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# ON THE GENERAL GAUSS SUMS AND THEIR FOURTH POWER MEAN

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**Abstract**

The main purpose of this paper is to study the fourth power mean of the general Gauss sums, and give two exact calculating formulae.

## 1. Introduction

For any Dirichlet character $\chi$ mod $q$, the classical Gauss sums are defined by

$$G(n, \chi) = \sum_{b=1}^{q} \chi(b) e\left(\frac{nb}{q}\right),$$

where $e(y) = e^{2\pi iy}$. The various properties of $G(n, \chi)$ appeared in many analytic number theory books (see references [1] and [2]). Perhaps the most famous properties of $G(n, \chi)$ are the following identities:

$$G(n, \chi^*) = \overline{\chi}^*(n)\tau(\chi^*) \qquad \text{and} \qquad |\tau(\chi^*)| = \sqrt{q},$$

where $\chi^*$ is a primitive character mod $q$, $\overline{\chi}^*$ is the conjugate character of $\chi^*$, and $\tau(\chi^*) = G(1, \chi^*)$. If $\chi$ is a nonprimitive character modulo $q$, then the value distribution of $\tau(\chi)$ is much irregular, even more is zero!

Let $q \geq 3$ be a positive integer. For any integer $n$ and positive integer $k$, we define the general $k$-th Gauss sums $G(n, k, \chi; q)$ as follows:

$$G(n, k, \chi; q) = \sum_{b=1}^{q} \chi(b) e\left(\frac{nb^k}{q}\right).$$

The summation is very important, because it is a generalization of the classical Gauss sums $G(n, \chi)$. But about the properties of $G(n, k, \chi; q)$, we know very little at present. The value of $|G(n, k, \chi; q)|$ is irregular as $\chi$ varies. One can only get some upper bound estimates. For example, for any integer $n$ with $(n, q) = 1$, from the gen-

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eral result of Cochrane and Zheng [3] we can deduce

$$\left|G(n, 2, \chi; q)\right| \le 2^{\omega(q)} q^{1/2},$$

where $\omega(q)$ denotes the number of distinct prime divisors of $q$. The case that $q$ is prime is due to Weil [5].

However, it is surprising that $G(n, k, \chi; q)$ enjoys many good value distribution properties in some problems of weighted mean value. Also for $k = 2$, the first author studied the hybrid mean value of Dirichlet $L$-functions and the general quadratic Gauss sums, and obtained several interesting asymptotic formulae as follows (see references [6] and [7]):

**Proposition 1.** *For any integer $n$ with $(n, p) = 1$, we have the asymptotic formulae*

$$\sum_{\chi \neq \chi_0} \left|G(n, 2, \chi; p)\right|^2 \cdot \left|L(1, \chi)\right| = C \cdot p^2 + O\left(p^{3/2} \ln^2 p\right)$$

*and*

$$\sum_{\chi \neq \chi_0} \left|G(n, 2, \chi; p)\right|^4 \cdot \left|L(1, \chi)\right| = 3 \cdot C \cdot p^3 + O\left(p^{5/2} \ln^2 p\right),$$

*where $L(s, \chi)$ denotes the Dirichlet $L$-function corresponding to the character $\chi$ modulo $p$,*

$$C = \prod_p \left[ 1 + \frac{\binom{2}{1}^2}{4^2 \cdot p^2} + \frac{\binom{4}{2}^2}{4^4 \cdot p^4} + \dots + \frac{\binom{2m}{m}^2}{4^{2m} \cdot p^{2m}} + \dots \right]$$

*is a constant, $\sum_{\chi \neq \chi_0}$ denotes the summation over all nonprincipal characters modulo $p$, $\prod_p$ denotes the product over all primes, and $\binom{2m}{m} = (2m)!/(m!)^2$.*

**Proposition 2.** *Let $p$ be an odd prime with $p \equiv 3 \mod 4$. Then for any fixed positive integer $n$ with $(n, p) = 1$, we have the asymptotic formula*

$$\sum_{\chi \neq \chi_0} \left|G(n, 2, \chi; p)\right|^6 \cdot \left|L(1, \chi)\right| = 10 \cdot C \cdot p^4 + O\left(p^{7/2} \ln^2 p\right).$$

