THE SCHUR INDICES OF THE CUSPIDAL UNIPOTENT CHARACTERS OF THE FINITE CHEVALLEY GROUPS $E_7(q)$

Dedicated to Professor Herbert Pahlings on his 65th birthday

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Abstract

We show that the two cuspidal unipotent characters of a finite Chevalley group $E_7(q)$ have Schur index 2, provided that $q$ is an even power of a (sufficiently large) prime number $p$ such that $p \equiv 1 \mod 4$. The proof uses a refinement of Kawanaka’s generalized Gelfand–Graev representations and some explicit computations with the CHEVIE computer algebra system.

1. Introduction

Throughout this paper, let $G$ be a simple algebraic group of adjoint type $E_7$. Assume that $G$ is defined over the finite field $\mathbb{F}_q$, with corresponding Frobenius map $F: G \rightarrow G$. There are precisely two cuspidal unipotent characters of $G^F$ denoted by $E_7[\pm \xi]$ where $\xi = \sqrt{-q}$; see the table in [2], §13.9.

The purpose of this paper is to determine the Schur index of $E_7[\pm \xi]$, at least if the characteristic of $\mathbb{F}_q$ is large enough. Modulo this condition on the characteristic, this completes the determination of the Schur indices of the unipotent characters of finite groups of Lie type; see [12], [5] and the references there.

By [4], Table 1, the character values of $E_7[\pm \xi]$ generate the field $\mathbb{Q}(\xi)$. Furthermore, by [4], Example 6.4, we already know that the Schur index is 1 if $p \not\equiv 1 \mod 4$ or if $q$ is not a square, where $p$ is the characteristic of $\mathbb{F}_q$. Thus, the remaining task is to determine the Schur index when $q$ is a square and $p \equiv 1 \mod 4$.

Theorem 1.1. Assume that $q$ is an even power of a (sufficiently large) prime $p$ such that $p \equiv 1 \mod 4$. Then the characters $E_7[\pm \xi]$ have Schur index 2.

Here, $p$ is “sufficiently large” if Lusztig’s results [11] on generalized Gelfand–Graev characters hold; it is conjectured that this is the case if $p$ is good for $G$.

The idea of the proof is as follows. We have already seen in [5], §4, that $E_7[\pm \xi]$ occur with multiplicity 1 in a generalized Gelfand–Graev character $\Gamma_u$, where $u$ is a

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Table 1. The weighted Dynkin diagram for the unipotent support of the cuspidal unipotent characters in type $E_7$

![Dynkin diagram]

certain unipotent element in $G$. Here, we shall use a refinement of the construction of $\Gamma_u$ to show that, under the given assumptions on $p$ and $q$, the characters $E_7[\pm \xi]$ occur with odd multiplicity in an induced character which cannot be realized over $\mathbb{Q}(\xi)$. By standard arguments on Schur indices, this implies that $E_7[\pm \xi]$ cannot be realized over $\mathbb{Q}(\xi)$. At some stage, the proof relies on the fact that, in Lusztig’s parametrization of the irreducible characters of $G^F$, the function $\Delta$ occurring in [10], Main Theorem 4.23, takes value $-1$ on the labels corresponding to $E_7[\pm \xi]$.

Furthermore, we rely on some explicit computations in $G^F$. However, we shall only use computations with the root system and the irreducible characters of the Weyl group of $G$, for which the CHEVIE system [6] is a convenient tool.

2. Generalized Gelfand–Graev characters for type $E_7$

A short summary of the construction of generalized Gelfand–Graev characters is given in [5], §2. Assume that $q$ is a power of a “good” prime $p \neq 2, 3$. Let $\Phi$ be the root system of $G$ with respect to a fixed maximally split torus $T$. Let $C$ be the unipotent class of $G$ whose weighted Dynkin diagram $d: \Phi \to \mathbb{Z}$ is given in Table 1. (The notation in that table also defines a labelling of the simple roots in the root system of $G$.) The class $C$ is the “unipotent support” of the two cuspidal unipotent characters of $G^F$; see [5], §4, and the references there.

Given the weight function $d: \Phi \to \mathbb{Z}$ specified by the diagram in Table 1, we define unipotent subgroups

$$U_{d,2} := \prod_{\alpha \in \Phi^+ \atop d(\alpha) \geq 2} X_\alpha$$

and

$$U_{d,1} := \prod_{\alpha \in \Phi^+ \atop d(\alpha) \geq 1} X_\alpha,$$

where $X_\alpha$ is the root subgroup in $G$ corresponding to the root $\alpha$. (It is understood that the products are taken in some fixed order.) The generalized Gelfand–Graev character associated with an element in $C^F$ is obtained by inducing a certain linear character from $U_{d,2}^F$. We have $C_G(u)/C_G(u^2) \cong \mathbb{Z}/2\mathbb{Z}$ for $u \in C$. Thus, $C^F$ splits into two classes in the finite group $G^F$. By Mizuno [13], Lemma 28, representatives of these
two $G^F$-classes are given by

$$y_{74} = x_{20}(1)x_{21}(1)x_{23}(1)x_{28}(1)x_{31}(1),$$

$$y_{75} = x_{20}(1)x_{21}(1)x_{28}(1)x_{24}(1)x_{23}(1)x_{25}(1)x_{36}(\xi),$$

where $\xi$ is a generator for the multiplicative group of $\mathbb{F}_q$ and where the subscripts correspond to the following roots in $\Phi^+$:

- $20 : \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$
- $21 : \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5,$
- $23 : \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6,$
- $24 : \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,$
- $25 : \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,$
- $28 : \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5,$
- $31 : \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,$
- $36 : \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7.$

(Attention: Here, we use the labelling of the roots as given by the CHEVIE system [6], which is slightly different from that of Mizuno.) We note that both $y_{74}$ and $y_{75}$ lie in $C \cap U_{d,2}^F$. Now let us fix

$$u \in \{y_{74}, y_{75}\} \subseteq C \cap U_{d,2}^F;$$

the above expressions show that

$$u = \prod_{\alpha \in \Phi^+} x_\alpha(\eta_\alpha) \quad \text{where } \eta_\alpha \in \mathbb{F}_q.$$ 

