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GROWTH PROPERTIES OF p -TH MEANS OF BIHARMONIC GREEN POTENTIALS IN THE UNIT BALL

Dedicated to Professor Masakazu Shiba on the occasion of his sixtieth birthday

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Abstract

Let u be a biharmonic Green potential on the unit ball \mathbf{B} of \mathbf{R}^n . We show that

$$\lim_{r \rightarrow 1} (1-r)^{n-2-(n-1)/p} \mathcal{M}_p(u, r) = 0$$

for p such that $1 \leq p < (n-1)/(n-4)$ in case $n \geq 5$ and $1 \leq p < \infty$ in case $n \leq 4$. Further, if $n \geq 5$ and $(n-1)/(n-4) \leq p < (n-1)/(n-5)$, then it is shown that

$$\liminf_{r \rightarrow 1} (1-r)^{n-2-(n-1)/p} \mathcal{M}_p(u, r) = 0.$$

Finally we show that these limits characterize biharmonic Green potentials among super-biharmonic functions on \mathbf{B} .

1. Introduction and statement of results

A function u on an open set $\Omega \subset \mathbf{R}^n$ ($n \geq 2$) is called biharmonic if $u \in C^4(\Omega)$ and $\Delta^2 u = 0$ on Ω , where Δ denotes the Laplacian and $\Delta^2 u = \Delta(\Delta u)$. We say that a lower semicontinuous and locally integrable function u on Ω is super-biharmonic in Ω if every point of Ω is a Lebesgue point of u and $\Delta^2 u$ is a nonnegative measure on Ω in the weak sense, that is,

$$\int_{\Omega} u(x) \Delta^2 \varphi(x) dx \geq 0 \quad \text{for all nonnegative } \varphi \in C_0^\infty(\Omega).$$

The open ball and the sphere centered at x with radius r are denoted by $B(x, r)$ and $S(x, r)$. We write $B(r) = B(0, r)$ and $S(r) = S(0, r)$. We also denote by \mathbf{B} and \mathbf{S} the unit ball $B(1)$ and the unit sphere $S(1)$. We write $x^* = |x|^{-2}x$, so that x^* is the inverse point of x relative to the unit sphere \mathbf{S} .

Let $G_2(x, y)$ denote the biharmonic Green function in the unit ball \mathbf{B} (cf. [8]), that

is,

$$G_2(x, y) = \begin{cases} \alpha_n \left(|x-y|^{4-n} - (|y||x-y^*|)^{4-n} - \frac{n-4}{2}(1-|x|^2)(1-|y|^2)(|y||x-y^*|)^{2-n} \right) & \text{in case } n \neq 2, 4, \\ \alpha_n \left(|x-y|^{4-n} \log \left(\frac{|x-y|}{|y||x-y^*|} \right)^2 + (1-|x|^2)(1-|y|^2)(|y||x-y^*|)^{2-n} \right) & \text{in case } n = 2, 4, \end{cases}$$

where $\alpha_n^{-1} = 2(4-n)(2-n)\sigma_n$ for $n \neq 2, 4$ and $\alpha_n^{-1} = (-1)^{n/2+1}8\sigma_n$ for $n = 2, 4$. Here σ_n denotes the surface measure of the unit sphere \mathbf{S} . For a nonnegative measure μ on \mathbf{B} , we define

$$G_2\mu(x) = \int_{\mathbf{B}} G_2(x, y) d\mu(y).$$

If μ has a density $f \in L^1_{\text{loc}}(\mathbf{B})$, then we write G_2f instead of $G_2\mu$. The function $G_2\mu$ is called a biharmonic Green potential if $G_2\mu \neq \infty$.

For a Borel measurable function u on $S(r)$, we define the average integral over $S(r)$ by

$$\mathcal{M}(u, r) = \int_{S(r)} u dS = \frac{1}{|S(r)|} \int_{S(r)} u dS,$$

where $|S(r)|$ denotes the surface measure of $S(r)$. For $p > 0$ and a Borel measurable function u on $S(r)$, define $\mathcal{M}_p(u, r) = \{\mathcal{M}(|u|^p, r)\}^{1/p}$.

Gardiner [5] and the second author [9] studied the limiting behavior of $\mathcal{M}_p(v, r)$ for (harmonic) Green potentials on \mathbf{B} . We also refer to Stoll [13, 14] for invariant potentials in the unit ball of \mathbf{C}^n .

In this paper we are concerned with biharmonic Green potentials on \mathbf{B} .