Let $n$ be any integer with $(n, p) = 1$. The first author [5] also obtained the following two identities:

$$\sum_{\chi \bmod p} |G(n, 2, \chi; p)|^4 = \begin{cases} (p-1)\left[3p^2 - 6p - 1 + 4\left(\frac{n}{p}\right)\sqrt{p}\right], & \text{if } p \equiv 1 \bmod 4; \\ (p-1)(3p^2 - 6p - 1), & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

and

$$\sum_{\chi \bmod p} \left|G(n, 2, \chi; p)\right|^6 = (p-1)(10p^3 - 25p^2 - 4p - 1), \text{ if } p \equiv 3 \bmod 4,$$

where $(n/p)$ is the Legendre symbol.

It is very natural to consider the calculating problem of the sum

$$\sum_{\chi \bmod q} \left|G(n, k, \chi; q)\right|^{2m},$$

and try to give some exact calculating formulae. For $m = 1$, we easily get

$$\begin{aligned}\sum_{\chi \bmod q} \left|G(n, k, \chi; q)\right|^2 &= \sum_{\chi \bmod q} \sum_{a=1}^{q} \chi(a) \mathrm{e}\left(\frac{na^k}{q}\right) \sum_{b=1}^{q} \overline{\chi}(b) \mathrm{e}\left(-\frac{nb^k}{q}\right) \\ &= \phi(q) \sum_{a=1}^{q}{}' \mathrm{e}\left(\frac{na^k}{q} - \frac{na^k}{q}\right) = \phi^2(q),\end{aligned}$$

where $\sum'^{q}_{a=1}$ denotes the summation over all $a$ such that $(a, q) = 1$. In this paper, we study the sum

$$\sum_{\chi \bmod q} \left|G(n, k, \chi; q)\right|^4,$$

and give two exact calculating formulae. That is, we shall prove the following two main Theorems.

**Theorem 1.** *Let $p$ be a prime with $3 \mid p - 1$, then we have the identity*

$$\sum_{\chi \bmod p} \left|G(1, 3, \chi; p)\right|^4 = 5p^3 - 18p^2 + 20p + 1 + \frac{U^5}{p} + 5pU - 5U^3 - 4U^2 + 4U,$$

*where $U = \sum_{a=1}^{p} e\left(a^3/p\right)$ is a real constant.*

**Theorem 2.** *Let $q \geq 3$ be a square-full number (i.e. $p \mid q$ if and only if $p^2 \mid q$), $n$, $k$ be integers with $(nk, q) = 1$ and $k \geq 1$. Then we have the identity*

$$\sum_{\chi \bmod q} \left|G(n, k, \chi; q)\right|^4 = q\phi^2(q) \prod_{p \mid q} (k, p-1)^2 \prod_{\substack{p \mid q \\ (k, p-1)=1}} \frac{\phi(p-1)}{p-1},$$

*where $\prod_{p \mid q}$ denotes the product over all prime divisors of $q$, and $\phi(q)$ is the Euler totient function.*

For general integers $m, k \geq 3$, whether there exist some exact calculating formulae for

$$\sum_{\chi \bmod q} \left|G(n, k, \chi; q)\right|^{2m}$$

is an open problem.

## 2. Some Lemmas

To complete the proof of the Theorems, we need following several lemmas.

**Lemma 1.** *Let $p$ be a prime with $3 \mid p-1$ and $\chi_1$ be a cubic character mod $p$, then we have the identity*

$$\sum_{b=1}^{p-1} \chi_1(b^3 - 1) = \frac{\tau^3(\chi_1)}{p} - 2.$$

Proof. For any integer $1 \leq a \leq p-1$, it is easy to show that

$$1 + \chi_1(a) + \chi_1^2(a) = \begin{cases} 3, & \text{if } a \text{ is a cubic residue mod } p; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

So that

$$\sum_{b=1}^{p-1} \chi_1(b^3 - 1) = \sum_{b=1}^{p-1} \left(1 + \chi_1(b) + \chi_1^2(b)\right)\chi_1(b-1).$$