Then we define a linear character $\varphi_u : U_{d,2}^F \to \mathbb{C}^\times$ by the formula

$$\varphi_u\left(\prod_{\alpha \in \Phi^+} x_\alpha(\xi_\alpha)\right) = \chi\left(\sum_{\alpha \in \Phi^+} c_\alpha \eta_\alpha \xi_\alpha\right) \quad \text{for all } \xi_\alpha \in \mathbb{F}_q,$$

where $c_\alpha \in \mathbb{F}_q$ are certain fixed constants (independent of the $\eta_\alpha$ and $\xi_\alpha$) and where $\chi : \mathbb{F}_q^+ \to \mathbb{C}^\times$ is a fixed non-trivial character of the additive group of $\mathbb{F}_q$; see [5], Definition 2.1, for more details. It will actually be convenient to choose $\chi$ in the following special way. Let $\chi_0 : \mathbb{F}_p^+ \to \mathbb{C}^\times$ be a fixed non-trivial character of the additive group of $\mathbb{F}_p$. Then we take $\chi$ to be

$$\chi := \chi_0 \circ \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$$

where $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q^+ \to \mathbb{F}_p^+$ is the trace map. Now we have

$$\text{Ind}_{U_{d,2}^F}^{G^F}(\varphi_u) = [U_{d,1}^F : U_{d,2}^F]^{1/2} \cdot \Gamma_u,$$

where $\Gamma_u$ is the generalized Gelfand–Graev character associated with $u$. We have seen in [5], Corollary 4.3, that

$$\langle E_7[\pm \xi], \Gamma_u \rangle_{G^F} = 1 \quad \text{for suitable } u \in \{y_{74}, y_{75}\}.$$
(Here, and throughout the paper, we denote by \((\ , \ )_A\) the standard inner product on the character ring of a finite group \(A\).)

We will now refine the construction of \(\Gamma_u\). The strategy for doing this has already been outlined in [5], §4. For this purpose, we shall assume from now on that

\[ q \text{ is an even power of } p. \]

Since \(G\) is simple of adjoint type, we have an \(\mathbb{F}_q\)-isomorphism

\[
h: \prod_{7 \text{ factors}} k^x \to T, \quad (x_1, \ldots, x_7) \mapsto h(x_1, \ldots, x_7),
\]

such that \(\alpha_i(h(x_1, \ldots, x_7)) = x_i \) for \(1 \leq i \leq 7\). In particular, we have \(T^F = \{h(x_1, \ldots, x_7) \mid x_i \in \mathbb{F}_q^x\}\). We shall set

\[
t := h(v^{1/2}, 1, 1, v^{1/2}, 1, v^{1/2}, 1) \in T^F
\]
as in the proof of [5], Lemma 4.1, where \(v\) is a generator for the multiplicative group of \(\mathbb{F}_p \subset \mathbb{F}_q\) and \(v^{1/2}\) is a square root of \(v\) in \(\mathbb{F}_q\). (The square root exists since \(q\) is an even power of \(p\).) Then \(t\) has the property that \(\alpha(t) = v\) for all roots \(\alpha\) involved in the expressions for \(y_{74}\) or \(y_{75}\) as products of root subgroup elements; furthermore, we have \(\alpha(t) = 1 \) for all roots \(\alpha\) such that \(d(\alpha) = 0\). The element \(t\) has order \(2(p - 1)\) and \(H := \langle t \rangle\) normalizes \(U_{d,2}\). We set

\[
s_1 := h(-1, 1, 1, -1, 1, -1, 1) = t^{p-1} \in T^F.
\]

Note that \(\alpha(s_1) = 1\) for all roots \(\alpha \in \Phi^+\) which are involved in the expressions of \(y_{74}\) and \(y_{75}\) as products of root subgroup elements. Thus, \(s_1\) fixes the character \(\varphi_u\), and so we can extend \(\varphi_u\) to \(U_{d,2}^F, \langle s_1 \rangle\). Actually, there are two such extensions which we denote by \(\tilde{\varphi}_u\) and \(\tilde{\varphi}'_u\). Their values are determined by

\[
\tilde{\varphi}_u(xs_1) = \varphi_u(x) \quad \text{and} \quad \tilde{\varphi}'_u(xs_1) = -\varphi_u(x) \quad \text{for all } x \in U_{d,2}^F.
\]

**Definition 2.1.** Let \(u \in \{y_{74}, y_{75}\}\). Then we set

\[
\psi_u := \text{Ind}_{U_{d,2}^F, \langle s_1 \rangle}^{U_{d,2}^F, H}(\tilde{\varphi}_u) \quad \text{and} \quad \psi'_u := \text{Ind}_{U_{d,2}^F, \langle s_1 \rangle}^{U_{d,2}^F, H}(\tilde{\varphi}'_u).
\]

Thus, we have

\[
\text{Ind}_{U_{d,2}^F, \langle s_1 \rangle}^{U_{d,2}^F, H} (\varphi_u) = \psi_u + \psi'_u \quad \text{and} \quad [U_{d,1}^F : U_{d,2}^F]^{1/2} \cdot \Gamma_u = \tilde{\Gamma}_u + \tilde{\Gamma}'_u,
\]

where

\[
\tilde{\Gamma}_u := \text{Ind}_{U_{d,2}^F, \langle s_1 \rangle}^{U_{d,2}^F, H} (\varphi_u) = \text{Ind}_{U_{d,2}^F, H}^{U_{d,2}^F} (\psi_u),
\]
The following result provides some crucial information concerning $\psi_{u}$ and $\psi'_{u}$.

**Proposition 2.2.** Recall that $q$ is an even power of $p$. Then, with the above notation, the following hold.

(a) Both $\psi_{u}$ and $\psi'_{u}$ are irreducible characters of $U_{d,2}^{F,H}$.
(b) $\psi_{u}$ can be realized over $\mathbb{Q}$.
(c) $\psi'_{u}$ is rational-valued but cannot be realized over $\mathbb{Q}$. In fact, $\psi'_{u}$ has non-trivial local Schur indices at $\infty$ and at the prime $p$.

