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THE ORIENTABILITY OF SMALL COVERS AND COLORING SIMPLE POLYTOPES

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Abstract

Small Cover is an n -dimensional manifold endowed with a \mathbf{Z}_2^n action whose orbit space is a simple convex polytope P . It is known that a small cover over P is characterized by a coloring of P which satisfies a certain condition. In this paper we shall investigate the topology of small covers by the coloring theory in combinatorics. We shall first give an orientability condition for a small cover. In case $n = 3$, an orientable small cover corresponds to a four colored polytope. The four color theorem implies the existence of orientable small cover over every simple convex 3-polytope. Moreover we shall show the existence of non-orientable small cover over every simple convex 3-polytope, except the 3-simplex.

0. Introduction

“Small Cover” was introduced and studied by Davis and Januszkiewicz in [5]. It is a real version of “Quasitoric manifold,” i.e., an n -dimensional manifold endowed with an action of the group \mathbf{Z}_2^n whose orbit space is an n -dimensional simple convex polytope. A typical example is provided by the natural action of \mathbf{Z}_2^n on the real projective space $\mathbf{R}P^n$ whose orbit space is an n -simplex. Let P be an n -dimensional simple convex polytope. Here P is *simple* if the number of codimension-one faces (which are called “*facets*”) meeting at each vertex is n , equivalently, the dual K_P of its boundary complex $\partial(P)$ is an $(n - 1)$ -dimensional simplicial sphere. In this paper we shall handle a convex polytope in the category of combinatorics. We denote the set of facets of P (or the set of vertices of K_P) by \mathcal{F} . Associated to a small cover M over P , there exists a function $\lambda: \mathcal{F} \rightarrow \mathbf{Z}_2^n$ called a “characteristic function” of M . A basic result in [5] is that small covers over P are classified by their characteristic functions (cf. [5, Proposition 1.8]). The characteristic function is a (face-)coloring of P (or a vertex-coloring of K_P), which satisfies a “linearly independent condition” (see §1). Coloring convex n -polytopes have been studied actively in combinatorics. We shall investigate the topological properties of small covers through the coloring theory in combinatorics specially when $n = 3$.

The notion of small cover can be generalized to the case where the base space is

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more general than a simple convex polytope. An n -dimensional nice manifold P with corners such that the dual complex K_P is a simplicial decomposition of $\partial(P)$, is called a *simple polyhedral complex*. For a coloring of P which satisfies the linearly independent condition, we can construct an n -dimensional manifold with \mathbf{Z}_2^n -action over P in a similar way. We call it a small cover over a simple polyhedral complex P .

In this paper we give a criterion when a small cover over a simple convex polytope is orientable (Theorem 1.7). In case $n = 3$, this criterion implies that an orientable three-dimensional small cover corresponds to a 4-colored simple convex 3-polytope. Therefore the existence of an orientable small cover over every simple convex 3-polytope is equivalent to the four color theorem (Corollary 1.8). Next we shall discuss the colorability of a 3-polytope making allowance for the linearly independent condition, and prove existence of non-orientable small cover over every simple convex 3-polytope, except the 3-simplex (Theorem 2.3). The proof of Theorem 2.3 is given in a way similar to the proof of classical five color theorem by Kempe. Moreover we shall discuss the existence of non-orientable small cover over 4-colorable simple polyhedral handlebody with a positive genus (Theorem 3.1).

1. The orientability of small covers

At first we shall recall the definition and basic results of small covers in [5] or [3]. An n -dimensional convex polytope is *simple* if the number of codimension-one faces (which are called *facets*) meeting at each vertex is n . Equivalently, P is simple if the dual of its boundary complex is a simplicial decomposition of $(n - 1)$ -dimensional sphere. We denote the simplicial complex dual to P by K_P . In this paper we shall handle polytopes in the category of combinatorics and understand that two polytopes are identical if they are combinatorially equivalent. Therefore a considerable structure of a polytope P is only its face structure, i.e., the dual complex K_P . The natural action of \mathbf{Z}_2^n on \mathbf{R}^n is called the *standard representation* and its orbit space is \mathbf{R}_+^n .

DEFINITION 1.1. A manifold M endowed with the action of the group \mathbf{Z}_2^n is a *small cover* over an n -dimensional simple convex polytope P if its orbit space is homeomorphic to P and M is locally isomorphic to the standard representation, i.e., there exists an automorphism θ of \mathbf{Z}_2^n such that for any point $x \in M$, there exists stable neighborhoods $V \subset M$ of x and $W \subset \mathbf{R}^n$ and θ -equivariant homeomorphism $f: V \rightarrow W$ ($f(gv) = \theta(g)f(v)$).

Let $\pi_1: M_1 \rightarrow P$ and $\pi_2: M_2 \rightarrow P$ be two small covers over P . An *equivalence over P* is an automorphism θ of \mathbf{Z}_2^n , together with a θ -equivariant homeomorphism $f: M_1 \rightarrow M_2$, which covers the identity on P .

EXAMPLE 1.2. The group \mathbf{Z}_2 acts on S^1 by a reflection, and its orbit space is an interval I . Taking the n -fold product, we have a \mathbf{Z}_2^n action on an n -dimensional

torus $T^n = S^1 \times \cdots \times S^1$, which is a small cover over the n -cube I^n .