From the properties of cubic character we know that

$$\chi_1^2 = \overline{\chi}_1, \qquad \chi_1(-1) = 1 \qquad \text{and} \qquad \overline{\tau(\chi_1)} = \tau(\overline{\chi}_1), \tag{2}$$

therefore

$$\sum_{b=1}^{p-1} \chi_1^2(b)\chi_1(b-1) = \sum_{b=1}^{p-1} \overline{\chi}_1(b)\chi_1(b-1) = \sum_{b=1}^{p-1} \chi_1(1-\overline{b}) = \sum_{b=1}^{p-1} \chi_1(b-1),$$

where $\overline{b}$ is the inverse of $b$ defined by $b\overline{b} \equiv 1 \mod p$ and $1 \leq \overline{b} \leq p-1$. So we have

$$\sum_{b=1}^{p-1} \chi_1(b^3 - 1) = 2\sum_{b=1}^{p-1} \chi_1(b-1) + \sum_{b=1}^{p-1} \chi_1\big(b(b-1)\big) = \sum_{b=1}^{p-1} \chi_1\big(b(b-1)\big) - 2. \tag{3}$$

Note that

$$\tau^2(\chi_1) = \sum_{b=1}^{p-1} \chi_1(b) e\left(\frac{b}{p}\right) \sum_{c=1}^{p-1} \chi_1(-c) e\left(-\frac{c}{p}\right) = \sum_{b=1}^{p-1}\sum_{c=1}^{p-1} \chi_1(bc)\chi_1(c) e\left(\frac{c(b-1)}{p}\right)$$
$$= \sum_{b=1}^{p-1}\sum_{c=1}^{p-1} \chi_1(b)\overline{\chi}_1(c) e\left(\frac{c(b-1)}{p}\right) = \overline{\tau(\chi_1)} \sum_{b=1}^{p-1} \chi_1\big(b(b-1)\big).$$

That is

$$\sum_{b=1}^{p-1} \chi_1\big(b(b-1)\big) = \frac{\tau^3(\chi_1)}{p}. \tag{4}$$

Now combining (3) and (4) we immediately get

$$\sum_{b=1}^{p-1} \chi_1(b^3-1) = \frac{\tau^3(\chi_1)}{p} - 2.$$

This completes the proof of Lemma 1. □

**Lemma 2.** *Let $p$ be a prime with $3 \mid p-1$ and $\chi_1$ be a cubic character* mod $p$, *then we have the following identities*

$$\begin{cases} \tau(\chi_1) + \overline{\tau(\chi_1)} = U; \\ \tau^2(\chi_1) + \overline{\tau^2(\chi_1)} = U^2 - 2p; \\ \tau^5(\chi_1) + \overline{\tau^5(\chi_1)} = U^5 + 5p^2U - 5pU^3. \end{cases}$$

Proof. From formula (1) we have

$$\sum_{b=1}^{p-1} e\left(\frac{b^3}{p}\right) = -1 + \tau(\chi_1) + \overline{\tau(\chi_1)},$$

therefore

$$\tau(\chi_1) + \overline{\tau(\chi_1)} = U.$$

Note that

$$\begin{aligned} U^2 &= \left(\tau(\chi_1) + \overline{\tau(\chi_1)}\right)^2 = \tau^2(\chi_1) + \overline{\tau^2(\chi_1)} + 2p, \\ U^3 &= \left(\tau(\chi_1) + \overline{\tau(\chi_1)}\right)^3 = \tau^3(\chi_1) + \overline{\tau^3(\chi_1)} + 3p\left(\tau(\chi_1) + \overline{\tau(\chi_1)}\right) \end{aligned}$$

and

$$U^5 = \left(\tau(\chi_1) + \overline{\tau(\chi_1)}\right)^5 = \tau^5(\chi_1) + \overline{\tau^5(\chi_1)} + 5p\left(\tau^3(\chi_1) + \overline{\tau^3(\chi_1)}\right) + 10p^2\left(\tau(\chi_1) + \overline{\tau(\chi_1)}\right),$$

So we easily get

$$\tau^2(\chi_1) + \overline{\tau^2(\chi_1)} = U^2 - 2p \quad \text{and} \quad \tau^5(\chi_1) + \overline{\tau^5(\chi_1)} = U^5 + 5p^2U - 5pU^3.$$

This proves Lemma 2. □

**Lemma 3.** *Let $q$ be a square-full number. Then for any nonprimitive character $\chi$ modulo $q$, we have the identity*

$$\tau(\chi) = G(\chi, 1) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right) = 0.$$