Proof. (see also the argument of Ohmori [14], p. 154.) Let

$$x := \prod_{\alpha \in B_{2}, \alpha \neq 0} x_{\alpha}(\xi_{\alpha}) \in U_{d,2}^{F}$$

and

$$y_{X} := \sum_{\alpha \in B_{2}, \alpha \neq 0} c_{\alpha} \eta_{\alpha} \xi_{\alpha},$$

where $\xi_{\alpha} \in F_q$. Then, as in the proof of [5], Proposition 2.3, we have

$$\varphi_{u}(t^{i}xt^{-i}) = \chi(\nu^{i}y_{X}) \quad \text{for } 1 \leq i \leq 2(p - 1).$$

In particular, this implies $\text{Stab}_{H}(\varphi_{u}) = (s_{1})$. Hence, by Clifford theory, the induced character

$$\text{Ind}_{U_{d,2}^{F,H}}^{U_{d,2}^{F,H}}(\varphi_{u}) = \psi_{u} + \psi'_{u}$$

has inner product 2. Thus, we $\psi_{u}$ and $\psi'_{u}$ must be irreducible, proving (a).

Next we prove (b). Using Mackey’s formula and relation (1), we have that

$$\text{Ind}_{U_{d,2}^{F,H}}^{U_{d,2}^{F,H}}(\varphi_{u})(x) = \sum_{i=1}^{2(p-1)} \varphi_{u}(t^{i}xt^{-i}) = \sum_{i=1}^{2(p-1)} \chi(\nu^{i}y_{X})$$

$$= \sum_{i=1}^{2(p-1)} \chi_{0}(\nu^{i} \text{Tr}_{q^{d}/q}(\gamma_{X})) = \left\{ \begin{array}{cl} 2(p-1) & \text{if } \text{Tr}_{q^{d}/q}(\gamma_{X}) = 0, \\ -2 & \text{if } \text{Tr}_{q^{d}/q}(\gamma_{X}) \neq 0. \end{array} \right.$$

In particular, this shows that the values are rational integers. Thus, $\psi_{u} + \psi'_{u}$ is rational-valued. Now assume, if possible, that $\psi_{u}$ is not rational-valued. Then the characters $\psi_{u}$ and $\psi'_{u}$ must be algebraically conjugate. Consequently, $\psi_{u}$ and $\psi'_{u}$ occur with the same multiplicity in every rational-valued character. Now, by the Mackey formula and Frobenius reciprocity, we have

$$\left\langle \psi'_{u}, \text{Ind}_{H}^{U_{d,2}^{F,H}}(1_{H}) \right\rangle_{U_{d,2}^{F,H}} = \left\langle \text{Ind}_{U_{d,2}^{F,H}}^{U_{d,2}^{F,H}}(\varphi'_{u}), \text{Ind}_{H}^{U_{d,2}^{F,H}}(1_{H}) \right\rangle_{U_{d,2}^{F,H}}$$

$$= \left\langle \text{Res}_{(g_{1})}^{U_{d,2}^{F,H}}(\varphi'_{u}), 1_{(g_{1})} \right\rangle_{U_{d,2}^{F,H}}.$$
= 0,
since $\tilde{\varphi}_u'(s_1) = -1$. (Here, the symbol $\mathbf{1}$ stands for the unit character.) By a similar argument, since $\tilde{\varphi}_u(s_1) = 1$, we also have

$$\left\langle \psi_u, \text{Ind}^{U_{d^2,2}^F|H}_{U_{d^2,2}^F|H}(1_H) \right\rangle_{U_{d^2,2}^F|H} = 1.$$  

Thus, $\psi_u$ and $\psi_u'$ do not occur with the same multiplicity in some rational-valued character, a contradiction. Thus, our assumption was wrong and so both $\psi_u$ and $\psi_u'$ are rational-valued. But then the above multiplicity 1 formula implies that $\psi_u$ can be realized over $\mathbb{Q}$, by a standard argument concerning Schur indices (see Isaacs [8], Corollary 10.2).

Finally, we prove (c). We begin by showing that the local Schur index at $\infty$ is non-trivial. In other words, we must show that $\psi_u'$ cannot be realized over $\mathbb{R}$. For this purpose, by a well-known criterion due to Frobenius and Schur (see Isaacs [8], Chapter 4), it is enough to show that

$$\frac{1}{|U_{d^2,2}^F|} \sum_{g \in U_{d^2,2}^F} \psi_u'(g^2) = -1.$$  

Now, in order to evaluate the above sum, we note that

$$\frac{1}{|U_{d^2,2}^F|} \sum_{g \in U_{d^2,2}^F} \psi_u(g^2) = 1,$$

since $\psi_u$ can be realized over $\mathbb{Q}$. Thus, it will be enough to show that

$$\frac{1}{|U_{d^2,2}^F|} \sum_{g \in U_{d^2,2}^F} \text{Ind}^{U_{d^2,2}^F|H}_{U_{d^2,2}^F|H}(\varphi_u)(g^2) = \frac{1}{|U_{d^2,2}^F|} \sum_{g \in U_{d^2,2}^F} (\psi_u + \psi_u')(g^2) = 0.$$  

Let $g \in U_{d^2,2}^F|H$ and write $g = xh$ where $x \in U_{d^2,2}^F$ and $h \in H$. Now the value of the above induced character on $g^2$ is zero unless $g^2 \in U_{d^2,2}^F$. Thus, we only need to consider elements $g = xh$ where $h = 1$ or $h = s_1$. So we must show that

$$\sum_{x \in U_{d^2,2}^F} \text{Ind}^{U_{d^2,2}^F|H}_{U_{d^2,2}^F|H} \left(\varphi_u\right)(x^2) + \sum_{x \in U_{d^2,2}^F} \text{Ind}^{U_{d^2,2}^F|H}_{U_{d^2,2}^F|H} \left(\varphi_u\right)(xs_1xs_1) = 0.$$  