Theorem 1.1. *Let $G_2\mu$ be a biharmonic Green potential on \mathbf{B} . If $1 \leq p < (n-1)/(n-4)$ in case $n \geq 5$ and $1 \leq p < \infty$ in case $n \leq 4$, then*

$$\lim_{r \rightarrow 1} (1-r)^{n-2-(n-1)/p} \mathcal{M}_p(G_2\mu, r) = 0.$$

Theorem 1.2. *Let $G_2\mu$ be a biharmonic Green potential on \mathbf{B} . If $n \geq 5$ and $(n-1)/(n-4) \leq p < (n-1)/(n-5)$, then*

$$\liminf_{r \rightarrow 1} (1-r)^{n-2-(n-1)/p} \mathcal{M}_p(G_2\mu, r) = 0.$$

Finally we give a characterization for a super-biharmonic function to be a biharmonic Green potential on \mathbf{B} .

Theorem 1.3. *Let u be a super-biharmonic function on \mathbf{B} . If u satisfies*

$$\liminf_{r \rightarrow 1} (1-r)^{-1} \mathcal{M}_1(u, r) = 0,$$

then it is a biharmonic Green potential on \mathbf{B} .

2. p -th means of biharmonic Green potentials

Throughout this paper, let M denote various constants independent of the variables in question.

We need the following fundamental estimates for the biharmonic Green function on the unit ball \mathbf{B} (cf. [1] and [7]).

Lemma 2.1. *There exist positive constants C_i , $1 \leq i \leq 4$, satisfying the following conditions:*

(1) *If $n \geq 5$, then for every $(x, y) \in \mathbf{B} \times \mathbf{B}$*

$$0 < C_1 \frac{(1-|x|^2)^2(1-|y|^2)^2}{|x-y|^{n-4}(|y||x-y^*|)^4} \leq G_2(x, y) \leq C_2 \frac{(1-|x|^2)^2(1-|y|^2)^2}{|x-y|^{n-4}(|y||x-y^*|)^4},$$

(2) *If $n = 4$, then for every $(x, y) \in \mathbf{B} \times \mathbf{B}$*

$$\begin{aligned} 0 &< C_1 \frac{(1-|x|^2)^2(1-|y|^2)^2}{(|y||x-y^*|)^4} \log \left(\frac{2|y||x-y^*|}{|x-y|} \right) \\ &\leq G_2(x, y) \leq C_2 \frac{(1-|x|^2)^2(1-|y|^2)^2}{(|y||x-y^*|)^4} \log \left(\frac{2|y||x-y^*|}{|x-y|} \right). \end{aligned}$$

(3) *If $n = 2, 3$, then for every $(x, y) \in \mathbf{B} \times \mathbf{B}$*

$$C_1 \frac{(1-|x|^2)^2(1-|y|^2)^2}{(|y||x-y^*|)^n} \leq G_2(x, y) \leq C_2 \frac{(1-|x|^2)^2(1-|y|^2)^2}{(|y||x-y^*|)^n}.$$

Furthe, in all cases,

$$C_3 \frac{(1-|x|^2)^2(1-|y|^2)^2}{(|y||x-y^*|)^n} \leq G_2(x, y) \leq C_4 \frac{(1-|x|^2)^2(1-|y|^2)^2}{|x-y|^n}.$$

By Lemma 2.1, we have the following result; see [7] and [10].

Corollary 2.2. *Let μ be a nonnegative measure on \mathbf{B} . Then $G_2\mu$ is a biharmonic Green potential if and only if*

$$(2.1) \quad \int_{\mathbf{B}} (1-|y|)^2 d\mu(y) < \infty.$$

Lemma 2.3. *If $(n-1)/n < p < \infty$ and $1/2 < r < 1$, then*

$$\mathcal{M}_p(G_2(\cdot, y), r) \leq M(1-r)^{2-n+(n-1)/p} \left(\frac{1-r}{|r-|y||} \right)^{n-(n-1)/p} (1-|y|)^2.$$

In particular, if $n = 2, 3$ and $(n-1)/n < p < \infty$, then

$$\mathcal{M}_p(G_2(\cdot, y), r) \leq M(1-r)^{2-n+(n-1)/p} \left(\frac{1-r}{1-r|y|} \right)^{n-(n-1)/p} (1-|y|)^2.$$

This follows from Lemma 2.1 and the fact that, if $\beta < 1-n$, then

$$(2.2) \quad \int_{S(r)} |x-y|^\beta dS(x) \leq M(r+|y|)^{1-n} |r-|y||^{\beta+n-1},$$

where M is a positive constant depending only on n and β .

Set $A(r) = \{y \in \mathbf{B} : (5r-1)/4 < |y| < (3r+1)/4\}$ for $0 < r < 1$.

