-l} (\nu+l-m)!^\sigma \, l!^{s-\sigma}$$

from (3.3) with $\varphi(x)$ replaced by $\psi(x)$. Hence, taking M_0 such that $MM_3 \le M_0$ and $C_{40} C_{14} T^l M^{m-2} M_3^m \le M_0$, we get (3.20). In the following, we prove (3.20) for any $\nu = [\rho \cdot \beta]_*$. Then, as see below, the inequalities (3.20) and (3.2)–(3.3) implies (6). So, it remains to estimate (3.20) for $[\rho \cdot \beta]_* > mn$. In order to do so, we shall employ the induction on ν.

Now, we assume $\nu = [\rho \cdot \beta]_* > mn$. Then, there exists j such that $\beta_j \ge m_j$. Set $\gamma = m_j e_j$ with a unit vector e_j with the j-th element $= 1$, and set $\beta' = \beta - \gamma$ and $\nu' = \rho \cdot \beta'$. Then, we have $\nu' = \nu - m$. Write

$$(3.21) \qquad \varphi(x)^{\nu-1} \partial_x^\beta P^l u = \partial_x^\gamma \varphi(x)^{\nu-1} \partial_x^{\beta'} P^l u + [\varphi(x)^{\nu-1}, \partial_x^\gamma] \partial_x^{\beta'} P^l u.$$

and estimate each term of the right hand side of the above identity.

Lemma 3.4. *Assume that* (3.20) *holds for* $\nu' < \nu$. *Then, we have*

$$(3.22) \qquad \|[\varphi(x)^{\nu-1}, \partial_x^\gamma] \partial_x^{\beta'} P^l u\|_{\lambda,n_0} \le C_{41} M_0^{\nu'+l} t^{1-\nu'-l-m} (\nu'+l)!^\sigma \, l!^{s-\sigma}.$$

Proof. Since

$$[\varphi(x)^{\nu-1}, \partial_x^\gamma] = - \sum_{\substack{\gamma' \le \gamma \\ \gamma' \ne 0}} \binom{\gamma}{\gamma'} \sum_{k=1}^{|\gamma'|} \frac{(\nu-1)!}{(\nu-1-k)!} \varphi_{\gamma',k}(x) \varphi(x)^{\nu-1-k} \partial_x^{\gamma-\gamma'}$$

holds with functions $\varphi_{\gamma',k}(x)$ independent of ν, we have

$$(3.23) \qquad \|[\varphi(x)^{\nu-1}, \partial_x^\gamma]\partial_x^{\beta'} P^l u\|_{\lambda,n_0}$$

$$\le C_{41}' \sum_{\substack{\gamma' \le \gamma \\ \gamma' \ne 0}} \sum_{k=1}^{|\gamma'|} \frac{(\nu-1)!}{(\nu-1-k)!} \|\varphi_{\gamma',k}(x)\varphi(x)^{\nu-1-k} \partial_x^{\beta'+\gamma-\gamma'} P^l u\|_{\lambda,n_0}.$$

Set $\mu = [\rho \cdot (\beta' + \gamma - \gamma')]_*$. Then, since $\mu \le \nu - |\gamma'| \le \nu' + m - k$, we have

$$\frac{(\nu-1)!}{(\nu-1-k)!} \|\varphi_{\gamma',k}(x)\varphi(x)^{\nu-1-k} \partial_x^{\beta'+\gamma-\gamma'} P^l u\|_{\lambda,n_0}$$

$$= \frac{(\nu-1)!}{(\nu-1-k)!} \|\{\varphi_{\gamma',k}(x)\varphi(x)^{\nu-k-\mu}\}\varphi(x)^{\mu-1} \partial_x^{\beta'+\gamma-\gamma'} P^l u\|_{\lambda,n_0}$$

$$\le \frac{(\nu'+m-1)!}{(\nu'+m-1-k)!} C_{41}'' M_0^{\mu+l+1-m} t^{1-\mu-l-m} (\mu+l-m)!^\sigma l!^{s-\sigma}$$