Now, since $U_{d^2,2}^F$ has odd order, the map $x \mapsto x^2$ defines a bijection of $U_{d^2,2}^F$ onto itself. Hence the first sum evaluates to

$$\sum_{x \in U_{d^2,2}^F} \text{Ind}^{U_{d^2,2}^F|H}_{U_{d^2,2}^F|H} \left(\varphi_u\right)(x^2) = |U_{d^2,2}^F| \cdot \left\langle \text{Ind}^{U_{d^2,2}^F|H}_{U_{d^2,2}^F|H} \left(\varphi_u\right), 1_{U_{d^2,2}^F|H} \right\rangle_{U_{d^2,2}^F|H} = 0,$$
Now consider the second sum. For this purpose, we note that $\alpha(S_1) = 1$ for all roots $\alpha \in \Phi^+$ which are involved in the expressions of $\gamma_{14}$ and $\gamma_{15}$ as products of root subgroup elements. Thus, if $\chi$ and $\gamma_\chi$ are as in (1), then we have

$$\gamma_{(\chi S_1)^2} = \sum_{\alpha \in \Phi^+} c_\alpha \eta_\alpha(\alpha(S_1) + 1) \xi_\alpha = 2\gamma_\chi = \gamma_{\chi^2}.$$

Using once more Mackey’s formula as at the beginning of this proof, we see that

$$\text{Ind}_{U_{d;2}}^{U_{d;2},H}(\psi'_u)(x^2) = \sum_{i=1}^{2(p-1)} \chi(i^2 \gamma_{x^2}) = \sum_{i=1}^{2(p-1)} \chi(i^2 \gamma_{(x S_1)^2}) = \text{Ind}_{U_{d;2}}^{U_{d;2},H}(\psi'_u)(x S_1, x S_1)$$

for all $x \in U_{d;2}$. Consequently, the second sum also equals 0. Thus, we have shown that $\psi'_u$ cannot be realized over $\mathbb{R}$. We shall now use some general properties of Schur indices; see Feit [3], §2, for references. First, since $\psi'_u$ is rational-valued but $\psi'_u$ cannot be realized over $\mathbb{R}$, the Schur index of $\psi'_u$ is 2 (by the Brauer–Speiser theorem; see [3], 2.4). Furthermore, there exists at least one prime number $l$ such that the $l$-local Schur index of $\psi'_u$ is 2 (by the Hasse sum formula; see [3], 2.15). Thus, it will be enough to show that the $l$-local Schur index of $\psi'_u$ is 1, for every prime $l \neq p$. Let $l$ be such a prime. If $l \neq 2$, then $\psi'_u$ is a character of $l$-defect 0 of $U_{d;2}^{F}$. So the $l$-local Schur index is 1 by [3], 2.10. Finally, if $l = 2$, then $\psi'_u$ is a character of 2-defect 1 and, hence, lies in a block with a cyclic defect group of order 2. Consequently, that block contains only two irreducible characters and so $\psi'_u$ remains irreducible as a 2-modular Brauer character. This implies again that the local Schur index is 1; see [3], 2.10.

3. A subgroup of type $D_6 \times A_1$

Our next aim is to compute the multiplicity of $E_7[\pm e]$ in $\tilde{\Gamma}_u$ and $\tilde{\Gamma}'_u$; see Definition 2.1. We already know that the multiplicity of $E_7[\mp e]$ in the sum $\tilde{\Gamma}_u + \tilde{\Gamma}'_u$ equals $[U_{d;1}^{F} : U_{d;2}^{F}]^{1/2}$, for suitable $u \in \{\gamma_{14}, \gamma_{15}\}$. We shall now try to compute the multiplicity in the difference $\tilde{\Gamma}_u - \tilde{\Gamma}'_u$. For this purpose, we take a closer look at the semisimple element $S_1$ and its centralizer. Let

$$G_1 := \langle T, X_\alpha \mid \alpha \in \Phi_1 \rangle \quad \text{where} \quad \Phi_1 := \{\alpha \in \Phi \mid \alpha(S_1) = 1\}.$$  

Using the CHEVIE function ReflectionSubgroup, we check that the root system $\Phi_1$ has type $D_6 \times A_1$; a system of simple roots in $\Phi_1$ is given by

$$\Pi_1 = \{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_{14}, \alpha_{18}, \alpha_{28}\}.$$
where
\[ \alpha_{14} := \alpha_1 + \alpha_3 + \alpha_4, \quad \alpha_{18} := \alpha_4 + \alpha_5 + \alpha_6, \quad \alpha_{28} := \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5. \]

(Here, the numbering of the roots is the same as that given by CHEVIE.) The corresponding Dynkin diagram and the restriction of the weight function \( d \) to \( \Pi_1 \) are given in Table 2. Furthermore, one can check, using CHEVIE (for example), that

\[ N_W(W_1) = \{ w \in W \mid w(\Phi_1) \subseteq \Phi_1 \} = W_1 \]

where \( W_1 := \langle w_\alpha \mid \alpha \in \Phi_1 \rangle \subset W \) is the Weyl group of \( G_1 \) (and where we denote by \( w_\beta \) the reflection with root \( \beta \), for any root \( \beta \in \Phi \)).

**Lemma 3.1.** We have \( C_G(S_1) = G_1 \); in particular, \( C_G(S_1) \) is connected.

Proof. By Carter [2], §3.5, we have \( C_G(S_1)^0 = G_1 \). Hence, \( G_1 \) is a normal subgroup in \( C_G(S_1) \). So it is enough to show that \( N_G(G_1) = G_1 \). Let \( g \in N_G(G_1) \). Then \( gTg^{-1} \) is a maximal torus in \( G_1 \) and so there exists some \( g_1 \in G_1 \) such that \( gTg^{-1} = g_1Tg_1^{-1} \). Thus, we have \( g_1^{-1}g \in N_G(T) \) and so \( g \in G_1N_G(T) \). Hence, we may assume without loss of generality that \( g \in N_G(T) \cap N_G(G_1) \). Now, for any \( g \in N_G(T) \cap N_G(G_1) \) and any \( \alpha \in \Phi_1 \), we have \( gX_{\alpha}g^{-1} = X_{w(\alpha)} \subseteq G_1 \), where \( w \) is the image of \( g \) in \( W = N_G(T)/T \). Thus, we have \( w(\Phi_1) \subseteq \Phi_1 \) and so \( w \in W_1 \) (see the above remarks). This implies \( g \in G_1 \), as required.