Lemma 2.4. *Let μ be a nonnegative measure on \mathbf{B} satisfying (2.1). If $(n-1)/n < p < \infty$, then*

$$\lim_{r \rightarrow 1} (1-r)^{n-2-(n-1)/p} \int_{\mathbf{B} \setminus A(r)} \mathcal{M}_p(G_2(\cdot, y), r) d\mu(y) = 0.$$

Proof. By Lemma 2.3, we obtain

$$\begin{aligned} & (1-r)^{n-2-(n-1)/p} \int_{\mathbf{B} \setminus A(r)} \mathcal{M}_p(G_2(\cdot, y), r) d\mu(y) \\ & \leq M \int_{\mathbf{B} \setminus A(r)} \left(\frac{1-r}{|r-|y||} \right)^{n-(n-1)/p} (1-|y|)^2 d\mu(y). \end{aligned}$$

Since $(1-r)/|r-|y|| \leq 4$ for $y \in \mathbf{B} \setminus A(r)$, Lebesgue's dominated convergence theorem implies that

$$\lim_{r \rightarrow 1} (1-r)^{n-2-(n-1)/p} \int_{\mathbf{B} \setminus A(r)} \mathcal{M}_p(G_2(\cdot, y), r) d\mu(y) = 0.$$

In case $n = 2$ and 3 , we can show by Lemma 2.3 that

$$\lim_{r \rightarrow 1} (1-r)^{n-2-(n-1)/p} \int_{\mathbf{B}} \mathcal{M}_p(G_2(\cdot, y), r) d\mu(y) = 0. \quad \square$$

Lemma 2.5. *Let $1/2 < r < 1$ and $y \in A(r)$. If $n \geq 5$, then*

$$\mathcal{M}_p(G_2(\cdot, y), r) \leq M(1-r)^{2-n+(n-1)/p}(1-|y|)^2$$

$$\times \begin{cases} 1 & \text{if } \frac{n-1}{n} < p < \frac{n-1}{n-4}, \\ \left(\log \frac{1-r}{|r-|y||}\right)^{1/p} & \text{if } p = \frac{n-1}{n-4}, \\ \left(\frac{1-r}{|r-|y||}\right)^{n-4-(n-1)/p} & \text{if } p > \frac{n-1}{n-4}. \end{cases}$$

If $n = 4$, then

$$\mathcal{M}_p(G_2(\cdot, y), r) \leq M(1-r)^{2-n+(n-1)/p}(1-|y|)^2.$$

Proof. For $y \in A(r)$ and $r > 1/2$, setting

$$I_1(y, r) = S(r) \setminus B\left(y, \frac{1-r}{2}\right) \quad \text{and} \quad I_2(y, r) = S(r) \cap B\left(y, \frac{1-r}{2}\right),$$

we write

$$\begin{aligned} \mathcal{M}_p(G_2(\cdot, y), r)^p &= \frac{1}{\sigma_n r^{n-1}} \left(\int_{I_1(y, r)} G_2(x, y)^p dS(x) + \int_{I_2(y, r)} G_2(x, y)^p dS(x) \right) \\ &= u_1(y) + u_2(y). \end{aligned}$$

Since $-np + n - 1 < 0$, we have

$$\begin{aligned} u_1(y) &\leq M(1-r)^{2p}(1-|y|)^{2p} \int_{I_1(y, r)} |x-y|^{-np} dS(x) \\ &\leq M(1-r)^{2p+n-1-np}(1-|y|)^{2p}. \end{aligned}$$

On the other hand, if $x \in I_2(y, r)$, then $1-r \leq |y||x-y^*| \leq 3(1-r)$. In case $n \geq 5$ we see that

$$G_2(x, y) \leq M(1-r)^{-2}(1-|y|)^2|x-y|^{4-n},$$

so that

$$\begin{aligned}
u_2(y) &\leq M(1-r)^{-2p}(1-|y|)^{2p} \int_{I_2(y,r)} |x-y|^{(4-n)p} dS(x) \\
&\leq M(1-r)^{-2p}(1-|y|)^{2p} |y-r|^{(4-n)p+n-1} \\
&\quad \times \begin{cases} \left(\frac{1-r}{|r-|y||}\right)^{n-1+(4-n)p} & \text{if } \frac{n-1}{n} < p < \frac{n-1}{n-4}, \\ \log \frac{1-r}{|r-|y||} & \text{if } p = \frac{n-1}{n-4}, \\ 1 & \text{if } p > \frac{n-1}{n-4}. \end{cases}
\end{aligned}$$

Since $(1-r)/|y-r| > 4$ on $A(r)$, we obtain the required inequality.