EXAMPLE 1.3. We have a usual \mathbf{Z}_2^n action on the real projective space $\mathbf{R}P^n$ as follows:

$$(g_1, \dots, g_n) \cdot [x_0, x_1, \dots, x_n] = [x_0, g_1x_1, \dots, g_nx_n].$$

This is a small cover over the n -simplex Δ^n .

Let $\pi: M \rightarrow P$ be a small cover over an n -dimensional simple convex polytope P . For a face F of P , the isotropy group at $x \in \pi^{-1}(\text{int } F)$ is independent of the choice of x , denoted by G_F . In particular, if F is a facet, G_F is a rank-one subgroup, hence, it is determined by a generator $\lambda(F) \in \mathbf{Z}_2^n$. In this way we obtain a function $\lambda: \mathcal{F} \rightarrow \mathbf{Z}_2^n$ where \mathcal{F} is the set of facets of P . This function is called *the characteristic function* of M . If F^{n-k} is a codimension- k face of P then $F = F_1 \cap \cdots \cap F_k$ where F_i 's are the facets which contain F , and G_F is the rank- k subgroup generated by $\lambda(F_1), \dots, \lambda(F_k)$. Therefore the characteristic function satisfies the following condition.

(*) If F_1, \dots, F_n are the facets meeting at a vertex of P , then $\lambda(F_1), \dots, \lambda(F_n)$ are linearly independent vectors of \mathbf{Z}_2^n .

In particular, the characteristic function $\lambda: \mathcal{F} \rightarrow \mathbf{Z}_2^n$ is a (face-)coloring of P (or a vertex-coloring of the dual graph K_P). We often call it a *linearly independent coloring* of P (or K_P). Conversely a coloring λ of P satisfying the linearly independent condition (*) determines a small cover $M(P, \lambda)$ over P whose characteristic function is the given λ . The construction of $M(P, \lambda)$ is as follows. For each point $p \in P$, let $F(p)$ be the unique face of P which contains p in its relative interior. We define an equivalence relation on $P \times \mathbf{Z}_2^n$ as follows:

$$(1) \quad (p, g) \sim (q, h) \Leftrightarrow p = q, \quad g^{-1}h \in G_{F(p)}$$

where $G_{F(p)}$ is the subgroup generated by $\lambda(F_1), \dots, \lambda(F_k)$ such that $F(p) = F_1 \cap \cdots \cap F_k$ ($F_i \in \mathcal{F}$). Then the quotient space $(P \times \mathbf{Z}_2^n) / \sim$ is $M(P, \lambda)$.

Theorem 1.4 ([5, Proposition 1.8]). *Let M be a small cover over P such that its characteristic function is $\lambda: \mathcal{F} \rightarrow \mathbf{Z}_2^n$. Then M is equivalent to $M(P, \lambda)$. In other words, the small cover is determined up to equivalence over P by its characteristic function.*

EXAMPLE 1.5. In case $n = 2$, P is a polygon and a characteristic function is a function $\lambda: \mathcal{F} \rightarrow \mathbf{Z}_2^2$ which satisfies the linearly independent condition (*). Let $\{e_1, e_2\}$ be a basis of \mathbf{Z}_2^2 . Since any pair of $\{e_1, e_2, e_1+e_2\}$ is linearly independent, λ is

just a 3-coloring of P . If (P, λ) is a k -gon colored by three colors (resp. two colors) then $M(P, \lambda)$ is the non-orientable surface $(k-2)\mathbf{R}P^2$ (resp. the orientable surface with genus $(k-2)/2$) endowed with a certain action of \mathbf{Z}_2^2 where $m\mathbf{R}P^2$ is the connected sum of m copies of $\mathbf{R}P^2$ (cf. [5, Example 1.20]).

REMARK 1.6. When an n -dimensional simple convex polytope P is s -colored ($s \geq n$), we understand that the image of the coloring function λ is a basis for \mathbf{Z}_2^s , and define the quotient space $Z(P, \lambda) = (P \times \mathbf{Z}_2^s) / \sim$ where the equivalence relation \sim is given in a way similar to (1). It is called the “manifold defined by the coloring λ ” in [7]. When $s = n$, $Z(P, \lambda)$ coincides with the small cover $M(P, \lambda)$, and is called the “pullback from the linear model” in [5]. In the special case $n = s = 3$, pullbacks from the linear model were studied by Izmistiev in details in [7].

Next we shall discuss the orientability condition of a small cover.

Theorem 1.7. *For a basis $\{e_1, \dots, e_n\}$ of \mathbf{Z}_2^n , a homomorphism $\epsilon: \mathbf{Z}_2^n \rightarrow \mathbf{Z}_2 = \{0, 1\}$ is defined by $\epsilon(e_i) = 1$ ($i = 1, \dots, n$). A small cover $M(P, \lambda)$ is orientable if and only if there exists a basis $\{e_1, \dots, e_n\}$ of \mathbf{Z}_2^n such that the image of $\epsilon\lambda$ is $\{1\}$.*