Let \( C_1 \) be the conjugacy class of \( y_{74} \) in \( G_1 \) and denote by \( d_1 : \Phi_1 \to \mathbb{Z} \) the corresponding weighted Dynkin diagram. Using the identification results in [1], Theorem 11.3.2, it is straightforward to check that, under the natural matrix representation of a group of type \( D_6 \times A_1 \), the elements \( y_{74} \) and \( y_{75} \) correspond to matrices with Jordan blocks of size 1, 1, 2, 5, 5 (where the block of size 2 comes from the \( A_1 \)-factor). Hence, using [2], §13.1, we see that \( d_1 \) is given by the restriction of \( d \) to \( \Phi_1 \), as specified in Table 2. Furthermore, we notice that the above roots can all be written as sums of roots in \( \Pi_1 \). Thus, we have

\[ y_{74}, y_{75} \in C_1 \cap U_{d_1,2}^F, \]
where $U_{d,2}$ is the unipotent subgroup of $G_1$ defined with respect to $d_1$.

**Lemma 3.2.** Let $u \in \{\gamma_74, \gamma_75\}$. Then we have $\dim \mathcal{B}_u^1 = 4$ (where $\mathcal{B}_u^1$ denotes the variety of Borel subgroups of $G_1$ containing $u$) and

$$C_{G_1}(u)/C_{G_1}(u)^\circ \cong C_{G}(u)/C_{G}(u)^\circ \cong \mathbb{Z}/2\mathbb{Z}.$$  

Proof. Let $u := \gamma_74$. The formula for $\dim \mathcal{B}_u^1$ follows from [2], §13.1. To prove the remaining statements, we note that

$$s_1 \in S := \{h(x, x^{-2}, x^{-2}, x^3, x^{-2}, x, 1) \mid x \in k^x\} \subseteq C_{G_1}(u).$$

Furthermore, one checks that $Z(G_1) = \{t \in T \mid \alpha(t) = 1 \text{ for all } \alpha \in \Phi_1\} = \langle s_1 \rangle$. Thus, since $S$ is connected, we have $Z(G_1) \subseteq C_{G_1}(u)^\circ$.

Now let $\pi : G_1 \to H_1$ be the adjoint quotient of $G_1$, where $H_1$ is a semisimple group of adjoint type $D_5 \times A_1$. Let $\bar{u}$ be the image of $u$ in $H_1$. Then, by Carter [2], §13.1, we know that $C_{H_1}(\bar{u})/C_{H_1}(\bar{u})^\circ \cong \mathbb{Z}/2\mathbb{Z}$. Furthermore, $\pi$ induces a surjective homomorphism

$$C_{G_1}(u)/C_{G_1}(u)^\circ \twoheadrightarrow C_{H_1}(\bar{u})/C_{H_1}(\bar{u})^\circ \cong \mathbb{Z}/2\mathbb{Z}$$

with kernel given by the image of $Z(G_1)$ in $C_{G_1}(u)/C_{G_1}(u)^\circ$. Since $Z(G_1) \subseteq C_{G_1}(u)^\circ$, that image is trivial and so the above surjective map is also injective. \qed

**Proposition 3.3.** Let $u \in \{\gamma_74, \gamma_75\} \subseteq C \cap U_{d,2}^F$. Then, as we already noted, we have $u \in C_1 \cap U_{d,2}^F$ and so the corresponding generalized Gelfand–Graev character $\Gamma_u^1$ of $G_1^F$ is well-defined. We have

$$\tilde{\Gamma}_u(y s_1) - \tilde{\Gamma}_u'(y s_1) = \Gamma_u^1(y) \quad \text{for all } y \in G_1^F \text{ unipotent},$$

Proof. By the Mackey formula, we have

$$\tilde{\Gamma}_u(y s_1) = \text{Res}_{G_1^F}^{G_1^F}(\tilde{\Gamma}_u)(y s_1) = \text{Res}_{G_1^F}^{G_1^F}\left(\text{Ind}_{U_{d,2},s_1}^{G_1^F}(\tilde{\varphi}_u)\right)(y s_1)$$

$$= \sum_{z} \text{Ind}_{U_{d,2},s_1}^{G_1^F}\left(\text{Res}_{U_{d,2},s_1}^{G_1^F}(\tilde{\varphi}_u)\right)(y s_1),$$

where $z$ runs over a set of representatives of the $(U_{d,2}^F, s_1, G_1^F)$-double cosets of $G_1^F$. Let us fix such a double coset representative, $z$ say. Assume that the value at $y s_1$ of the corresponding induced character in the above sum is non-zero. Then $y s_1$ must be $G_1^F$-conjugate to an element in the subgroup $(U_{d,2}^F, s_1, G_1^F) \cap G_1^F$. Consequently, $s_1$ must be $G_1^F$-conjugate to an element in that subgroup. Since $\langle s_1 \rangle$ is a Sylow 2-subgroup of $U_{d,2}^F, s_1$, we conclude that all elements of order 2 in $U_{d,2}^F, s_1$ are of the form $x s_1 x^{-1}$.
where \( x \in U_{d,2}^F \). Thus, we have \( c^{-1} x c = z^{-1} x s_1 x^{-1} z \) for some \( c \in G_1^F \) and some \( x \in U_{d,2}^F \). Consequently, \( x^{-1} z c^{-1} \in C_G(s_1)^F = G_1^F \) and so \( z \in x G_1^F c \in U_{d,2}^F . G_1^F \). Thus, \( z \) represents the trivial double coset and so we can take \( z = 1 \). Using the fact that

\[
U_{d,2}^F \cap G_1^F = U_{d,2}^F \times \langle s_1 \rangle
\]