Similarly, in case $n=4$, we find

$$\begin{aligned}
u_2(y) &\leq M(1-r)^{(2-n)p}(1-|y|)^{2p} \int_{I_2(y,r)} \left(\log \frac{2(1-r)}{|x-y|}\right)^p dS(x) \\
&\leq M(1-r)^{(2-n)p}(1-|y|)^{2p}(1-r)^{n-1}.
\end{aligned}$$

Hence it follows that

$$\mathcal{M}_p(G_2(\cdot, y), r) \leq M(1-r)^{2-n+(n-1)/p}(1-|y|)^2. \quad \square$$

3. Proofs of Theorems 1.1 and 1.2

In this section, we give proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let $1 \leq p < (n-1)/(n-4)$ when $n \geq 5$ and $1 \leq p < \infty$ when $n \leq 4$. By applying Minkowski's inequality for integrals, we have

$$\begin{aligned}
\mathcal{M}_p(G_2\mu, r) &\leq \int_{\mathbf{B}} \mathcal{M}_p(G_2(\cdot, y), r) d\mu(y) \\
(3.1) \quad &= \int_{\mathbf{B} \setminus A(r)} \mathcal{M}_p(G_2(\cdot, y), r) d\mu(y) + \int_{A(r)} \mathcal{M}_p(G_2(\cdot, y), r) d\mu(y).
\end{aligned}$$

Thus Theorem 1.1 follows from Lemmas 2.4 and 2.5. \square

Proof of Theorem 1.2. First, we give a proof in case $(n-1)/(n-4) < p < (n-1)/(n-5)$. Set $\beta = n-4 - (n-1)/p$ and $dv(x) = (1-|x|^2) d\mu(x)$. Here note that $0 < \beta < 1$. By Lemmas 2.3, 2.5 and (3.1), we see that

$$(3.2) \quad (1-r)^{n-2-(n-1)/p} \mathcal{M}_p(G_2\mu, r) \leq o(1) + M(1-r)^\beta \int_{A(r)} |y-r|^{-\beta} dv(y).$$

Hence it suffices to show that

$$(3.3) \quad \liminf_{r \rightarrow 1} (1-r)^\beta \int_{A(r)} ||y| - r|^{-\beta} d\nu(y) = 0.$$

For this purpose, we see that

$$\int_{1-2^{-j+1}}^{1-2^{-j}} ||y| - r|^{-\beta} dr \leq M2^{-j(1-\beta)}.$$

Hence it follows that

$$\begin{aligned} & \int_{1-2^{-j+1}}^{1-2^{-j}} \left((1-r)^\beta \int_{A(r)} ||y| - r|^{-\beta} d\nu(y) \right) \frac{dr}{1-r} \\ & \leq 2^{-j(\beta-1)} \int_{\{y: 2^{-j-1} < 1-|y| < 2^{-j+2}\}} \left(\int_{1-2^{-j+1}}^{1-2^{-j}} ||y| - r|^{-\beta} dr \right) d\nu(y) \\ & \leq M\nu(\{y: 2^{-j-1} < 1-|y| < 2^{-j+2}\}). \end{aligned}$$

Since $\nu(\mathbf{B}) < \infty$, we can find a sequence $\{r_j\}$ such that $2^{-j} < 1-r_j < 2^{-j+1}$ and

$$\lim_{j \rightarrow \infty} (1-r_j)^\beta \int_{A(r_j)} ||y| - r_j|^{-\beta} d\nu(y) = 0,$$

which implies (3.3). Thus the case $(n-1)/(n-4) < p < (n-1)/(n-5)$ now follows from (3.2) and (3.3).

Next, we deal with the case $p = (n-1)/(n-4)$. We see that

$$(1-r)^{n-2-(n-1)/p} \mathcal{M}_p(G_2\mu, r) \leq o(1) + M \int_{A(r)} \left(\log \frac{1-r}{||y| - r|} \right)^{1/p} d\nu(y).$$

In the same way as above, we have

$$\begin{aligned} & \int_{1-2^{-j+1}}^{1-2^{-j}} \left(\int_{A(r)} \left(\log \frac{1-r}{||y| - r|} \right)^{1/p} d\nu(y) \right) \frac{dr}{1-r} \\ & \leq 2^j \int_{\{y: 2^{-j-1} < 1-|y| < 2^{-j+2}\}} \left(\int_{1-2^{-j+1}}^{1-2^{-j}} \left(\log \frac{1-r}{||y| - r|} \right)^{1/p} dr \right) d\nu(y) \\ & \leq M\nu(\{y: 2^{-j-1} < 1-|y| < 2^{-j+2}\}), \end{aligned}$$

which implies that the left hand-side is zero by letting $j \rightarrow \infty$. Thus the theorem is established. \square

REMARK 3.1. Our theorems are best possible as to the power of $1-r$.