$$\le \{2^k(\nu'+l)\cdots(\nu'+l+1-k)\}$$

$$\times C_{41}'' M_0^{\nu'-k+l+1} t^{1-(\nu'+m-k)-l} (\nu'-k+l)!^\sigma l!^{s-\sigma}$$

$$\le 2^k C_{41}'' M_0^{\nu'+l} t^{1-\nu'-l-m} (\nu'+l)!^\sigma l!^{s-\sigma}.$$

Combining this with (3.23), we get (3.22). $\qquad\qquad \square$

Next, we estimate the H_{λ,n_0}-norm of the first term of (3.21) by using the semi-ellipticity of A and (3.1). Then, we have

$$(3.24) \quad \|\partial_x^\gamma \varphi(x)^{\nu-1} \partial_x^{\beta'} P^l u\|_{\lambda,n_0} \le \|\varphi(x)^{\nu-1} \partial_x^{\beta'} P^l u\|_{\lambda,n_0+m}$$

$$\le C_{42}\{\|A\varphi(x)^{\nu-1} \partial_x^{\beta'} P^l u\|_{\lambda,n_0} + \|\varphi(x)^{\nu-1} \partial_x^{\beta'} P^l u\|_{\lambda,n_0}\}$$

$$\le C_{42}\{\|\varphi(x)^{\nu-1} \partial_x^{\beta'} A P^l u\|_{\lambda,n_0} + \|[A, \varphi(x)^{\nu-1}]\partial_x^{\beta'} P^l u\|_{\lambda,n_0}$$

$$+ C_{43}\|\varphi(x)^{\nu'-1} \partial_x^{\beta'} P^l u\|_{\lambda,n_0}\}$$

$$\le C_{42}\Big\{t^{-1}\|\varphi(x)^{\nu-1} \partial_x^{\beta'} P^{l+1} u\|_{\lambda,n_0} + t^{-1}\sum_{k=1}^n \|\varphi(x)^{\nu-1} \partial_x^{\beta'} x_k \partial_{x_k} P^l u\|_{\lambda,n_0}$$

$$+ \|\varphi(x)^{\nu-1} \partial_x^{\beta'} (P+1)^l \{f(u)\}\|_{\lambda,n_0}$$

$$+ \|[A, \varphi(x)^{\nu-1}]\partial_x^{\beta'} P^l u\|_{\lambda,n_0}\Big\}$$

$$+ C_{42}C_{43}M_0^{\nu'+l+1-m}t^{1-\nu'-l}(\nu'+l-m)!^\sigma l!^{s-\sigma}.$$

We estimate each term in the last member of (3.24).

Lemma 3.5. *Assume that* (3.20) *holds for* $\nu' < \nu$. *Then, we have*

$$(3.25) \qquad \|[A, \varphi(x)^{\nu-1}]\partial_x^{\beta'} P^l u\|_{\lambda,n_0} \le C_{44}M_0^{\nu'+l}t^{1-\nu'-l-m}(\nu'+l)!^\sigma l!^{s-\sigma}.$$

We can prove this lemma by the same method as in the proof of Lemma 3.4.

Lemma 3.6. *Let* σ *satisfy* $\sigma \ge s/m$. *Assume that* (3.20) *holds for* $\nu' < \nu$. *Then, the inequality*

$$(3.26) \qquad \|\varphi(x)^{\nu-1}\partial_x^{\beta'} P^{l+1} u\|_{\lambda,n_0} \le C_{45}M_0^{\nu'+l}t^{2-\nu'-l-m}(\nu'+l)!^\sigma l!^{s-\sigma}$$

holds.