(where \( U_{d,2} \subseteq G_1 \) is the unipotent subgroup defined with respect to the weighted Dynkin diagram \( d_1 : \Phi_1 \to \mathbb{Z} \)) we find that

\[
\tilde{\Gamma}_u(y s_1) = \text{Ind}_{U_{d,2}^F \times \langle s_1 \rangle}^{G_1^F} (\varphi_u^1 \otimes 1_{\langle s_1 \rangle})(y s_1)
\]

where \( \varphi_u^1 \) denotes the restriction of \( \varphi_u \) to \( U_{d,2}^F \). Since \( s_1 \) is in the center of \( G_1 \), it is readily checked that

\[
\tilde{\Gamma}_u(y s_1) = \frac{1}{2} \varphi_u(s_1) \text{Ind}_{U_{d,2}^F}^{G_1^F} (\varphi_u^1)(y) = \frac{1}{2} \text{Ind}_{U_{d,2}^F}^{G_1^F} (\varphi_u^1)(y).
\]

By a completely analogous argument, we also obtain that

\[
\Gamma'_u(y s_1) = \frac{1}{2} \varphi'_u(s_1) \text{Ind}_{U_{d,2}^F}^{G_1^F} (\varphi'_u^1)(y) = -\frac{1}{2} \text{Ind}_{U_{d,2}^F}^{G_1^F} (\varphi'_u^1)(y).
\]

Thus, it remains to check that

\[
\Gamma_u^1 = \text{Ind}_{U_{d,2}^F}^{G_1^F} (\varphi_u^1).
\]

For this purpose, we must show that \( \varphi_u^1 \) indeed is the linear character of \( U_{d,2}^F \) required in the definition of \( \Gamma_u^1 \). Now, the definition of \( \Gamma_u^1 \) requires the choice of a non-degenerate bilinear form and of an opposition automorphism on the Lie algebra of \( G_1 \). However, the Lie algebra of \( G_1 \) is naturally contained in the Lie algebra of \( G \), with compatible Cartan decompositions. Thus, the chosen bilinear form and the chosen opposition automorphism restrict to the Lie algebra of \( G_1 \), and this implies that \( \varphi_u^1 \) is the required linear character of \( U_{d,2}^F \). \( \square \)

A formula of this kind has been stated (without proof) by Kawanaka in [9], Lemma 2.3.5; see also the Ph. D. thesis of Wings [16], §3.2.1.

**Remark 3.4.** Let \( g \in G^F \) and write \( g = g_s g_u = g_u g_s \) where \( g_s \in G^F \) is semisimple and \( g_u \in G^F \) is unipotent. Assume that \( g_s \) is not conjugate to \( s_1 \) in \( G^F \). Then we have

\[
(\tilde{\Gamma}_u - \Gamma_u')(g) = 0.
\]

Indeed, if the value is non-zero, then \( g \) must be \( G^F \)-conjugate to an element in \( U_{d,2}^F \cdot \langle s_1 \rangle \). But then \( g_s \) will also be \( G^F \)-conjugate to an element in that subgroup. Using a Sylow argument as in the above proof, we see that either \( g_s = 1 \) or \( g_s \) is
$G^F$-conjugate to $s_1$, as claimed. Furthermore, if $g_s = 1$, then it is readily checked that $\tilde{\Gamma}_u'(g) = \tilde{\Gamma}_u''(g)$.

Thus, in order to compute the scalar product of $E_7[\pm \xi]$ with $\tilde{\Gamma}_u - \tilde{\Gamma}_u''$, it will be enough to know the values of $E_7[\pm \xi]$ on elements of the form $ys_1$ where $y \in G_1^F$ is unipotent. Furthermore, since $E_7[\xi]$ and $E_7[-\xi]$ are complex conjugate and since $\tilde{\Gamma}_u$ and $\tilde{\Gamma}_u'$ are rational-valued, it will actually be enough to consider the sum $E_7[\xi] + E_7[-\xi]$. Now, by Lusztig [10], Main Theorem 4.23, we have

$$E_7[\xi] + E_7[-\xi] = R_{512_u} - R_{512_u'}.$$  

(Note that the function $\Delta$ occurring in [10], 4.23, takes value $-1$ on the labels corresponding to the characters $E_7[\pm \xi]$.) Here, $512_u$, $512_u'$ are the two irreducible characters of $W$ of degree 512 and $R_{512_u}$, $R_{512_u'}$ are the corresponding “almost characters”, as defined by Lusztig [10], (3.7). For any $\phi \in \text{Irr}(W)$, we have

$$R_\phi := \frac{1}{|W|} \sum_{w \in W} \phi(w) R_{T_w,1};$$

here, $T_w \subseteq G$ is an $F$-stable maximal torus obtained from $T$ by twisting with $w$ and $R_{T_w,1}$ is the Deligne–Lusztig generalized character associated with the trivial character of $T_w^F$. Similarly, for any $\psi \in \text{Irr}(W_1)$, we denote by $R_{\psi}^{1}$ the corresponding almost character of $G_1^F$.

**Lemma 3.5.** Let $\phi \in \text{Irr}(W)$ and write

$$\text{Res}_{W_1}^W(\phi) = \sum_{\psi \in \text{Irr}(W_1)} m(\phi, \psi) \psi \quad \text{where} \quad m(\phi, \psi) \in \mathbb{Z}_{\geq 0}.$$  

Let $y \in G_1^F$ be a unipotent element. Then we have

$$R_\phi(ys_1) = \sum_{\psi \in \text{Irr}(W_1)} m(\phi, \psi) R_{\psi}^{1}(y).$$