To show this, for $p \geq 1$ and $\delta > 0$, we give an example of biharmonic Green potential v satisfying

$$(3.4) \quad \lim_{r \rightarrow 1} (1-r)^{n-2-(n-1)/p-\delta} \mathcal{M}_p(v, r) = \infty.$$

Letting

$$f(x) = (1 - |x|)^{-3+\delta/2} |x - e_1|^{1-n},$$

where $e_1 = (1, 0, \dots, 0)$, we consider the potential

$$v(x) = \int_{\mathbf{B}} G_2(x, y) f(y) dy.$$

Then $\int_{\mathbf{B}} (1 - |x|)^2 f(x) dx < \infty$. Further, in case $n \geq 5$, we have

$$\begin{aligned} v(x) &\geq \int_{B(x, (1-|x|)/2)} G_2(x, y) f(y) dy \\ &\geq M(1 - |x|)^{-3+\delta/2} |x - e_1|^{1-n} \int_{B(x, (1-|x|)/2)} |x - y|^{4-n} dy \\ &= M(1 - |x|)^{1+\delta/2} |x - e_1|^{1-n}, \end{aligned}$$

so that

$$\begin{aligned} \mathcal{M}_p(v, r) &\geq M(1-r)^{1+\delta/2} \left(\int_{S(r)} |x - e_1|^{p(1-n)} dS(x) \right)^{1/p} \\ &\geq M(1-r)^{1+\delta/2+1-n+(n-1)/p} \\ &= M(1-r)^{2-n+(n-1)/p+\delta/2}. \end{aligned}$$

Thus (3.4) follows. We can show the remaining case in the same manner.

4. Proof of Theorem 1.3

First we study the spherical means of super-biharmonic functions.

Lemma 4.1 (cf. [6]). *Let u be a super-biharmonic function on \mathbf{B} . Then*

$$u(x) \geq \frac{1}{r_2^2 - r_1^2} \left(r_2^2 \int_{S(x, r_1)} u dS - r_1^2 \int_{S(x, r_2)} u dS \right)$$

whenever $x \in \mathbf{B}$ and $0 < r_1 < r_2 < 1 - |x|$.

Proof. Denote by K_2 the fundamental solution for the operator Δ^2 in \mathbf{R}^n , that is,

$$K_2(x) = \begin{cases} \alpha_n |x|^{4-n} & \text{in case } n \neq 2, 4, \\ 2\alpha_n |x|^{4-n} \log |x| & \text{in case } n = 2, 4. \end{cases}$$

For $x \in \mathbf{B}$ and $0 < r < 1 - |x|$, we can find a biharmonic function h in $B(x, r)$ such that

$$u(y) = \int_{B(x,r)} K_2(z-y) d\mu(z) + h(y)$$

for every $y \in B(x, r)$, where $\mu = \Delta^2 u$. By the Almansi expansion, we have

$$h(x) = \frac{r_2^2}{r_2^2 - r_1^2} \int_{S(x,r_1)} h(y) dS(y) - \frac{r_1^2}{r_2^2 - r_1^2} \int_{S(x,r_2)} h(y) dS(y)$$

for every $0 < r_1 < r_2 < r$. Hence we have only to show that

$$(4.1) \quad K_2(x) \geq \frac{r_2^2}{r_2^2 - r_1^2} \int_{S(r_1)} K_2(x-y) dS(y) - \frac{r_1^2}{r_2^2 - r_1^2} \int_{S(r_2)} K_2(x-y) dS(y)$$

for every $x \in \mathbf{R}^n$ and $0 < r_1 < r_2$. We define

$$g_2(x) = K_2(x) - \frac{r_2^2}{r_2^2 - r_1^2} \int_{S(r_1)} K_2(x-y) dS(y) + \frac{r_1^2}{r_2^2 - r_1^2} \int_{S(r_2)} K_2(x-y) dS(y)$$

and

$$g_1(x) = -\Delta g_2(x).$$

We see that

$$g_1(x) = K_1(x) - \frac{r_2^2}{r_2^2 - r_1^2} \int_{S(r_1)} K_1(x-y) dS(y) + \frac{r_1^2}{r_2^2 - r_1^2} \int_{S(r_2)} K_1(x-y) dS(y)$$

where $K_1(x) = (n-2)^{-1} \sigma_n^{-1} |x|^{2-n}$ if $n > 2$ and $K_1(x) = \sigma_2^{-1} \log(1/|x|)$ if $n = 2$. Note that $g_i(x) = g_i(x')$ for $|x| = |x'|$ and $g_i \in C^{2(i-1)}(\mathbf{R}^n \setminus \{0\})$. Further we see that $g_i(x) = 0$ for $|x| \geq r_2$, $g_2(0) = \infty$ if $n \geq 4$ and $g_2(0) > 0$ if $n = 2, 3$. Setting $t = K_1(x)$, we define $f_i(t) = g_i(x)$. Then