Proof.   Note $\nu' \ge m$. Then, since $\sigma \ge s/m$, we have

$$(l+1)^{s-\sigma} \le (l+1)^{(m-1)\sigma}$$
$$\le \prod_{k=1}^{m-1}(l+1+\nu'-m+k) = \prod_{k=2}^{m}(\nu'+l+k-m).$$

Hence, we get from (3.20)

$$\|\varphi(x)^{\nu-1}\partial_x^{\beta'} P^{l+1} u\|_{\lambda,n_0} \le C_{45}'\|\varphi(x)^{\nu'-1}\partial_x^{\beta'} P^{l+1} u\|_{\lambda,n_0}$$
$$\le C_{45}'M_0^{\nu'+(l+1)+1-m}t^{1-\nu'-(l+1)}(\nu'+l+1-m)!^\sigma (l+1)!^{s-\sigma}$$
$$\le C_{45}'M_0^{\nu'+l}t^{-\nu'-l}(\nu'+l)!^\sigma l!^{s-\sigma}.$$

This yields (3.26) from $m \ge 2$. $\square$

Lemma 3.7. *Assume that* (3.20) *holds for* $\nu' < \nu$. *Then, we have*

$$(3.27) \qquad \|\varphi(x)^{\nu-1}\partial_x^{\beta'} x_k \partial_{x_k} P^l u\|_{\lambda,n_0} \le C_{46}M_0^{\nu'+l}t^{2-\nu'-l-m}(\nu'+l)!^\sigma l!^{s-\sigma}.$$

Proof.   Write

$$\|\varphi(x)^{\nu-1}\partial_x^{\beta'} x_k \partial_{x_k} P^l u\|_{\lambda,n_0} \le \|\varphi(x)^{\nu-1} x_k \partial_{x_k}\partial_x^{\beta'} P^l u\|_{\lambda,n_0} + \beta_k\|\varphi(x)^{\nu-1}\partial_x^{\beta'} P^l u\|_{\lambda,n_0}$$

and set $\mu = [\rho \cdot \beta' + \rho_k]_*$. Then, since $\rho_k = m/m_k$ and $m_k \ge 2$, we have $\mu \le \nu' + m - 1$. Hence, we obtain from the boundedness of supp φ

$$\|\varphi(x)^{\nu-1}\partial_x^{\beta'} x_k \partial_{x_k} P^l u\|_{\lambda,n_0}$$

$$\leq C_{46}'\{\|\varphi(x)^{\mu-1}\partial_{x_k}\partial_x^{\beta'}P^lu\|_{\lambda,n_0} + \beta_k\|\varphi(x)^{\nu'-1}\partial_x^{\beta'}P^lu\|_{\lambda,n_0}\}$$

$$\leq C_{46}'\{M_0^{\mu+l+1-m}t^{1-\mu-l}(\mu+l-m)!^\sigma \, l!^{s-\sigma}$$

$$+ \beta_k M_0^{\nu'+l+1-m}t^{1-\nu'-l}(\nu'+l-m)!^\sigma \, l!^{s-\sigma}\}$$

$$\leq C_{46}''M_0^{\nu'+l}t^{2-\nu'-l-m}(\nu'+l)!^\sigma \, l!^{s-\sigma}.$$

This yields (3.27). □

Lemma 3.8. *Let $f(u)$ be a function satisfying* $(A.1)_\sigma$ *and assume that* (3.20) *holds for $\nu' < \nu$. Then, an inequality*

$$(3.28) \qquad \|\varphi(x)^{\nu-1}\partial_x^{\beta'}(P+1)^l\{f(u)\}\|_{\lambda,n_0} \leq C_{47}M_0^{\nu'+l}t^{1-\nu'-l-m}(\nu'+l)!^\sigma \, l!^{s-\sigma}$$

holds.

Proof. Since we have

$$\|\varphi(x)^{\nu-1}\partial_x^{\beta'}(P+1)^l\{f(u)\}\|_{\lambda,n_0} \leq \sum_{l'=0}^{l}\binom{l}{l'}\|\varphi(x)^{\nu-1}\partial_x^{\beta'}P^{l'}\{f(u)\}\|_{\lambda,n_0},$$

we may only prove

$$(3.29) \qquad \|\varphi(x)^{\nu-1}\partial_x^{\beta'}P^l\{f(u)\}\|_{\lambda,n_0} \leq C_{48}M_0^{\nu'+l}t^{1-\nu'-l-m}(\nu'+l)!^\sigma \, l!^{s-\sigma}$$