Proof. Let us calculate the n -dimensional integral homology group $H_n(M; \mathbf{Z})$ of a small cover $M = M(P, \lambda)$. The combinatorial structure of P defines a natural cellular decomposition of $M = (P \times \mathbf{Z}_2^n) / \sim$. We denote by (C_k, ∂_k) the chain complex associated with this cellular decomposition. In particular, C_n and C_{n-1} are the free abelian groups generated by $\{P\} \times \mathbf{Z}_2^n = \{(P, g) \mid g \in \mathbf{Z}_2^n\}$ and $\{\mathcal{F} \times \mathbf{Z}_2^n\} / \sim = \{[F, g] \mid F \in \mathcal{F}, g \in \mathbf{Z}_2^n\}$, respectively, where the equivalence class of $\mathcal{F} \times \mathbf{Z}_2^n$ is defined by the equivalence relation $(F, g) \sim (F, \lambda(F) + g)$. We give an orientation on a facet F_i such that $\partial(P) = F_1 + \dots + F_f$ where $f = \#\mathcal{F}$. Under these notations if $X = \sum_{g \in \mathbf{Z}_2^n} n_g(P, g) \in C_n$ ($n_g \in \mathbf{Z}$) is an n -cycle of M then

$$\partial_n(X) = \left[\sum_{g \in \mathbf{Z}_2^n} n_g \sum_{i=1}^f (F_i, g) \right] = \sum_{[F_i, g] \in (\mathcal{F} \times \mathbf{Z}_2^n) / \sim} (n_g + n_{\lambda(F_i)+g}) [F_i, g] = 0.$$

Therefore $X \in \ker \partial_n$ if and only if $n_g = -n_{\lambda(F)+g}$ for any facet F and $g \in \mathbf{Z}_2^n$. The latter is equivalent to $n_g = (-1)^k n_{\lambda(F_{i_1}) + \dots + \lambda(F_{i_k}) + g}$ for any facets F_{i_1}, \dots, F_{i_k} and $g \in \mathbf{Z}_2^n$. Suppose that there is no basis of \mathbf{Z}_2^n such that $\epsilon\lambda \equiv 1$. It means that for each set of facets F_{i_1}, \dots, F_{i_n} such that $e_j = \lambda(F_{i_j})$ ($1 \leq j \leq n$) is a basis of \mathbf{Z}_2^n , there exists a facet F such that $\epsilon\lambda(F) = 0$, i.e., $\lambda(F) = e_{j_1} + \dots + e_{j_k}$ where k is an even number. Then $n_g = (-1)^k n_{e_{j_1} + \dots + e_{j_k} + g} = n_{\lambda(F)+g} = -n_g$ that is $n_g = 0$ for any $g \in \mathbf{Z}_2^n$. Thus $H_n(M; \mathbf{Z}) = \ker \partial_n = 0$, and M is non-orientable. On the other hand, when there exists a basis of \mathbf{Z}_2^n such that $\epsilon\lambda \equiv 1$, for any $g \in \mathbf{Z}_2^n$, the parity of $k = k(g)$ does not depend on the choice of F_{i_j} 's such that $g = \lambda(F_{i_1}) + \dots + \lambda(F_{i_k})$. In fact, if $\lambda(F_{i_1}) + \dots + \lambda(F_{i_k}) =$

$\lambda(F_{j_1}) + \dots + \lambda(F_{j_l})$ then $\epsilon\lambda(F_{i_1}) + \dots + \epsilon\lambda(F_{i_k}) = \epsilon\lambda(F_{j_1}) + \dots + \epsilon\lambda(F_{j_l})$, therefore $k \equiv l \pmod 2$. Then $H_n(M; \mathbf{Z}) = \ker \partial_n \cong \mathbf{Z}$ is generated by $X = \sum_{g \in \mathbf{Z}_2^n} (-1)^{k(g)} (P, g)$, and M is orientable. \square

We call a linearly independent coloring which satisfies the orientability condition in Theorem 1.7 *an orientable coloring* of P (or K_P). In case $n = 2$, it is easy to see that an orientable coloring of a polygon is just a 2-coloring of P (see Example 1.5). In case $n = 3$, a three-dimensional small cover $M(P, \lambda)$ is orientable if and only if there exists a basis $\{\alpha, \beta, \gamma\}$ of \mathbf{Z}_2^3 such that the image of λ is contained in $\{\alpha, \beta, \gamma, \alpha + \beta + \gamma\}$. Since each triple of $\{\alpha, \beta, \gamma, \alpha + \beta + \gamma\}$ is linearly independent, the orientable coloring of a 3-polytope P is just a 4-coloring of P . By the four color theorem ([1]), we obtain the following corollary (in fact, the corollary below is equivalent to the four color theorem).

Corollary 1.8. *There exists an orientable small cover over every simple convex 3-polytope.*

REMARK 1.9. Although there exists a small cover over every three-dimensional simple convex polytope, for each integer $n \geq 4$, there exists an n -dimensional simple convex polytope Q which admits no small cover (cf. [5, Nonexample 1.22]). In fact, a cyclic polytope C_k^n defined as the convex hull of $k \geq n + 1$ points on a curve $\gamma(t) = (t, t^2, \dots, t^n)$ is a simplicial polytope such that the one-skeleton of C_k^n is a complete graph when $n \geq 4$ (see [2, §13]). Let Q_k^n be the simple polytope dual to C_k^n . Since the chromatic number of Q_k^n is k , the polytope Q_k^n admits no small cover whenever $k \geq 2^n$.

2. Existence of non-orientable small covers

We call a linearly independent coloring which does not satisfy the orientability condition in Theorem 1.7 *a non-orientable coloring*. In this section we shall discuss the existence of a non-orientable coloring over a simple polytope in case $n = 3$. We shall recall and use some notions of 3-polytopes and the graph theory. For further details see [2] or [6].