Proof (see Theorem 7.2 of [4]). □

**Lemma 4.** *Let $p$ be a prime, $k$, $\alpha$ and $\beta$ be positive integers with $(k, p) = 1$ and $\alpha \geq \beta \geq 2$, $a$ be any integer with $(a, p) = 1$. Then we have the identity*

$$\sum_{c=1}^{p^\alpha}{}' e\left(\frac{ac^k}{p^\beta}\right) = 0.$$

Proof. Let $d = (k, p-1)$ and $\chi_2$ be a $d$th-order character mod $p$. Then we have

$$\sum_{c=1}^{p^\alpha}{}' \mathrm{e}\left(\frac{ac^k}{p^\beta}\right) = p^{\alpha-\beta} \sum_{c=1}^{p^\beta}{}' \mathrm{e}\left(\frac{ac^k}{p^\beta}\right) = p^{\alpha-\beta} \sum_{c=1}^{p^\beta}{}' \left[1 + \chi_2(c) + \cdots + \chi_2^{d-1}(c)\right] \mathrm{e}\left(\frac{ac}{p^\beta}\right)$$

$$= p^{\alpha-\beta} \left( \sum_{c=1}^{p^\beta}{}' \mathrm{e}\left(\frac{c}{p^\beta}\right) + \overline{\chi_2}(a) \sum_{c=1}^{p^\beta} \chi_2(c) \mathrm{e}\left(\frac{c}{p^\beta}\right) + \cdots + \overline{\chi_2}^{d-1}(a) \sum_{c=1}^{p^\beta} \chi_2^{d-1}(c) \mathrm{e}\left(\frac{c}{p^\beta}\right) \right).$$

From the properties of Dirichlet characters and Lemma 3 we can get

$$\sum_{c=1}^{p^\alpha}{}' \mathrm{e}\left(\frac{ac^k}{p^\beta}\right) = 0.$$

This proves Lemma 4. □

**Lemma 5.** *Let $p$ be a prime, $k$ and $\alpha$ be positive integers with $(k, p) = 1$ and $\alpha \geq 2$, $n$ be any integer with $(n, p) = 1$. Let $d = (k, p-1)$, then we have the identity*

$$\sum_{\chi \bmod p^\alpha} \left| G(n, k, \chi, p^\alpha) \right|^4 = \begin{cases} p^\alpha \phi^2(p^\alpha) \cdot \dfrac{\phi(p-1)}{p-1}, & \text{if } d = 1; \\ d^2 p^\alpha \phi^2(p^\alpha), & \text{if } d > 1. \end{cases}$$

Proof. If $d = 1$, then from Lemma 3 we have

$$\sum_{\chi \bmod p^\alpha} \left| G(n,k,\chi,p^\alpha) \right|^4 = \sum_{\chi \bmod p^\alpha} \left| \sum_{b=1}^{p^\alpha} \chi^k(b) \mathrm{e}\left(\frac{nb^k}{p^\alpha}\right) \right|^4 = \sum_{\chi \bmod p^\alpha} |\tau(\chi)|^4$$

$$= p^{2\alpha}\phi\big(\phi(p^\alpha)\big) = p^\alpha \phi^2(p^\alpha) \cdot \frac{\phi(p-1)}{p-1}.$$

On the other hand, if $d > 1$, then $p > 2$. From the properties of Dirichlet characters mod $p^\alpha$ we may get

$$\sum_{\chi \bmod p^\alpha} \left| G(n,k,\chi,p^\alpha) \right|^4 = \phi(p^\alpha) \sum_{b=1}^{p^\alpha}{}' \left| \sum_{c=1}^{p^\alpha}{}' \mathrm{e}\left(\frac{nc^k(b^k-1)}{p^\alpha}\right) \right|^2$$

$$= d\phi^3(p^\alpha) + \phi(p^\alpha) \sum_{\substack{b=1 \\ p^\alpha \nmid b^k-1}}^{p^\alpha}{}' \left| \sum_{c=1}^{p^\alpha}{}' \mathrm{e}\left(\frac{nc^k(b^k-1)}{p^\alpha}\right) \right|^2$$