$$f_1(t) = \begin{cases} 0 & \text{if } K_1(\infty) < t \leq K_1(r_2), \\ -\frac{r_1^2}{r_2^2 - r_1^2} (t - K_1(r_2)) & \text{if } K_1(r_2) < t \leq K_1(r_1), \\ t - t_0 & \text{if } t > K_1(r_1), \end{cases}$$

where $t_0 = (r_2^2 K_1(r_1) - r_1^2 K_1(r_2)) / (r_2^2 - r_1^2)$ and $K_1(r) = K_1(x)$ with $|x| = r$. Hence we see that $f_1(t) < 0$ on $K_1(r_2) < t < t_0$ and $f_1(t) > 0$ on $t > t_0$. Since $f_2''(t) = -c(t) f_1(t)$ with $c(t) > 0$ and $\lim_{t \rightarrow \infty} f_2(t) > 0$, we obtain $f_2(t) > 0$ for $t > K_1(r_2)$. Thus (4.1) follows. \square

Lemma 4.2. *Let u be a super-biharmonic function on \mathbf{B} . Then*

$$\lim_{r \rightarrow 1} \mathcal{M}(u, r) \text{ exists in } (-\infty, \infty].$$

In particular, if $\liminf_{r \rightarrow 1} \mathcal{M}(u^+, r) = 0$, then

$$\mathcal{M}(u, r) \leq M(1 - r^2) \text{ for } r_0 < r < 1.$$

Proof. Let u be a super-biharmonic function on \mathbf{B} and $\mu = \Delta^2 u$. For $0 < t < 1$, there exists a biharmonic function h_t on $B(t)$ such that

$$u(x) = \int_{\overline{B(t)}} G_2(x, y) d\mu(y) + h_t(x) \quad (x \in B(t)).$$

Set

$$v_t(x) = \begin{cases} h_{(1+t)/2}(x) + \int_{\overline{B((1+t)/2)} \setminus \overline{B(t)}} G_2(x, y) d\mu(y) & \text{if } x \in B((1+t)/2), \\ u(x) - \int_{\overline{B(t)}} G_2(x, y) d\mu(y) & \text{if } x \in \mathbf{B} \setminus \overline{B(t)}. \end{cases}$$

Then v_t is well defined. Further v_t is biharmonic on $B(t)$, super-biharmonic on \mathbf{B} , $v_t(0) < \infty$ and

$$(4.2) \quad u(x) = v_t(x) + \int_{\overline{B(t)}} G_2(x, y) d\mu(y) \quad (x \in \mathbf{B}).$$

In view of Lemma 2.3, we see that

$$(4.3) \quad \int_{S(r)} \left(\int_{\overline{B(t)}} G_2(x, y) d\mu(y) \right) dS(x) \leq M(1 - r)^2 \frac{\mu(\overline{B(t)})}{r - t}$$

for $t < r < 1$,

where M is a positive constant independent of t and r . By Lemma 4.1, we have

$$\mathcal{M}(v_t, r_2) \geq \frac{r_2^2}{r_1^2} \mathcal{M}(v_t, r_1) - \frac{r_2^2 - r_1^2}{r_1^2} v_t(0)$$

for $0 < r_1 < r_2 < 1$, which implies that

$$(4.4) \quad \liminf_{r_2 \rightarrow 1} \mathcal{M}(v_t, r_2) \geq \frac{1}{r_1^2} \mathcal{M}(v_t, r_1) - \frac{1 - r_1^2}{r_1^2} v_t(0) > -\infty$$

for $0 < r_1 < 1$. Hence we have

$$\liminf_{r_2 \rightarrow 1} \mathcal{M}(v_t, r_2) \geq \limsup_{r_1 \rightarrow 1} \mathcal{M}(v_t, r_1).$$

In view of (4.4), we see that $\lim_{r \rightarrow 1} \mathcal{M}(v_t, r)$ exists in $(-\infty, \infty]$, and so $\lim_{r \rightarrow 1} \mathcal{M}(u, r)$ exists in $(-\infty, \infty]$.