For a simple convex 3-polytope P , we set p_k the number of k -cornered facets of P . Then the numbers of facets, edges and vertices of P are $f_2 = \sum_{k \geq 3} p_k$, $f_1 = \sum_{k \geq 3} k p_k / 2$ and $f_0 = \sum_{k \geq 3} k p_k / 3$, respectively. Since the Euler number $f_0 - f_1 + f_2$ of $\partial(P)$ is two, we obtain immediately the following formula.

Lemma 2.1. *For any simple convex 3-polytope P ,*

$$(2) \quad 3p_3 + 2p_4 + p_5 = 12 + \sum_{k \geq 7} (k - 6)p_k.$$

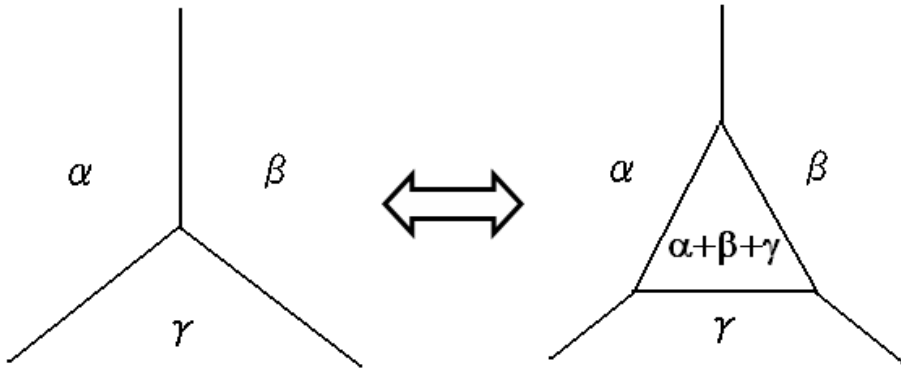


Fig. 1.

The lemma implies a well-known fact that each simple convex 3-polytope has a facet which has less than six edges.

We introduce an operation “*blow up*” for a vertex of a simple 3-polytope P (see Fig. 1). We cut around a vertex of P and create a new triangular facet there. The reverse operation of the blow up is called a “*blow down*.” Notice that for any simple convex polytope except the 3-simplex, two triangular facets must not adjoin each other. Therefore except for the 3-simplex, the blow down for any triangular facet is possible. For the dual simplicial complex K_P , a blow up is operated for a 2-simplex of K_P and a blow down is operated for a vertex of K_P with degree three, respectively. The blow up can be done keeping the linearly independence. In fact, we can assign $\alpha + \beta + \gamma$ to the new triangle where α , β and γ are the colors assigned to three facets adjacent to the triangle. The blow up operation corresponds to *the equivariant connected sum* of $\mathbf{R}P^3$ at the fixed point of small covers corresponding to the vertex (see [5, 1.11]). When P is endowed with a linearly independent coloring, the blow down for a triangular facet can not be done keeping the linearly independence. From the above fact we obtain the following lemma immediately.

Lemma 2.2. *If a convex polytope P has a non-orientable coloring then each polytope obtained by blowing up P has also a non-orientable coloring.*

Theorem 2.3. *There exists a non-orientable small cover over every simple convex 3-polytope, except the 3-simplex.*

Proof. Let P be a simple convex polytope but not the 3-simplex. Operating the blow downs for triangular facets of P over again, P can be transformed to a polytope P' which does not have a triangular facet or is the triangular prism. In the latter case although the triangular prism can be transformed to the 3-simplex further by the blow down, we stop the operation because the 3-simplex is excepted from this the-

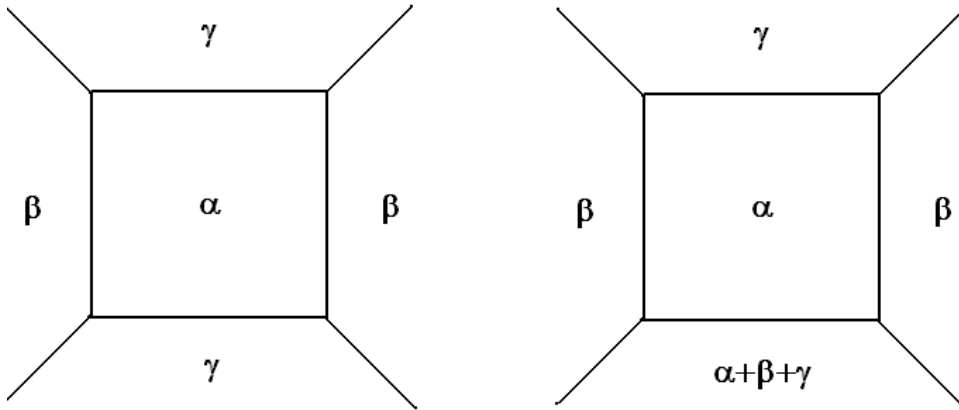


Fig. 2.

orem. By Lemma 2.2, a non-orientable coloring of P' leads to that of P . Therefore we can assume that P does not have a triangular facet or P is the triangular prism. We assume the four color theorem, and shall prove that some facets of 4-colored polytope can be repainted making allowance for the linearly independent condition and construct a non-orientable coloring. Here we assume that P is colored by four colors $\{\alpha, \beta, \gamma, \alpha + \beta + \gamma\}$ for some basis $\{\alpha, \beta, \gamma\}$ of \mathbf{Z}_2^3 . (When P is colored by only three colors, we can repaint a facet and assume that P is 4-colored if necessary.) The case that P has a quadrilateral facet is immediate. In fact, the 4-coloring around a quadrilateral facet F must be the following situation: a center quadrilateral F is colored by α and two facets adjacent to F are colored by β and the rest two facets adjacent to F are colored by one color γ or two colors γ and $\alpha + \beta + \gamma$, respectively (see Fig. 2). In both cases we can repaint the center quadrilateral by $\alpha + \beta$ instead of α , and produce the non-orientable coloring. In particular, the triangular prism has a non-orientable coloring because it has a quadrilateral facet.