in order to prove (3.28). So, in the following, we prove (3.29). For β'' satisfying $m \leq \nu'' = [\rho \cdot \beta'']_* < \nu$ we have from (3.20), $m \geq 3$ and $\nu'' \leq [\rho \cdot \beta''] + 1$

$$\|\varphi(x)^{\nu''-1}\partial_x^{\beta''}P^lu\|_{\lambda,n_0} \leq M_0^{\nu''+l+1-m}t^{1-\nu''-l}(\nu''+l-m)!^\sigma \, l!^{s-\sigma}$$

$$\leq M_0^{[\rho\cdot\beta'']+l+2-m}t^{-[\rho\cdot\beta'']-l}([\rho\cdot\beta'']+1+l-m)!^\sigma \, l!^{s-\sigma}$$

$$\leq C_{49}M_0^{[\rho\cdot\beta'']+l-1}t^{-[\rho\cdot\beta'']-l}([\rho\cdot\beta'']+l)! \, ([\rho\cdot\beta'']+l-1)!^{\sigma-1} \, l!^{s-\sigma}$$

$$\times \frac{1}{(|\beta''|+l+1)^2},$$

since $(|\beta''|+l+1)^2 \leq C_{49}([\rho\cdot\beta'']+l-1)([\rho\cdot\beta'']+l)$ holds for some constant C_{49}. This yields

$$(3.30) \quad \|\varphi(x)^{(\nu''-1)\vee 1}\partial_x^{\beta''}P^lu\|_{\lambda,n_0}$$

$$\leq C_{49}M_0^{[\rho\cdot\beta'']+l-1}t^{-[\rho\cdot\beta'']-l}([\rho\cdot\beta'']+l)! \, ([\rho\cdot\beta'']+l-1)!^{\sigma-1} \, l!^{s-\sigma}$$

$$\times \frac{1}{(|\beta''|+l+1)^2},$$

for β'' satisfying $m \leq \nu'' = [\rho \cdot \beta'']_* < \nu$. Moreover, from (3.2) and (3.3) we get (3.30) for β'' satisfying $1 \leq \nu'' = [\rho \cdot \beta'']_* < m$ with an appropriate constant M_0. We also use

$$(3.31) \qquad \|P^l u\|_{\lambda,n_0} \leq C_{49} M_0^{l-1} t^{-l} l! \, (l-1)!^{\sigma-1} l!^{s-\sigma} \cdot \frac{1}{(|\beta''|+l+1)^2},$$

which is guaranteed in the previous section.

Now, we use the differentiation of composite function. Then, writing $\tilde{\varphi}_0(x) = 1$ and $\tilde{\varphi}_\beta(x) = \varphi(x)$ for $\beta \neq 0$ we have

$$\|\varphi(x)^{\nu-1} \partial_x^{\beta'} P^l \{f(u)\}\|_{\lambda,n_0}$$

$$\leq \sum_{k=1}^{|\beta'|+l} \sum_{k'+k''=k} \sum_{\substack{\beta^1+\cdots+\beta^k=\beta' \\ l_1+\cdots+l_k=l \\ |\beta^j|+l_j\geq 1}} \frac{\beta'! \, l!}{k'! \, k''!} \left\|\varphi(x)^{\nu-\bar{\nu}^k(\tilde{\beta}^k)-1} \partial_u^{k'} \partial_{\bar{u}}^{k''} f(u)\right.$$

$$\times \prod_{j=1}^{k'} \frac{1}{\beta^j! \, l_j!} \tilde{\varphi}_{\beta^j}(x)^{(\nu_j-1)\vee 1} \partial_x^{\beta^j} P^{l_j} u$$

$$\times \left. \prod_{j=k'+1}^{k} \frac{1}{\beta^j! \, l_j!} \tilde{\varphi}_{\beta^j}(x)^{(\nu_j-1)\vee 1} \partial_x^{\beta^j} P^{l_j} \bar{u}\right\|_{\lambda,n_0}$$