$$= d\phi^3(p^\alpha) + \phi(p^\alpha)\Psi. \tag{5}$$

By Lemma 4 we have

$$\Psi = \sum_{\beta=0}^{\alpha-1} \sum_{\substack{b=1 \\ (b^k-1,p^\alpha)=p^\beta}}^{p^\alpha}{}' \left| \sum_{c=1}^{p^\alpha}{}' \mathrm{e}\left(\frac{nc^k(b^k-1)/p^\beta}{p^{\alpha-\beta}}\right) \right|^2$$

$$= \sum_{\substack{b=1 \\ (b^k-1,p^\alpha)=p^{\alpha-1}}}^{p^\alpha}{}' \left| \sum_{c=1}^{p^\alpha}{}' \mathrm{e}\left(\frac{nc^k(b^k-1)/p^{\alpha-1}}{p}\right) \right|^2$$

$$= \sum_{\substack{b=1 \\ p^{\alpha-1} | b^3-1}}^{p^\alpha}{}' \left| \sum_{c=1}^{p^\alpha}{}' \mathrm{e}\left(\frac{nc^k(b^k-1)/p^{\alpha-1}}{p}\right) \right|^2 - \sum_{\substack{b=1 \\ p^{\alpha} | b^k-1}}^{p^\alpha}{}' \left| \sum_{c=1}^{p^\alpha}{}' \mathrm{e}\left(\frac{nc^k(b^k-1)/p^{\alpha-1}}{p}\right) \right|^2$$

$$= \Omega - d\phi^2(p^\alpha), \tag{6}$$

where

$$\Omega = \sum_{\substack{b=1 \\ p^{\alpha-1} | b^3-1}}^{p^\alpha}{}' \left| \sum_{c=1}^{p^\alpha}{}' \mathrm{e}\left(\frac{nc^k(b^k-1)/p^{\alpha-1}}{p}\right) \right|^2.$$

Let $g$ be a primitive root mod $p^\alpha$, then we have

$$\begin{aligned}\Omega &= p^{2(\alpha-1)} \sideset{}{'}\sum_{\substack{b=1\\ p^{\alpha-1}|b^k-1}}^{p^\alpha} \left|\sum_{c=1}^{p-1} e\left(\frac{nc^k(b^k-1)/p^{\alpha-1}}{p}\right)\right|^2 \\ &= p^{2(\alpha-1)} \sum_{\substack{l=0\\ p^{\alpha-1}|g^{lk}-1}}^{\phi(p^\alpha)-1} \left|\sum_{c=1}^{p-1} e\left(\frac{nc^k(g^{lk}-1)/p^{\alpha-1}}{p}\right)\right|^2 \\ &= p^{2(\alpha-1)} \sum_{\substack{l=0\\ \phi(p^{\alpha-1})|lk}}^{\phi(p^\alpha)-1} \left|\sum_{c=1}^{p-1} e\left(\frac{nc^k(g^{lk}-1)/p^{\alpha-1}}{p}\right)\right|^2 .\end{aligned}$$

Let $lk = s\phi(p^{\alpha-1})$, where $0 \le s \le dp-1$. Note that $g^{\phi(p^{\alpha-1})} \equiv 1 \bmod p$,

$$g^{s\phi(p^{\alpha-1})} - 1 = \left(g^{\phi(p^{\alpha-1})} - 1\right)\left(g^{\phi(p^{\alpha-1})(s-1)} + \cdots + g^{\phi(p^{\alpha-1})} + 1\right)$$

and

$$g^{\phi(p^{\alpha-1})(s-1)} + \cdots + g^{\phi(p^{\alpha-1})} + 1 \equiv s \bmod p,$$

