Ishii, A. and Kanenobu, T.
Osaka J. Math.
42 (2005), 273–290

DIFFERENT LINKS WITH THE SAME LINKS-GOULD INVARIANT

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(Received December 9, 2003)

Abstract

Using the skein relations for the Links-Gould (LG) invariant discovered by the first author, we give two constructions to provide a pair of different knots or links sharing the same LG invariant. Using this, we give examples of arbitrarily many 2-bridge knots with the same LG invariant. These knots also share the same HOMFLY and Kauffman polynomials. Further, for 2-bridge links we construct a similar example, which also share the same 2-variable Alexander polynomial. We also give a non-amphicheiral hyperbolic knot whose chirality is not undetected by the LG invariant.

1. Introduction

Links and Gould [23] have derived the Links-Gould 2-variable polynomial invariant, or the LG invariant for short, of an oriented link from the one-parameter family of four dimensional representations of the quantum superalgebra $U_q[gl(2|1)]$. De Wit, Kauffman, and Links [5] have then given an explicit form of the R-matrix and associated with a family of four dimensional representations. They found that the LG invariant detects chirality of some links where the HOMFLY and Kauffman polynomials fail and that it does not detect the noninvertibility or mutation of links. Successively, De Wit [4] has given a method for the automatic evaluation of the LG invarinat for links given as a closed braid with string index at most 5. He showed that the LG invariant can distinguish all prime knots with up to 10 crossings and also it can detect the chirality of those that are chiral. Recently, the first author [8] has succeeded in classifying infinitely many knots sharing the same Jones and HOMFLY polynomials which had given by the second author [12, 13]. For the Jones, HOMFLY, and Kauffman polynomials, we refer the reader to [10, 20, 22].

On the analogy of the formula for the LG invariant given in [5, p. 170], the first author [7] has discovered another formula for the LG invariant (Proposition 3.1). In this paper, we make use of these two skein relations together with some properties given in Proposition 3.2 to evaluate the LG invariant. Given a link that contains two tangles, we may obtain another link by exchanging these two tangles. In [13, 14, 15],

The second author was partially supported by Grant-in-Aid for Scientific Research (B) (No. 14340027), Japan Society for the Promotion of Science.

calculating the difference of the HOMFLY and Kauffman polynomials of the two links, the second author has constructed exmples of knots and links sharing the same HOMFLY or Kauffman polynomials. We apply such a construction to the LG invariant; we calculate the difference of the LG invariants of such two links (Lemma 3.3), and then give two constructions to provide a pair of knots or links sharing the same LG invariant (Theorems 4.1 and 5.1).

De Wit [4, p. 325] asked whether the LG invariant always distinguishes between nonmutant prime knots. Applying Theorem 4.1, we construct arbitrarily many 2-bridge knots with the same LG invariant (Theorem 6.1), which are given as a set of amphicheiral, fibered 2-bridge knots that share the same HOMFLY and Kauffman polynomials in [15, Theorem 1]. Further, we construct a similar set of 2-bridge links (Theorem 6.2), which also share the same 2-variable Alexander polynomial; as before this is the set given in [15, Theorem 2]. It is well-known that a 2-bridge knot or link is prime and has no mutant.

Next, applying Theorem 5.1, we construct links whose LG invariants are the same as those of mirror images (Proposition 7.1), and give an example of chiral hyperbolic knot whose chirality is undetected by the LG invariant (Example 7.2), where we use the computer program SnapPea to show that the knot is hyperbolic and chiral.