with $\nu_j = [\rho \cdot \beta^j]_*$, and $\bar{\nu}^k(\tilde{\beta}^k) = \sum_{\{j\,;\,\beta^j\neq 0\}} \{(\nu_j-1)\vee 1\}$ for $\tilde{\beta}^k = (\beta^1,\ldots,\beta^k)$. Since we can prove $0 \leq \nu - \bar{\nu}^k(\tilde{\beta}^k) \leq k-1$ there exists a constant C_{50} such that $\|\varphi^{\nu-\bar{\nu}^k(\tilde{\beta}^k)-1} v\|_{\lambda,n_0} \leq C_{50}^k \|v\|_{\lambda,n_0}$ for $v \in H_{\lambda,n_0}$. Hence, using Proposition 1.2 and Proposition 1.6 we have from (3.30) and (3.31)

$$\|\varphi(x)^{\nu-1} \partial_x^{\beta'} P^l \{f(u)\}\|_{\lambda,n_0} \leq C_{51} \sum_{k=1}^{n} k \cdot \beta'! \, l! \, M_1^k k!^{\sigma-1}$$

$$\times C_{50}^k \sum_{\substack{\beta^1+\cdots+\beta^k=\beta' \\ l_1+\cdots+l_k=l \\ |\beta^j|+l_j\geq 1}} C_2^{k-1} \prod_{j=1}^{k} \frac{1}{\beta^j! \, l_j!} C_{49} M_0^{[\rho\cdot\beta^j]+l_j-1} t^{-[\rho\cdot\beta^j]-l_j} ([\rho\cdot\beta^j]+l_j)!$$

$$\times ([\rho\cdot\beta^j]+l_j-1)!^{\sigma-1} l_j!^{s-\sigma}/(|\beta^j|+l_j+1)^2.$$

Note $[\rho\cdot\beta^1]+\cdots+[\rho\cdot\beta^k] \leq [\rho\cdot\beta']_* = \nu'$. Then, we have from Lemma 1.8

$$\|\varphi(x)^{\nu-1} \partial_x^{\beta'} P^l \{f(u)\}\|_{\lambda,n_0}$$

$$\leq C_{50}' C_2^{-1} M_0^{\nu'+l} t^{-\nu'-l} \sum_{k=1}^{l+|\beta'|} k \left(\frac{M_1 C_2 C_{49} C_{50}}{M_0}\right)^k (\nu'+l)!^{\sigma-1} l!^{s-\sigma}$$

$$\times \sum_{\substack{\mu_1+\cdots+\mu_\nu=|\beta'|+l \\ \mu_j\geq 1}} \left\{ \beta'! \, l! \sum_{\substack{\beta^1+\cdots+\beta^k=\beta' \\ l_1+\cdots+l_k=l \\ |\beta^j|+l_j=\mu_j}} \prod_{j=1}^k \frac{(|\beta^j|+l_j)!}{l_j! \, \beta^j!} \right.$$

$$\times \left. \prod_{j=1}^k \frac{([\rho\cdot\beta^j]+l_j)!}{(|\beta^j|+l_j)!} \frac{1}{(\mu_j+1)^2} \right\},$$

$$\leq C'_{50} C_2^{-1} M_0^{\nu'+l} t^{-\nu'-l} (\nu'+l)!^{\sigma-1} l!^{s-\sigma} \sum_{k=1}^{l+|\beta'|} k \left(\frac{M_1 C_2 C_{49} C_{50}}{M_0} \right)^k$$

$$\times \sum_{\substack{\mu_1+\cdots+\mu_\nu=|\beta'|+l \\ \mu_j\geq 1}} (|\beta'|+l)! \frac{(\nu'+l)!}{(|\beta'|+l)!} \prod_{j=1}^k \frac{1}{(\mu_j+1)^2}$$

$$\leq C'_{50} C_2^{-1} C_8^{-1} M_0^{\nu'+l} t^{1-\nu'-l-m} (\nu'+l)!^{\sigma} l!^{s-\sigma}$$

$$\times \sum_{k=1}^{l+|\beta'|} k \left(\frac{M_1 C_2 C_{49} C_{50} C_8}{M_0} \right)^k \cdot \frac{1}{(|\beta'|+l)^2}.$$

This proves (3.29) and hence (3.28) if we take M_0 such that $M_1 C_2 C_{49} C_{50} C_8 < M_0$. $\square$

Now, we are prepared to prove Theorem 4.