Suppose that P has no triangle and quadrilateral. By Lemma 2.1, P must have a pentagonal facet F . We can assume that the 4-coloring around F is in the following situation: the center pentagon F is colored by α and the adjacent five facets F_1, \dots, F_5 are 3-colored by $\beta, \gamma, \beta, \gamma$ and $\alpha + \beta + \gamma$, respectively (see Fig. 3). Here we shall repaint some facets of P and construct a non-orientable coloring in a way similar to the proof of the classical Five Color Theorem by Kempe using the “Kempe chain” (cf. [8] or [6]). First we consider the $\{\alpha, \beta\}$ -chain containing the pentagon F , i.e., the connected component of facets colored by α or β , which contains F . If the $\{\alpha, \beta\}$ -chain has no elementary cycle containing F then we divide it by the edge $F \cap F_3$ into two chains, and the one side which contains F_3 can be repainted by $\alpha + \gamma$ and $\beta + \gamma$ instead of α and β , respectively. If the $\{\alpha, \beta\}$ -chain has an elementary cycle containing F then F_2 and F_4 belong to a different component of $\{\gamma, \alpha + \beta + \gamma\}$ -chain respectively, because of the Jordan curve theorem. Therefore the one side of them can

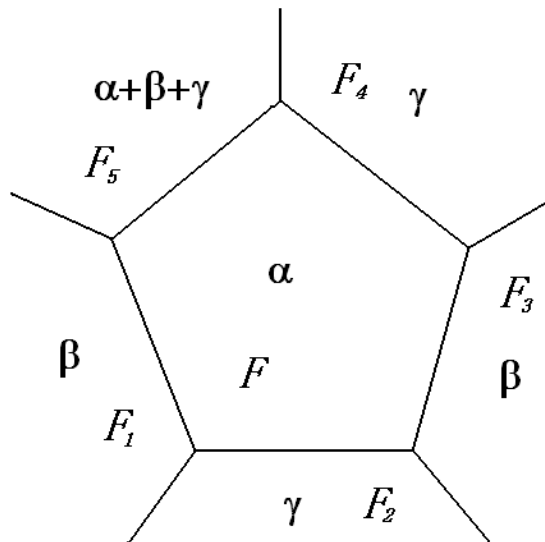


Fig. 3.

be repainted by $\alpha + \gamma$ and $\beta + \gamma$ instead of γ and $\alpha + \beta + \gamma$, respectively. In both cases the repainted polytope is five or six-colored and the new coloring also satisfies the linearly independent condition. Therefore we obtain a non-orientable coloring of P . \square

3. Coloring simple polyhedral handlebodies

Let P be an n -dimensional nice manifold with corners. We say that P is a *simple polyhedral complex* if its dual complex K_P is a simplicial decomposition of $\partial(P)$. This condition implies that any intersection of two faces is a face of P . We can characterize a simple polyhedral complex by a pair of a manifold P and a simplicial decomposition K of $\partial(P)$. In fact a simplicial decomposition K of $\partial(P)$ determines the polyhedral structure of P as follows. For each simplex $\sigma \in K$, let F_σ denote the geometric realization of the poset $K_{\geq \sigma} = \{\tau \in K \mid \sigma \leq \tau\}$. We say that F_σ is a *codimension- k face* of P if σ is a $(k - 1)$ -simplex of K . Then its dual complex K_P is clearly same as K .

We call a facet-coloring of P (or a vertex-coloring of K_P) simply a *coloring* of P (or K_P). We denote by \mathcal{F} the set of facets of P (or the set of vertices of K_P). A function $\lambda: \mathcal{F} \rightarrow \mathbf{Z}_2^n$ is called a linearly independent coloring of P (or K_P) if λ satisfies for P the condition (*) in §1. We put $M(P, \lambda) = (P \times \mathbf{Z}_2^n) / \sim$, where the equivalence relation \sim is defined as (1) in §1. We have a \mathbf{Z}_2^n -action on an n -dimensional manifold $M(P, \lambda)$ with the orbit space P . Conversely an n -dimensional manifold M endowed with a locally standard \mathbf{Z}_2^n -action whose orbit space is homeomorphic to P determines

a characteristic function $\lambda: \mathcal{F} \rightarrow \mathbf{Z}_2^n$. Then M is equivalent to $M(P, \lambda)$ if the restriction on $\pi^{-1}(\text{int } P)$ of the projection $\pi: M \rightarrow P$ is a trivial covering. We will say that $M(P, \lambda)$ is a *small cover* over P . (Warning: In [5], for each $(n-1)$ -dimensional simplicial complex K , the simple polyhedral complex P_K is the cone on K . Then $M(P_K, \lambda)$ can be defined in a similar way, however, it is not always a manifold, and therefore it is called a “ \mathbf{Z}_2^n -space” in [5].)