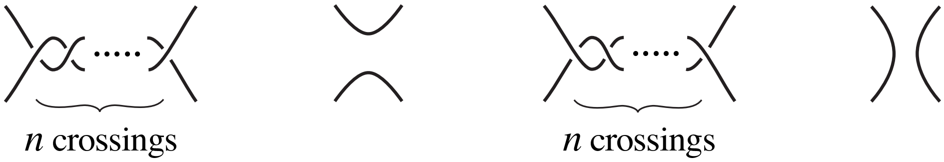
Lastly, we discuss a certain relation between the LG invariant and the Conway polynomial. We will show: if the LG invariant of a knot or link is obtained recursively by using the skein relations in Proposition 3.1 and that the LG invariant of the trivial knot is one (algebraic links are such ones [7]), then the Conway polynomial is recovered from the LG invariant (Propsotion 8.2). The first author [9] has generalized this to any link. As an application of this, there exist arbitrarily many 2-bridge knots with the same Kauffman polynomial but distinct LG invariants (Corollary 8.3).

This paper consisits of eight sections. In Section 2, we give some definitions for tangles and 3-braids. In Section 3, we give the skein relations and some properties for the LG invariant and prove above-mentioned Lemma 3.3, which is a key in this paper. In Sections 4 and 5, we prove Theorems 4.1 and 5.1, which provide a pair of knots or links sharing the same LG invariant. In Section 6, we prove Theorems 6.1 and 6.2, which give arbitrarily many 2-bridge knots or links with the same LG invariant. In Section 7, we give examples of knots and links whose chirality is undetected by the LG invariant. In Section 8, we discuss a relation between the LG invariant and the Conway polynomial.

ACKNOWLEDGEMENTS. The authors would like to thank David De Wit for helpful comments.

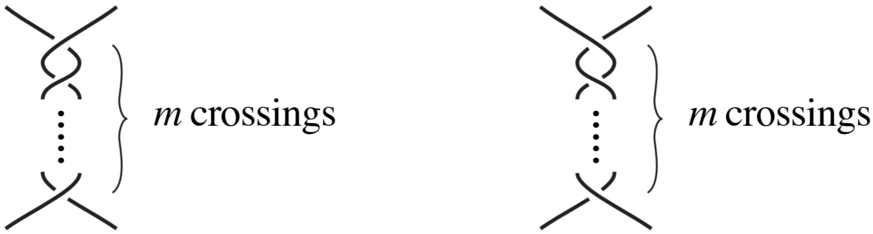
2. Preliminaries

A *tangle* in a link is a region in the projection plane surrounded by a circle such that the link crosses the circle exactly four times. We suppose such four points occur



(a) (b) (c) (d)

Fig. 1. (a) The n tangle. (b) The 0 tangle. (c) The $-n$ tangle. (d) The ∞ tangle.



(a) (b)

Fig. 2. (a) The $1/m$ tangle. (b) The $-1/m$ tangle.

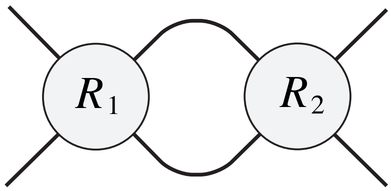


Fig. 3. The tangle sum $R_1 + R_2$.

in the four compass directions NW, NE, SW, and SE; cf. [3, p. 331] and [1, Sect. 2.3]. We define the integral tangle or n tangle, $n \in \mathbb{Z}$, and the ∞ tangle as in Fig. 1, where $n > 0$. Further, we define the $1/m$ tangle, $m \in \mathbb{Z} \setminus \{0\}$, as in Fig. 2, where $m > 0$.

Given a tangle R, we denote by $\rho_X R$, $\rho_Y R$, and $\rho_Z R$ the tangles obtained by rotating R through 180° about a horizontal axis, a vertical axis, and an axis perpendicular to the projection plane, respectively. In addition, we denote by μR the mirror image of R.

Given two tangles R_1 and R_2, we define the sum $R_1 + R_2$ as shown in Fig. 3.