we have

$$\begin{aligned}\Omega &= p^{2(\alpha-1)} \sum_{s=0}^{dp-1} \left|\sum_{c=1}^{p-1} e\left(\frac{nc^k(g^{s\phi(p^{\alpha-1})}-1)/p^{\alpha-1}}{p}\right)\right|^2 \\ &= p^{2(\alpha-1)} \cdot d(p-1)^2 + dp^{2(\alpha-1)} \sum_{s=1}^{p-1} \left|\sum_{c=1}^{p-1} e\left(\frac{nc^k \cdot s \cdot (g^{\phi(p^{\alpha-1})}-1)/p^{\alpha-1}}{p}\right)\right|^2 \\ &= d\phi^2(p^\alpha) + dp^{2(\alpha-1)} \sum_{s=1}^{p-1} \left|\sum_{c=1}^{p-1} e\left(\frac{c^k \cdot s}{p}\right)\right|^2 = dp^{2(\alpha-1)} \sum_{s=1}^{p} \left|\sum_{c=1}^{p-1} e\left(\frac{c^k \cdot s}{p}\right)\right|^2 \\ &= dp^{2(\alpha-1)} \sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{s=1}^{p} e\left(\frac{s(c^k-d^k)}{p}\right) = dp^{2(\alpha-1)} \sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{s=1}^{p} e\left(\frac{sd^k(c^k-1)}{p}\right) \\ &= d^2 p^{2(\alpha-1)} p(p-1) = d^2 p^\alpha \phi(p^\alpha). \qquad (7)\end{aligned}$$

So for $d > 1$, from (5), (6) and (7) we get

$$\sum_{\chi \bmod p^\alpha} \left|G(n,k,\chi,p^\alpha)\right|^4 = d\phi^3(p^\alpha) + d^2 p^\alpha \phi^2(p^\alpha) - d\phi^3(p^\alpha) = d^2 p^\alpha \phi^2(p^\alpha).$$

This completes the proof of Lemma 5. □

**Lemma 6.** *Let $n$, $k$, $q_1$ and $q_2$ be integers with $(q_1, q_2) = 1$. Then for any character $\chi \bmod q_1q_2$, we have the identity*

$$\left|G(n, k, \chi; q_1q_2)\right| = \left|G(nq_2^{k-1}, k, \chi_1; q_1)\right| \cdot \left|G(nq_1^{k-1}, k, \chi_2; q_2)\right|,$$

*where $\chi = \chi_1\chi_2$ with $\chi_1 \bmod q_1$ and $\chi_2 \bmod q_2$.*

Proof. Since $(q_1, q_2) = 1$, so if $a$ and $b$ pass through a complete residue system mod $q_1$ and $q_2$ respectively, then $aq_2 + bq_1$ passes through a complete residue system mod $q_1q_2$. Note that $\chi = \chi_1\chi_2$ with $\chi_1 \bmod q_1$ and $\chi_2 \bmod q_2$ we have

$$\begin{aligned}
\left|G(n, k, \chi; q_1q_2)\right| &= \left|\sum_{b=1}^{q_1q_2} \chi(b)e\left(\frac{nb^k}{q_1q_2}\right)\right| \\
&= \left|\sum_{a=1}^{q_1}\sum_{b=1}^{q_2} \chi_1(aq_2 + bq_1)\chi_2(aq_2 + bq_1)e\left(\frac{n(aq_2 + bq_1)^k}{q_1q_2}\right)\right| \\
&= \left|\sum_{a=1}^{q_1} \chi_1(aq_2)e\left(\frac{n(aq_2)^k}{q_1q_2}\right)\right| \cdot \left|\sum_{b=1}^{q_2} \chi_2(bq_1)e\left(\frac{n(bq_1)^k}{q_1q_2}\right)\right| \\
&= \left|\sum_{a=1}^{q_1} \chi_1(a)e\left(\frac{nq_2^{k-1}a^k}{q_1}\right)\right| \cdot \left|\sum_{b=1}^{q_2} \chi_2(b)e\left(\frac{nq_1^{k-1}b^k}{q_2}\right)\right|,
\end{aligned}$$

where we have used $|\chi_1(q_2)| = |\chi_2(q_1)| = 1$. This proves Lemma 6. □

### 3. Proof of the theorems

In this section, we complete the proof of the Theorems. Let $p$ be a prime with $3 \mid p - 1$ and $\chi_1$ be a cubic character mod $p$, from (1) and (2) we have