**Proof.** The character formula for $R_{T_w,1}$ (see [2], Theorem 7.2.8) shows that

$$R_{T_w,1}(ys_1) = \frac{|C_w(\psi)|}{|W_1|} \sum_{\substack{n \in W_1 \\
 \text{and} \ n \sim \omega}} R_{T_{\psi}^{1},1}(y)$$

where the relation $\sim$ means conjugacy in $W$. (Here, $R_{T_w,1}^{1}$ denotes a Deligne–Lusztig generalized character of $G_1^F$.) Thus, we have

$$R_\phi(ys_1) = \frac{1}{|W|} \sum_{\substack{n \in W_1 \\
 \text{and} \ n \sim \omega}} \frac{|C_w(\psi)|}{|W_1|} \phi(n) R_{T_{\psi}^{1},1}(y).$$
\[ = \frac{1}{|W|} \sum_{w_1 \in W_1} \left( \frac{1}{|W|} \sum_{w \in W \atop w \sim w_1} |C_W(w)| \phi(w) \right) \Gamma^1_{u_1,y_1}(y) \]

Now, we have \( \phi(w) = \phi(w_1) \) and \( |C_W(w)| = |C_W(w_1)| \) for all \( w_1 \in W_1 \) such that \( w \sim w_1 \). Thus, we have

\[ \frac{1}{|W|} \sum_{w \in W \atop w \sim w_1} |C_W(w)| \phi(w) = \frac{|C_W(w_1)|}{|W|} \phi(w_1) \sum_{w \in W \atop w \sim w_1} 1 = \phi(w_1). \]

Writing \( \phi(w_1) = \sum_{\psi} m(\phi, \psi) \psi(w_1) \), we obtain the desired expression. \( \square \)

**Corollary 3.6.** With the notation of Proposition 3.3 and Lemma 3.5, we have

\[ \langle R_\phi, \hat{\Gamma}_u - \hat{\Gamma}^v_{u_1} \rangle_{GF} = \sum_{\psi \in \text{Irr}(W_1)} m(\phi, \psi) \langle R_\psi, \Gamma^1_{u_1} \rangle_{GF}, \]

for any \( \phi \in \text{Irr}(W) \) and \( u \in \{y_{74}, y_{75}\} \subseteq C_1 \cap U^F_{d_1,2} \).

Proof. Immediate from Proposition 3.3, Remark 3.4 and Lemma 3.5. \( \square \)

We now need some explicit information concerning the restriction of characters from \( W \) to \( W_1 \). Using the CHEVIE function *InductionTable*, we compute that

\[ \text{Res}^W_{W_1}(512_d) \otimes \varepsilon = \langle [21, 3] \boxtimes 1 \rangle + \text{sum of } \psi \text{ where } \psi \in \text{Irr}(W_1) \text{ and } a_\psi > 4, \]

\[ \text{Res}^W_{W_1}(512_d^v) \otimes \varepsilon = \langle [2, 31] \boxtimes 1 \rangle + \text{sum of } \psi \text{ where } \psi \in \text{Irr}(W_1) \text{ and } a_\psi > 4. \]

Here, \( 1 \) denotes the unit character on the \( A_1 \)-factor of \( W_1 \) and \( \varepsilon \) denotes the sign character of \( W_1 \). The characters of the \( D_6 \)-factor are denoted by \( [\lambda, \mu] \) where \( \lambda \) and \( \mu \) are partitions such that \( |\lambda| + |\mu| = 6 \). The \( a \)-invariant of a character is defined as in Lusztig [10], (4.1); in CHEVIE, these \( a \)-invariants are obtained by the function *LowestPowerGenericDegrees*. We have

\[ a_\psi = 4 \quad \text{for } \psi = [21, 3] \boxtimes 1 \text{ and } \psi = [2, 31] \boxtimes 1. \]

With these explicit formulas, we can now prove the following result.

**Proposition 3.7.** Assume that the characteristic \( p \) is large enough, such that Lusztig’s formula in [11], Theorem 7.5, for the values of a generalized Gelfand– Graev holds for \( \Gamma^1_{u_1} \). By [5], Corollary 4.3, there exists some \( u \in \{y_{74}, y_{75}\} \) such that \( \langle E_7[\pm \xi], \Gamma^1_{u_1} \rangle_{GF} = 1 \). For this element \( u \), we have

\[ \langle E_7[\pm \xi], \hat{\Gamma}_u - \hat{\Gamma}^v_{u_1} \rangle_{GF} = -1. \]
Proof. We have already mentioned in the remarks preceding Lemma 3.5 that
\[ E_7[x] + E_7[-x] = R_{512_a} - R_{512_b}. \]

Since \( \Gamma_u \) and \( \Gamma'_u \) are rational-valued (see Proposition 2.2), we have
\[
\langle E_7[x], \Gamma_u - \Gamma'_u \rangle_{GF} = \frac{1}{2} \langle E_7[x], E_7[-x], \Gamma_u - \Gamma'_u \rangle_{GF} = \frac{1}{2} \langle R_{512_a} - R_{512_b}, \Gamma_u - \Gamma'_u \rangle_{GF}. \]

Now let \( \psi \in \text{Irr}(W_1) \) be a constituent in the restriction of \( 512_a \) or \( 512_b \) from \( W \) to \( W_1 \). Then, by Corollary 3.6, we must compute the scalar product \( \langle R_{\psi}, \Gamma'_u \rangle_{GF} \).

Let \( D \) denote the Alvis–Curtis–Kawanaka duality operation on the character ring of \( G_1 \); see Lusztig [10], (6.8). We have \( D(R_{\psi}) = R_{\psi \otimes \psi} \) and so
\[
\langle R_{\psi}, \Gamma'_u \rangle_{GF} = \langle D(R_{\psi}), D(\Gamma'_u) \rangle_{GF} = \langle R_{\psi \otimes \psi}, D(\Gamma'_u) \rangle_{GF}. \]

Now, in order to evaluate the above scalar product, it is enough to know the values of \( R_{\psi \otimes \psi} \) on the unipotent elements of \( G_1. \) By Shoji’s algorithm [15] and by [11], Corollary 10.9, we know that \( R_{\psi \otimes \psi}(y) = 0 \) if \( \dim \mathfrak{B}_u < a_{\psi \otimes \psi}. \) On the other hand, we have \( D(\Gamma'_u)(y) = 0 \) if \( \dim \mathfrak{B}_u < \dim \mathfrak{B}_y. \) (This follows from [11]; see the remarks in [4], (2.4).) Thus, the above scalar product is zero if \( a_{\psi \otimes \psi} > \dim \mathfrak{B}_u = 4. \) Taking into account the explicit information concerning the restrictions of \( 512_a \) and \( 512_b \) from \( W \) to \( W_1 \), we conclude that

\[
\langle E_7[x], \Gamma_u - \Gamma'_u \rangle_{GF} = \frac{1}{2} \langle R_{[21,3] \otimes 1} - R_{[2,31] \otimes 1}, D(\Gamma'_u) \rangle_{GF}. \]