When P is an orientable simple polyhedral complex, the orientability condition of $M(P, \lambda)$ is same as the condition in Theorem 1.7. Therefore we may generalize the notion of (*non-*) *orientable coloring* of P (or K_P) to this case. Henceforth, we take P as a simple polyhedral handlebody with genus $g > 0$, i.e., a handlebody P together with a simplicial decomposition K_P of the orientable closed surface Σ_g with genus g . In this case, the formula (2) in Lemma 2.1 is generalized to

$$(3) \quad \sum_{k \geq 3} (k - 6)p_k = 12(g - 1)$$

where p_k is the number of k -cornered facets of P (or vertices of K_P whose degree is k). In the rest of this section we shall prove the following theorem.

Theorem 3.1. *Let P be a 4-colorable simple polyhedral handlebody with genus $g > 0$ (equivalently, there exists an orientable small cover over P). If P has sufficiently many facets then there also exists a non-orientable small cover over P .*

Assume that P is colored by four colors $\{\alpha, \beta, \gamma, \alpha + \beta + \gamma\}$ for some basis $\{\alpha, \beta, \gamma\}$ of \mathbf{Z}_2^3 . We shall repaint some facets of P and construct a non-orientable coloring. By the same reason as in the proof of Theorem 2.3, when P has a quadrilateral facet, the construction of non-orientable coloring is immediate.

Next we consider two operations “*blow down*” and “*blow up*” introduced in §2 (see Fig. 1). We can define these operations for a simple polyhedral handlebody P (or a simplicial decomposition of an orientable surface K_P) in a similar way. The blow up can be always done for any vertex of P together with a linearly independent coloring. Notice that for any simple polyhedral handlebody with a positive genus, two triangular facets must not adjoin each other. Therefore the blow down can be always done for any triangular facet of P . Operating the blow down for triangular facets of P one after another, we can reduce P to a simple polyhedral complex P' which has no triangular facet. As we have already seen in Lemma 2.2, a non-orientable coloring on P' can be extended on P . In the course of this process if a quadrilateral facet appears, we can also construct a non-orientable coloring of P . We assume that a quadrilateral facet does not appear during the reduction from P to P' . Generally if we operate blow up for a vertex of a triangular facet then a quadrilateral facet will be created. By the above assumption P must be obtained by blow up for original vertices of P' (but not for new vertices born by blow up). Therefore the number of facets of P is

at most the sum of numbers of facets and vertices of P' . Consequently it is sufficient to prove Theorem 3.1 in the case that P does not have a quadrilateral or a triangle. In fact if the following proposition holds for any simple polyhedral handlebody which has more than N facets but not a quadrilateral and a triangle then Theorem 3.1 holds for any simple polyhedral handlebody which has more than $N + M$ facets where M is the maximum of numbers of vertices of simple polyhedral handlebodies which do not have more than N facets and a quadrilateral and a triangle.

Proposition 3.2. *Let P be a 4-colorable simple polyhedral handlebody with genus $g > 0$ such that P does not have a quadrilateral or a triangle (equivalently its dual K_P is a simplicial decomposition of an orientable surface Σ_g with genus $g > 0$ such that K_P does not have a vertex with degree three or four). If P has sufficiently many facets then P has a non-orientable coloring.*

For a subset A of vertices of a simplicial complex K , we denote by Γ_A the subgraph of one-skeleton K^1 generated by A (which is called the *section subgraph*). We need the following lemma instead of the Jordan curve theorem.

Lemma 3.3. *Let K be a simplicial decomposition of the orientable closed surface Σ_g with genus $g > 0$, and (A, B) be a division of vertices of K ($A \sqcup B = V(K)$) such that the section subgraphs Γ_A, Γ_B are both connected and have no cycle of length three. When $2g + 1$ edges are removed from $\Gamma_A \cup \Gamma_B$, either Γ_A or Γ_B is disconnected.*

Proof. Because Γ_A and Γ_B have no cycle of length three, each 2-simplex Δ of K intersects both of Γ_A and Γ_B , i.e., all vertices and only one edge of Δ belong to $\Gamma_A \cup \Gamma_B$. Therefore the number of 2-simplices of K and the number of edges of K which do not belong to $\Gamma_A \cup \Gamma_B$ coincide. Thus $\chi(\Gamma_A \cup \Gamma_B) = \chi(K) = 2 - 2g$ where $\chi(G)$ is the Euler number of G . Since Γ_A and Γ_B are both one-dimensional connected subcomplexes of K , this means that the first Betti number of $\Gamma_A \cup \Gamma_B$ is $2g$, thus the lemma follows. \square