Moreover, assume that $\liminf_{r \rightarrow 1} \mathcal{M}(u^+, r) = 0$. By Lemma 4.1, we have

$$(r_2^2 - r_1^2)v_t(0) \geq r_2^2 \mathcal{M}(v_t, r_1) - r_1^2 \mathcal{M}(v_t, r_2) \geq r_2^2 \mathcal{M}(v_t, r_1) - r_1^2 \mathcal{M}((v_t)^+, r_2)$$

for $0 < r_1 < r_2 < 1$. Since $(v_t)^+ \leq u^+$ and $\liminf_{r_2 \rightarrow 1} \mathcal{M}(u^+, r_2) = 0$, we have $\liminf_{r_2 \rightarrow 1} \mathcal{M}((v_t)^+, r_2) = 0$. Hence we obtain

$$(4.5) \quad \mathcal{M}(v_t, r_1) \leq v_t(0)(1 - r_1^2).$$

Combining (4.3) and (4.5) we conclude that

$$\mathcal{M}(u, r) \leq M(1 - r)^2 \quad \text{for } t < r < 1,$$

where M is independent of r . The claim follows. \square

REMARK 4.3. Let u be a super-biharmonic function on \mathbf{B} satisfying $u(0) < \infty$ and $\liminf_{r \rightarrow 1} \mathcal{M}(u^+, r) = 0$. Then

$$\mathcal{M}(u, r) \leq u(0)(1 - r^2) \quad \text{for } 0 < r < 1.$$

Corollary 4.4. Let u be a super-biharmonic function on \mathbf{B} satisfying

$$(4.6) \quad \liminf_{r \rightarrow 1} \mathcal{M}(u^+, r) = 0.$$

Then the following are equivalent.

- (1) $\liminf_{r \rightarrow 1} (1 - r)^{-1} \mathcal{M}_1(u, r) = 0$ (resp. $< \infty$).
- (2) $\liminf_{r \rightarrow 1} (1 - r)^{-1} \mathcal{M}(u^-, r) = 0$ (resp. $< \infty$).

Lemma 4.5. Let u be a super-biharmonic function on \mathbf{B} and $\mu = \Delta^2 u$. Suppose u satisfies

$$\liminf_{r \rightarrow 1} (1 - r)^{-1} \mathcal{M}_1(u, r) < \infty.$$

Then (2.1) is satisfied and u is of the form

$$u(x) = G_2 \mu(x) + (1 - |x|^2)h(x),$$

where h is harmonic on \mathbf{B} satisfying

$$\sup_{r < 1} \mathcal{M}_1(h, r) < \infty.$$

Proof. By considering the function $v_{1/2}$ in the proof of Lemma 4.2, we may assume that $u(0) < \infty$. Take a function v_t as in the proof of Lemma 4.2. First we show that

$$(4.7) \quad \inf_{0 < t < 1} v_t(0) > -\infty.$$

By Corollary 4.4, we see that $\liminf_{r \rightarrow 1} \mathcal{M}(u^+, r) = 0$, and so (4.5) holds. Using (4.3) and (4.5), we establish

$$\begin{aligned} v_t(0) &\geq \frac{1}{1-r^2} \mathcal{M}(v_t, r) \\ &\geq -\frac{1}{1-r^2} \left(\mathcal{M}_1(u, r) + \int_{S(r)} \left(\int_{B(t)} G_2(x, y) d\mu(y) \right) dS(x) \right) \\ &\geq -\frac{1}{1-r^2} \left(\mathcal{M}_1(u, r) + M(1-r)^2 \frac{\mu(\overline{B}(t))}{r-t} \right) \\ &= -\frac{1}{1+r} \left((1-r)^{-1} \mathcal{M}_1(u, r) + M(1-r) \frac{\mu(\overline{B}(t))}{r-t} \right) \end{aligned}$$

for $t < r < 1$. Letting $r \rightarrow 1$, we obtain

$$v_t(0) \geq -\frac{1}{2} \liminf_{r \rightarrow 1} (1-r)^{-1} \mathcal{M}_1(u, r),$$

which implies (4.7).

By Lemma 2.1, we have

$$\begin{aligned} C_3 \int_{\overline{B}(t)} (1-|y|^2)^2 d\mu(y) &\leq \int_{\overline{B}(t)} G_2(0, y) d\mu(y) \\ &= u(0) - v_t(0) \\ &\leq u(0) - \inf_{0 < t < 1} v_t(0) < \infty \end{aligned}$$

with the positive constant C_3 in Lemma 2.1. Hence (2.1) follows. In view of Corollary 2.2, $G_2\mu$ is a biharmonic Green potential, and hence there exists a biharmonic function v on \mathbf{B} such that $u = G_2\mu + v$ on \mathbf{B} . By Theorem 1.1, we see that

$$(4.8) \quad \liminf_{r \rightarrow 1} (1-r)^{-1} \mathcal{M}_1(v, r) < \infty.$$

We note that v has an Almansi representation (see [3] and [11]):

$$v(x) = g(x) + (1-|x|^2)h(x)$$

where g and h are harmonic on \mathbf{B} . If $|x| < r < 1$, then we apply the Poisson integrals of both sides and find that

$$\liminf_{r \rightarrow 1} (1-r)^{-1} |g(x) + (1-r^2)h(x)| < \infty.$$

This shows that g is identically zero in \mathbf{B} . □

The above proof also gives the following.