Let L_R be an oriented link diagram that contains a tangle R. For another tangle S, we denote by L_S the link obtained from L_R by replacing R with S. If S is the n tangle with $n \in \mathbb{Z} \cup \{\infty\}$ (resp. $1/m$ tangle with $m \in \mathbb{Z} \setminus \{0\}$), we denote the link L_S by L_n (resp. $L_{1/m}$). In the following, we shall use a similar notation. The four links

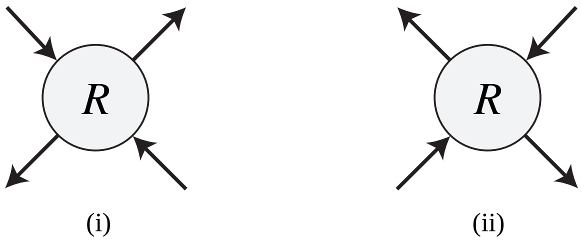


Fig. 4. An oriented tangle R.

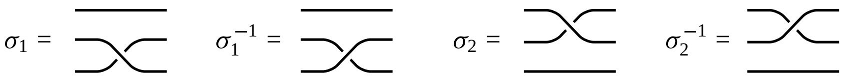


Fig. 5. Elementary 3-braids.

L_R, $L_{\rho_X R}$, $L_{\rho_Y R}$, and $L_{\rho_Z R}$ are called *mutants* of one another; cf. [22, p. 29]. Note also that two links L_{R+S} and L_{S+R}, that is, the links containing the tangle sums $R + S$ and $S + R$, respectively, are mutants. In the following, if we consider an oriented tangle, we will always assume it is oriented as shown in Fig. 4(i) or (ii).

A 3-braid is an element of the 3-braid group generated by the elementary 3-braids σ_1, σ_2 as shown in Fig. 5.

3. Skein relations for the LG invariant

We denote the LG invariant of an oriented link L by $\mathrm{LG}(L; t_0, t_1)$, or $\mathrm{LG}(L)$ for short, which is a 2-variable polynomial in variables t_0 and t_1; see [6, 7]. In [4], the LG invariant is given as a polynomial in variables P and q, where $P = \sqrt{t_0/t_1}$ and $q = \sqrt{t_0 t_1}$.

Let L_R be an oriented link diagram that contains a tangle R. Then among the oriented links $L_{-1/2}$, L_0, L_∞, L_{-2}, L_2, we have the skein relations [7, Theorem 3.1]:

Proposition 3.1.

(1) $\mathrm{LG}(L_{-1/2}) + (1 - t_0 - t_1)\,\mathrm{LG}(L_0) + (t_0 t_1 - t_0 - t_1)\,\mathrm{LG}(L_\infty) + t_0 t_1\,\mathrm{LG}(L_{-2}) = 0;$

(2) $$\mathrm{LG}(L_{-1/2}) + (t_0 t_1 - t_0 - t_1 + 2)\,\mathrm{LG}(L_0) - (t_0 t_1 - t_0 - t_1 + 2)\,\mathrm{LG}(L_\infty) - \mathrm{LG}(L_2) = 0;$$

(3) $\mathrm{LG}(L_2) + t_0 t_1\,\mathrm{LG}(L_{-2}) = (t_0 t_1 + 1)\,\mathrm{LG}(L_0) - 2(t_0 - 1)(t_1 - 1)\,\mathrm{LG}(L_\infty).$

The first relation (1) is equivalent to a relation originally due to De Wit et al. [5]. By analogy with this, the first author then has discovered the second relation (2). Subtracting (1) from (2), we obtain the third relation (3).

The LG invariant satisfies the following properties [4, 5]; cf. [7, Proposition 3.3].