For a four-colored simple polyhedral handlebody P (or a simplicial decomposition K_P of a surface Σ_g), we consider a division of facets \mathcal{F} into two Kempe chains in a way similar to the proof of Theorem 2.3, e.g., $\mathcal{F} = A \sqcup B$ where A (resp. B) is a set of vertices which are colored by α or β (resp. γ or $\alpha + \beta + \gamma$). In this case $\{\alpha, \beta\}$ -chain of P (resp. $\{\gamma, \alpha + \beta + \gamma\}$ -chain) corresponds to the section subgraph Γ_A (resp. Γ_B) of K_P . We notice that there are three ways to divide \mathcal{F} into two Kempe chains, i.e., $\{\alpha, \beta\} \sqcup \{\gamma, \alpha + \beta + \gamma\}$, $\{\alpha, \gamma\} \sqcup \{\beta, \alpha + \beta + \gamma\}$ and $\{\alpha, \alpha + \beta + \gamma\} \sqcup \{\beta, \gamma\}$. If an $\{\alpha, \beta\}$ -chain is disconnected then one of its connected component can be repainted by $\alpha + \gamma$ and $\beta + \gamma$ instead of α and β , respectively, and we obtain a non-orientable

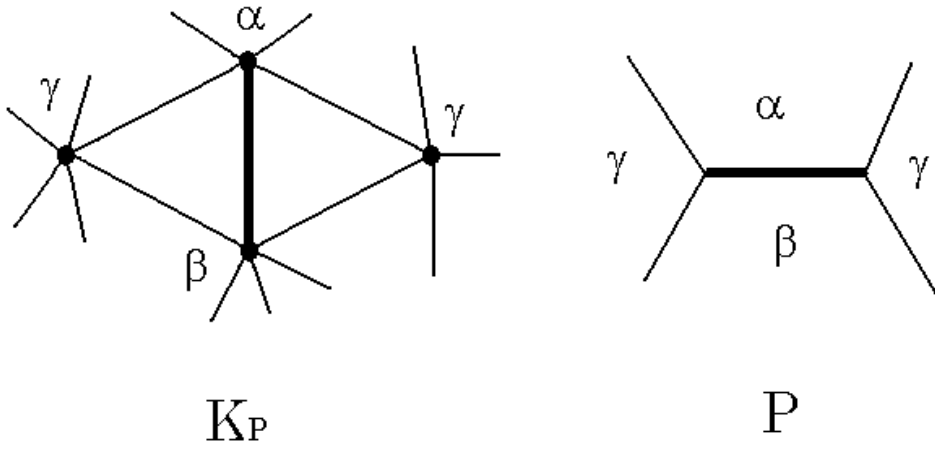


Fig. 4.

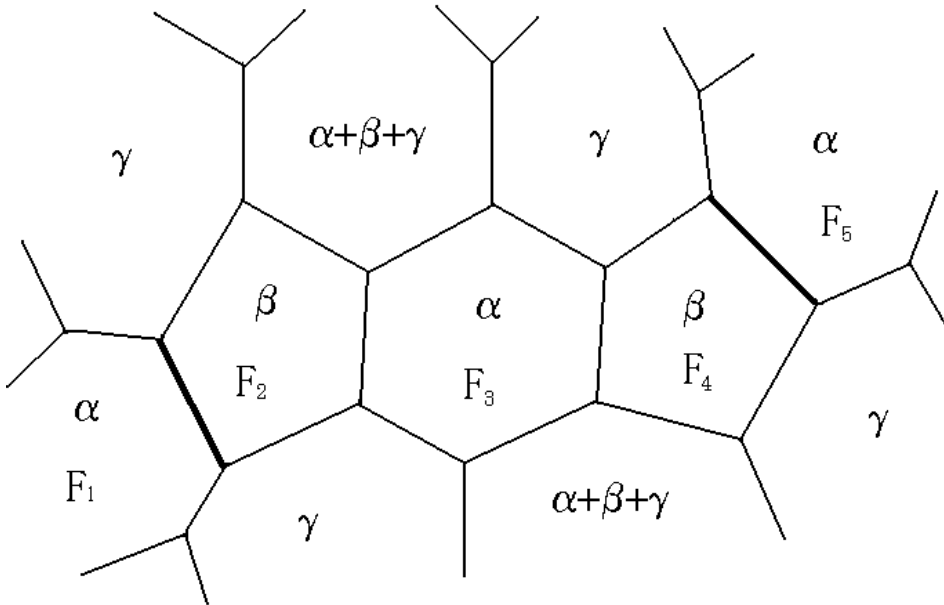


Fig. 5.

coloring of P . Assume that every chain is connected. Then each division of \mathcal{F} into two chains satisfies the condition in Lemma 3.3.

In order to divide a connected Kempe-chain into two components we introduce a notion of a *cuttable edge* of K_P (or P). An edge of K_P is called a *cuttable edge* (of type $(\{\alpha, \beta\}, \gamma)$) when its star subcomplex of K_P (i.e., the subcomplex generated

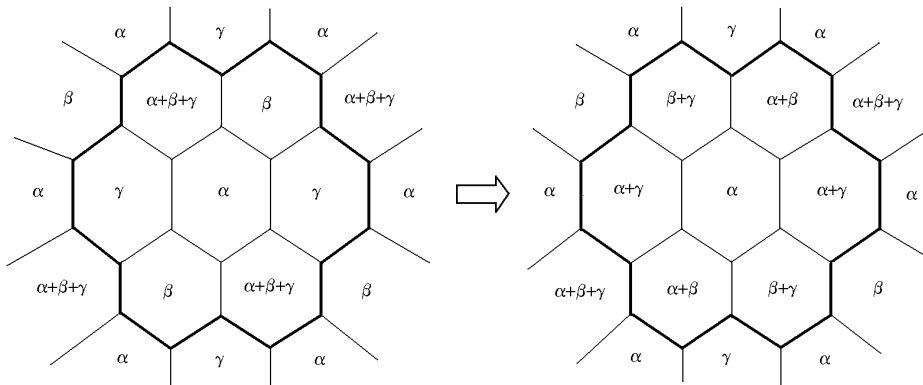


Fig. 6.