Lemma 4.6. *If u is a biharmonic function on \mathbf{B} satisfying*

$$\liminf_{r \rightarrow 1} (1-r)^{-1} \mathcal{M}_1(u, r) = 0,$$

then $u \equiv 0$ on \mathbf{B} .

Proof of Theorem 1.3. By Lemma 4.5, we see that u is of the form

$$u(x) = G_2\mu(x) + v(x),$$

where v is biharmonic on \mathbf{B} and

$$\int_{\mathbf{B}} (1-|y|)^2 d\mu(y) < \infty.$$

By Theorem 1.1 we find

$$\liminf_{r \rightarrow 1} (1-r)^{-1} \mathcal{M}_1(v, r) = 0.$$

Hence we see from Lemma 4.6 that $v \equiv 0$. □

5. Remarks

Suppose u is super-biharmonic on \mathbf{B} , and set $\mu = \Delta^2 u \geq 0$. Further, suppose there exists a sequence $\{r_j\}$, $0 < r_1 < r_2 < \dots < s < 1$, tending to 1 such that $\{u(r_j \cdot) dS\}$ converges weak-star to some finite Borel measure ν on \mathbf{S} and

$$\liminf_{r \rightarrow 1} (1-r)^{-1} \mathcal{M}((u - F[\nu])^-, r) < \infty.$$

Here $F[\nu](x) = \int_{\mathbf{S}} F(x, y) d\nu(y)$ with $F(x, y) = D_{\mathbf{n}(y)}(\Delta_y G_2(x, y))$, where $D_{\mathbf{n}(y)}$ denotes differentiation with respect to the outward unit normal. Then, as in Abkar and Hedenmalm [1], u can be represented as

$$u(x) = G_2\mu(x) + F[\nu](x) + (1-|x|^2)P[\lambda](x),$$

where $P[\lambda]$ denotes the Poisson integral of some finite Borel measure λ on \mathbf{S} and

$$\int_{\mathbf{B}} (1 - |x|^2)^2 d\mu(x) < \infty.$$

References

- [1] A. Abkar and H. Hedenmalm: *A Riesz representation formula for super-biharmonic functions*, Ann. Acad. Sci. Fenn. Math. **26** (2001), 305–324.
- [2] D.H. Armitage and S.J. Gardiner: *Classical Potential Theory*, Springer-Verlag London, 2001.
- [3] N. Aronszajn, T.M. Creese and L.J. Lipkin: *Polyharmonic functions*, Clarendon Press, 1983.
- [4] T. Futamura and Y. Mizuta: *Isolated singularities of super-polyharmonic functions*, Hokkaido Math. J. **33** (2004), 675–695.
- [5] S.J. Gardiner: *Growth properties of p -th means of potentials in the unit ball*, Proc. Amer. Math. Soc. **103** (1988), 861–869.
- [6] W. Haußmann and O. Kounchev: *Definiteness of the Peano kernel associated with the polyharmonic mean value property*, J. London Math. Soc. **62** (2000), 149–160.
- [7] W.K. Hayman and P.B. Kennedy: *Subharmonic functions*, Academic Press, London-New York, 1976.
- [8] W.K. Hayman and B. Korenblum: *Representation and uniqueness theorems for polyharmonic functions*, J. Anal. Math. **60** (1993), 113–133.
- [9] Y. Mizuta: *Growth properties of p -th hyperplane means of Green potentials in a half space*, J. Math. Soc. Japan, **44** (1992), 1–12.
- [10] Y. Mizuta: *Potential theory in Euclidean spaces*, Gakkōtōsyō, Tokyo, 1996.
- [11] M. Nicolesco: *Recherches sur les fonctions polyharmoniques*, Ann. Sci. École Norm. Sup. **52** (1935), 183–220.
- [12] P. Pizetti: *Sulla media deivalori che una funzione dei punti dello spazio assume alla superficie di una sfera*, Rend. Lincei **5** (1909), 309–316.
- [13] M. Stoll: *Rate of growth of p th means of invariant potentials in the unit ball of \mathbf{C}^n* , J. Math. Anal. Appl. **143** (1989), 480–499.
- [14] M. Stoll: *Rate of growth of p th means of invariant potentials in the unit ball of \mathbf{C}^n , II*, J. Math. Anal. Appl. **165** (1992), 374–398.

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