Now [21, 3] \( \otimes \) 1 and [2, 31] \( \otimes \) 1 lie in the same family of characters of \( W_1 \); see [10], Chapter 4. The Fourier matrix (which has size \( 4 \times 4 \)) for that family shows that
\[
R_{[21,3] \otimes 1} - R_{[2,31] \otimes 1} = -\rho_1 - \rho_2 \]
where \( \rho_1 \) and \( \rho_2 \) are unipotent characters of \( G_1. \) Now, we can explicitly compute the unipotent support of these two characters; see [11], §11, or [7], §3.C. This involves the knowledge of the Springer correspondence for \( G_1 \). Using the description of that correspondence in [2], §13.3, we find that \( \rho_1 \) and \( \rho_2 \) have unipotent support \( C_1. \) Thus, by the formula in [7], Remark 3.8, we have

\[
\langle \rho_i, D(\Gamma_{y_3}) + D(\Gamma_{y_3}'), G_1 \rangle = \langle D(\rho_i), \Gamma_{y_3} + \Gamma'_{y_3} \rangle_{G_1} = 1 \quad \text{for } i = 1, 2. \]

Note that \( C_{G_1}(y_3)/C_{G_1}(y_3)^{\circ} \cong \mathbb{Z}/2\mathbb{Z} \) by Lemma 3.2 and that \( D(\rho_1), D(\rho_2) \) are actual characters in the present situation; see [10], (6.8.2). Now we have \( u \in \{y_4, y_5\} \) and
we would like to show that

\[
\langle D(\rho_i), \Gamma_{u|G}^{\gamma_i} \rangle_{u|G} = \langle \rho_i, D(\Gamma_{u|G}^{\gamma_i}) \rangle_{u|G} = 1 \quad \text{for } i = 1, 2.
\]

This can be seen as follows. Fix \(i \in \{1, 2\}\). Since \(D(\rho_i)\) is an actual character, we certainly have \(\langle D(\rho_i), \Gamma_{u|G}^{\gamma_i} \rangle_{u|G} \geq 0\). Hence, using (2), the latter scalar product equals 0 or 1. Assume, if possible, that the scalar product is zero. Then the scalar product of \(-\rho_1 - \rho_2\) with \(D_G(\Gamma_{u|G}^{\gamma_i})\) would be \(-1\) or 0. Consequently, the scalar product in (1) would be \(-1/2\) or 0. Thus, the only possibility is that the scalar product in (1) equals 0. But this would mean that

\[
\langle E_7[\pm \xi], \Gamma_u + \Gamma_u^{\gamma_i} \rangle_{u|G} = \langle U_{d,1}^F : U_{d,2}^F \rangle^{1/2} \langle E_7[\pm \xi], \Gamma_u \rangle_{u|G} = \langle U_{d,1}^F : U_{d,2}^F \rangle^{1/2}
\]

is an even number, which is not true. So, our assumption was wrong and (3) holds. Inserting this into (1), we obtain the desired result.

\[\square\]

4. Proof of Theorem 1.1

By [5], Corollary 4.3, the Schur index of \(E_7[\pm \xi]\) is at most 2. Hence, we only need to show that \(E_7[\pm \xi]\) cannot be realized over \(\mathbb{Q}(\xi)\). Now, we have

\[
\langle E_7[\pm \xi], \Gamma_u \rangle_{u|G} = 1 \quad \text{for suitable } u \in \{y_{74}, y_{75}\}.
\]

So, using the formulas in Definition 2.1, we obtain that

\[
\langle E_7[\pm \xi], \Gamma_u + \Gamma_u^{\gamma_i} \rangle_{u|G} = \langle U_{d,1}^F : U_{d,2}^F \rangle^{1/2} = q^m \quad \text{for some } m \geq 1.
\]

Combining this with Proposition 3.7 and using Frobenius reciprocity, this yields

\[
\langle \text{Res}_{U_{d,2}^F}^{^H} (E_7[\pm \xi]), \psi_u \rangle_{U_{d,2}^F} = \langle E_7[\pm \xi], \Gamma_u \rangle_{u|G} = \frac{1}{2} (q^m + 1).
\]

Since \(p \equiv 1 \mod 4\), we also have \(q \equiv 1 \mod 4\) and so the above scalar product is an odd number. Now assume, if possible, that \(E_7[\pm \xi]\) can be realized over \(\mathbb{Q}(\xi)\). Then the restriction of \(E_7[\pm \xi]\) to \(U_{d,2}^F \cdot H\) can also be realized over \(\mathbb{Q}(\xi)\). Thus, by a standard argument on Schur induces ([8], Corollary 10.2), the Schur index of \(\psi_u^\gamma\) over \(\mathbb{Q}(\xi)\) divides the above odd number. Since the Schur index of \(\psi_u^\gamma\) over \(\mathbb{Q}(\xi)\) is at most 2 (see Proposition 2.2), it must be one. Thus, \(\psi_u^\gamma\) can be realized over \(\mathbb{Q}(\xi)\).

Now, since \(q\) is a square, we have \(\xi = \sqrt{-1}\). Furthermore, since \(p \equiv 1 \mod 4\), we have \(\sqrt{-1} \in \mathbb{Q}_p\) (the field of \(p\)-adic numbers). Hence \(\psi_u^\gamma\) can be realized over \(\mathbb{Q}_p\), contradicting Proposition 2.2(c). Thus, our assumption was wrong and so \(E_7[\pm \xi]\) cannot be realized over \(\mathbb{Q}(\xi)\).

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