by simplices which contain the edge) is three-colored, i.e., the both end vertices of the edge are colored by $\{\alpha, \beta\}$ and others are colored by only one color γ . Similarly an edge of P is called a cutable edge when the dual edge is a cutable edge of K_P (see Fig. 4). A cutable edge of type $(\{\alpha, \beta\}, \gamma)$ is an edge of $\{\alpha, \beta\}$ -chain. If there exist cutable edges of a same type $(\{\alpha, \beta\}, \gamma)$ such that an $\{\alpha, \beta\}$ -chain becomes to be disconnected when they are removed, then one of its connected component can be repainted and we can construct a non-orientable coloring of P . For example, for a four-coloring of P shown in Fig. 5, $\{\alpha, \beta\}$ -chain is the set of facets F_i 's and $F_1 \cap F_2$ and $F_4 \cap F_5$ are cutable edges of the same type $(\{\alpha, \beta\}, \gamma)$. Here a component $F_2 \cup F_3 \cup F_4$ of $\{\alpha, \beta\}$ -chain between two cutable edges can be repainted by $\alpha + \gamma$ and $\beta + \gamma$ instead of α and β , respectively, and we can construct a non-orientable coloring of P . We remark that the edge $F \cap F_3$ in Fig. 3 in the proof of Theorem 2.3 is a cutable edge which divides connected chain into two components. When P has more than $12g$ cutable edges, there exists a division $\mathcal{F} = A \sqcup B$ into two chains (Γ_A, Γ_B) such that $\Gamma_A \cup \Gamma_B$ has more than $4g$ cutable edges because there are three ways to divide \mathcal{F} into two Kempe chains. Here there are at most two types of cutable edges contained in Γ_A (or Γ_B), respectively. Then either of Γ_A or Γ_B becomes to be disconnected when cutable edges of a same type are removed because of Lemma 3.3. Therefore P can be repainted as a non-orientable coloring when P has more than $12g$ cutable edges.

Denote the number of facets of P with k -corners by p_k . By assumption $p_3 = p_4 = 0$. We notice that a facet with k -corners has at least two cutable edges if k is not a multiple of three. Therefore if

$$(4) \quad \sum_{k \not\equiv 0 \pmod{3}} p_k > 12g$$

then there exist more than $12g$ cutable edges and P can be repainted as a non-

orientable coloring. If there exists a hexagonal facet of P such that the six facets adjacent to it are all hexagonal, then the seven facets can be repainted as a non-orientable coloring as shown in Fig. 6 or they have at least two cutable edges. This is the case if $p_6 > \sum_{k \neq 6} kp_k$ because a k -cornered facet ($k \neq 6$) adjacent to at most k hexagonal facets. More generally, if $p_6 - \sum_{k \neq 6} kp_k > 7(t - 1)$ then P can be repainted as a non-orientable coloring or there exist more than t hexagonal facets each of which has at least two cutable edges. Therefore, if the above inequality hold for $t = 12g - \sum_{k \equiv 0 \pmod{3}} p_k$, i.e.

$$(5) \quad p_6 > \sum_{k \neq 6} kp_k + 7 \left(12g - \sum_{k \equiv 0 \pmod{3}} p_k - 1 \right)$$

then P can be repainted as a non-orientable coloring.

Values of p_k 's, which do not satisfy the above inequities (4) and (5), are bounded. In fact, it follows from (3) and inequalities opposite to (4) and (5) that

$$\begin{aligned} \sum_k p_k &\leq \sum_{k \neq 6} (k+1)p_k + 7 \left(12g - \sum_{k \equiv 0 \pmod{3}} p_k - 1 \right) && \text{from (5)}^\dagger \\ &= -p_5 + \sum_{k > 6} (k-6)p_k + 7 \sum_{k \equiv 0 \pmod{3}, k \geq 9} p_k + 84g - 7 \\ &= 7 \sum_{k \equiv 0 \pmod{3}, k \geq 9} p_k + 96g - 19 && \text{from (3)} \\ &\leq \frac{7}{3} \sum_{k \geq 7} (k-6)p_k + 96g - 19 \\ &= \frac{7}{3} (p_5 + 12g - 12) + 96g - 19 && \text{from (3)} \\ &= \frac{7}{3} p_5 + 124g - 47 \\ &\leq \frac{7}{3} \sum_{k \equiv 0 \pmod{3}} p_k + 124g - 47 \\ &\leq 152g - 47 && \text{from (4)}^\dagger \end{aligned}$$

where (4)[†] and (5)[†] are the inequalities opposite to (4) and (5), respectively. Therefore the proof of Theorem 3.1 is completed.

REMARK 3.4. The 4-colorability of graphs embedded into an orientable surface is an interesting problem. For example, we have the following conjecture (cf. [4, Conjecture 1.1]): *every simplicial decomposition of an orientable surface such that all vertices have even degree and all non-contractible cycles are sufficiently large*

is 4-colorable. In case that $g = 1$ and a graph satisfies a special condition called “6-regular,” the 4-colorability of toroidal 6-regular graph was studied in [4].

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