Proposition 3.2. (i) *For the trivial knot U*, $\mathrm{LG}(U) = 1$.
(ii) *If L is a split link, then* $\mathrm{LG}(L) = 0$.
(iii) *Let L_1 and L_2 be links and $L_1 \# L_2$ be their connected sum. Then* $\mathrm{LG}(L_1 \# L_2) = \mathrm{LG}(L_1)\,\mathrm{LG}(L_2)$.
(iv) *Let L be a link and $L!$ be its mirror image. Then* $\mathrm{LG}(L!; t_0, t_1) = \mathrm{LG}(L; t_0^{-1}, t_1^{-1})$.
(v) *Let L be a link and $-L$ be the link obtained from L by reversing the orientation of every component. Then* $\mathrm{LG}(-L; t_0, t_1) = \mathrm{LG}(L; t_0, t_1)$. *This further implies the symmetry:* $\mathrm{LG}(L; t_0, t_1) = \mathrm{LG}(L; t_1, t_0)$.
(vi) $\mathrm{LG}(L)$ *is unchanged by mutation of L.*

In this paper, we study the LG invariant through the skein relations (1), (2), and (3) together with these properties.

Let R be an oriented tangle as in Fig. 4(i) or (ii). We say that R is *LG decomposable* if the LG invariant of any oriented link L_R that contains R is expressed in terms of those of L_0, L_2 and L_∞; that is, $\mathrm{LG}(L_R)$ is expressed in the form

$$\mathrm{LG}(L_R) = f\,\mathrm{LG}(L_0) + g\,\mathrm{LG}(L_2) + h\,\mathrm{LG}(L_\infty), \tag{4}$$

where $f,\ g,\ h \in \mathbb{Z}[t_0^{\pm 1}, t_1^{\pm 1}]$. It is easy to see that the $2n$ tangle with $n \in \mathbb{Z}$ and the $1/(2m)$ tangle with $m \in \mathbb{Z} \setminus \{0\}$ are such tangles. Further, if oriented tangles R and S are LG decomposable, then so are the sum $R+S$, the mirror image μR, and the tangle obtained by rotating R through 90° about an axis perpendicular to the projection plane by the skein relations in Proposition 3.1. In particular, an oriented algebraic tangle in the sense of Conway [3] is LG decomposable; cf. [7].

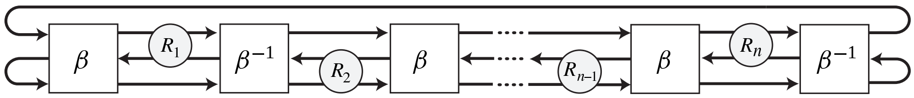
Let L_{R_1,R_2} be an oriented link containing two tangles R_1 and R_2 which are oriented as in Figs. 4(i) or (ii). Suppose that R_1 and R_2 are LG decomposable; that is, for any link M_{R_i} that contains R_i, $\mathrm{LG}(M_{R_i})$ is of the form

$$\mathrm{LG}(M_{R_i}) = f_i\,\mathrm{LG}(M_0) + g_i\,\mathrm{LG}(M_2) + h_i\,\mathrm{LG}(M_\infty), \tag{5}$$

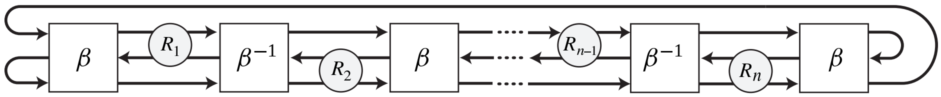
where $f_i,\ g_i,\ h_i \in \mathbb{Z}[t_0^{\pm 1}, t_1^{\pm 1}]$ ($i = 1,\ 2$). We consider the difference of the LG invariants of the links L_{R_1,R_2} and L_{R_2,R_1}, where L_{R_2,R_1} is obtained from L_{R_1,R_2} by exchanging the tangles R_1 and R_2.

Lemma 3.3.

$$\begin{aligned}\mathrm{LG}(L_{R_1,R_2}) - \mathrm{LG}(L_{R_2,R_1}) = {} & (f_1 g_2 - g_1 f_2)\left(\mathrm{LG}(L_{0,2}) - \mathrm{LG}(L_{2,0})\right) \\ & + (f_1 h_2 - h_1 f_2)\left(\mathrm{LG}(L_{0,\infty}) - \mathrm{LG}(L_{\infty,0})\right) \\ & + (g_1 h_2 - h_1 g_2)\left(\mathrm{LG}(L_{2,\infty}) - \mathrm{LG}(L_{\infty,2})\right).\end{aligned} \tag{6}$$



n is odd.



n is even.