$$\begin{aligned}
\sum_{\chi \bmod p} \left|G(1, 3, \chi; p)\right|^4 &= \sum_{\chi \bmod p} \left|\sum_{b=1}^{p-1} \chi(b)\mathrm{e}\left(\frac{b^3}{p}\right)\right|^4 \\
&= (p-1)\sum_{b=1}^{p-1}\left|\sum_{c=1}^{p-1} \mathrm{e}\left(\frac{c^3(b^3-1)}{p}\right)\right|^2 \\
&= (p-1)\sum_{b=1}^{p-1}\left|\sum_{c=1}^{p-1}(1 + \chi_1(c) + \overline{\chi}_1(c))\mathrm{e}\left(\frac{c(b^3-1)}{p}\right)\right|^2 \\
&= (p-1)\sum_{b=1}^{p-1}\left|\sum_{c=1}^{p-1} \mathrm{e}\left(\frac{c(b^3-1)}{p}\right) + \overline{\chi}_1(b^3-1)\tau(\chi_1) + \chi_1(b^3-1)\overline{\tau(\chi_1)}\right|^2
\end{aligned}$$

$$= 3(p-1)^3 + (p-1) \sum_{\substack{b=1 \\ p\nmid b^3-1}}^{p-1} \left|\overline{\chi}_1(b^3-1)\tau(\chi_1) + \chi_1(b^3-1)\overline{\tau(\chi_1)} - 1\right|^2$$

$$= 3(p-1)^3 + (p-1)(p-4)(2p+1) + (p-1)\left[\tau^2(\chi_1)\sum_{b=1}^{p-1}\chi_1(b^3-1)\right.$$

$$\left. + \overline{\tau^2(\chi_1)}\sum_{b=1}^{p-1}\overline{\chi}_1(b^3-1) - 2\tau(\chi_1)\sum_{b=1}^{p-1}\overline{\chi}_1(b^3-1) - 2\overline{\tau(\chi_1)}\sum_{b=1}^{p-1}\chi_1(b^3-1)\right]$$

$$= 3(p-1)^3 + (p-1)(p-4)(2p+1) + (p-1)\Psi,$$

where

$$\Psi = \tau^2(\chi_1)\sum_{b=1}^{p-1}\chi_1(b^3-1) + \overline{\tau^2(\chi_1)}\sum_{b=1}^{p-1}\overline{\chi}_1(b^3-1) - 2\tau(\chi_1)\sum_{b=1}^{p-1}\overline{\chi}_1(b^3-1)$$

$$- 2\overline{\tau(\chi_1)}\sum_{b=1}^{p-1}\chi_1(b^3-1).$$

By Lemma 1 and Lemma 2 we get

$$\Psi = \frac{\tau^5(\chi_1) + \overline{\tau^5(\chi_1)}}{p} - 4\left[\tau^2(\chi_1) + \overline{\tau^2(\chi_1)}\right] + 4\left[\tau(\chi_1) + \overline{\tau(\chi_1)}\right]$$

$$= \frac{U^5}{p} + 5pU - 5U^3 - 4U^2 + 8p + 4U.$$

Therefore

$$\sum_{\chi \bmod p} \left|G(1,3,\chi;p)\right|^4 = 5p^3 - 18p^2 + 20p + 1 + \frac{U^5}{p} + 5pU - 5U^3 - 4U^2 + 4U.$$

This proves Theorem 1.

Let $q \geq 3$ be a square-full number, $n$, $k$ be any integers with $(nk, q) = 1$ and $k \geq 1$. Let $q = \prod_{i=1}^{r} p_i^{\alpha_i}$ be the factorization of $q$ into prime powers and $\chi = \prod_{i=1}^{r} \chi_i$, where $\chi_i$ be a character mod $p_i^{\alpha_i}$. From Lemma 5 and Lemma 6 we have

$$\sum_{\chi \bmod q} \left|G(n,k,\chi;q)\right|^4 = \prod_{i=1}^{r}\left[\sum_{\chi_i \bmod p_i^{\alpha_i}} \left|G\left(n\left(\frac{q}{p_i^{\alpha_i}}\right)^{k-1}, k, \chi_i; p_i^{\alpha_i}\right)\right|^4\right]$$

$$= \prod_{i=1}^{r}\left[(k, p_i-1)^2 p_i^{\alpha_i}\phi^2(p_i^{\alpha_i})\right] \prod_{\substack{i=1 \\ (k,p_i-1)=1}}^{r} \frac{\phi(p_i-1)}{p_i-1}$$

$$= q\phi^2(q)\prod_{p|q}(k, p-1)^2 \prod_{\substack{p|q\\(k,p-1)=1}} \frac{\phi(p-1)}{p-1}.$$

This completes the proof of Theorem 2.

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