Proof of Theorem 4. Let $\varphi(x)$ be a C_0^∞-function. Then, from (3.2) and (3.3) with $2 \leq \nu \leq mn$ we have for β satisfying $|\rho\cdot\beta| \leq mn$

$$(3.32) \qquad \|\varphi(x)\partial_x^\beta u\|_{\lambda,n_0} \leq M_0^{\nu+1} t^{-\nu} \nu!^\sigma \qquad (\nu = [\rho\cdot\beta]_*).$$

Now, let $\nu > mn$. Then, if we assume (3.20) holds for β satisfying $[\rho\cdot\beta]_* < \nu$, we have from (3.21), (3.24) and Lemmas 3.4–3.8

$$\|\varphi(x)^{\nu-1}\partial_x^\beta P^l u\|_{\lambda,n_0} \leq \{C_{41} + C_{42}(C_{45} + C_{46} + C_{47} + C_{44} + C_{43})\}$$
$$\times M_0^{\nu'+l} t^{1-\nu'-l-m} (\nu'+l)!^\sigma l!^{s-\sigma}.$$

with $\nu' = \nu - m = [\rho\cdot\beta]_* - m$. Retake the constant M_0 so large such that $C_{41}+C_{42}(C_{45}+C_{46}+C_{47}+C_{44}+C_{43}) \leq M_0$. Then, we have (3.20) for β satisfying $|\rho\cdot\beta| = \nu$, and hence, by the induction, we have (3.20) for any β with $|\rho\cdot\beta| \geq mn$. This implies (3.32) for any β. Finally, we use $[\rho\cdot\beta]_*!^\sigma \leq C_{52}^{|\beta|+1} \beta!^{\rho\sigma}$. Then, we get (6) from (3.32). $\square$

References

[1]   N. Hayashi, P.I. Naumkin and P.N. Pipolo: *Analytic smoothing effect for some derivative nonlinear Schrödinger equations*, Tsukuba J. Math. **24** (2000), 21–34.

[2]   N. Hayashi, P.I. Naumkin and H. Uchida: *Analytic smoothing effects of global small solutions to the elliptic-hyperbolic Davey-Stewartson system*, Advances in Differential Equations **7** (2002), 469–492.

[3]   E. Kaikina, K. Kato, P.I. Naumkin and T. Ogawa: *Wellposedness and analytic smoothing effect for the Benjamin-Ono equation*, preprint.

[4]   K. Kajitani: *Smoothing effect in Gevrey classes for Schrödionger equations* I, preprint.

[5]   K. Kajitani and Y. Dan: *Smoothing effect and exponential time decay of solutions of Schrödinger equations*, Proc. Japan Acad. **78** (2002), 92–95.

[6]   K. Kajitani and S. Wakabayashi: *Analyticity smoothing effect for Schrödinger type equations with variable coefficients*, Direct and Inverse Problems of Mathematical Physics 2000, 185–219.

[7]   K. Kato and K. Taniguchi: *Gevrey regularizing effect for nonlinear Schrödinger equations*, Osaka J. Math. **33** (1996), 863–880.

[8]   H. Kumano-go: Pseudo-differential operators, The MIT Press, Cambridge and London, 1982.

[9]   Y. Morimoto, L. Robbiano and C. Zuilly: *Remark on the analytic smoothing for the Schrödinger equation*, Indiana Univ. Math. J. **49** (2000), 1563–1579.

[10] H. Uchida: *Analytic of solutions to nonlinear Schrödinger equations*, SUT Jour. Math. **37** (2001), 105–135.

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