Fig. 6. $K[\beta; R_1, R_2, \ldots, R_{n-1}, R_n]$.

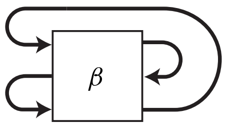


Fig. 7. $K[\beta]$.

Proof. Using (5), we have

$$
\begin{aligned}
\mathrm{LG}(L_{R_1,R_2}) &= f_1\,\mathrm{LG}(L_{0,R_2}) + g_1\,\mathrm{LG}(L_{2,R_2}) + h_1\,\mathrm{LG}(L_{\infty,R_2}) \\
&= f_1 f_2\,\mathrm{LG}(L_{0,0}) + f_1 g_2\,\mathrm{LG}(L_{0,2}) + f_1 h_2\,\mathrm{LG}(L_{0,\infty}) \\
&\quad + g_1 f_2\,\mathrm{LG}(L_{2,0}) + g_1 g_2\,\mathrm{LG}(L_{2,2}) + g_1 h_2\,\mathrm{LG}(L_{2,\infty}) \\
&\quad + h_1 f_2\,\mathrm{LG}(L_{\infty,0}) + h_1 g_2\,\mathrm{LG}(L_{\infty,2}) + h_1 h_2\,\mathrm{LG}(L_{\infty,\infty}).
\end{aligned} \tag{7}
$$

Similarly, we have

$$
\begin{aligned}
\mathrm{LG}(L_{R_2,R_1}) &= f_1 f_2\,\mathrm{LG}(L_{0,0}) + g_1 f_2\,\mathrm{LG}(L_{0,2}) + h_1 f_2\,\mathrm{LG}(L_{0,\infty}) \\
&\quad + f_1 g_2\,\mathrm{LG}(L_{2,0}) + g_1 g_2\,\mathrm{LG}(L_{2,2}) + h_1 g_2\,\mathrm{LG}(L_{2,\infty}) \\
&\quad + f_1 h_2\,\mathrm{LG}(L_{\infty,0}) + g_1 h_2\,\mathrm{LG}(L_{\infty,2}) + h_1 h_2\,\mathrm{LG}(L_{\infty,\infty}).
\end{aligned} \tag{8}
$$

Then we obtain (6). □

4. The LG invariant of the link $K[\beta; R_1, R_2, \ldots, R_n]$

For a 3-braid β and tangles $R_1, R_2, \ldots, R_{n-1}, R_n$, we define a class of oriented links $K[\beta; R_1, R_2, \ldots, R_{n-1}, R_n]$ as shown in Fig. 6; if $n = 0$, we interpret it as the 2-bridge link $K[\beta]$ as shown in Fig. 7. In particular, if each R_i is an integral tangle, then $K[\beta; R_1, R_2, \ldots, R_n]$ is a 2-bridge link.

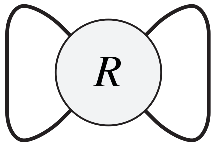


Fig. 8. The link $D(R)$.

We say that a 3-braid is *strongly amphicheiral* if it is of the form

$$(\sigma_2^{p_1}\sigma_1^{p_2}\cdots)(\cdots\sigma_2^{-p_2}\sigma_1^{-p_1}); \tag{9}$$

the closure of such a 3-braid is strongly amphicheiral in the sense of [26]. Note that if a strongly amphicheiral 3-braid β can be oriented as in Fig. 6 or 7, then it is easy to see that β is a pure braid, that is, each strand of β joins the horizontal two end points. Also the knot $K[\beta]$ is strongly amphicheiral.

In this section, we will prove the following theorem.

Theorem 4.1. *If β is a strongly amphicheiral pure 3-braid and each of tangles $R_1, R_2, \dots, R_{n-1}, R_n$ is LG decomposable, then*

$$\mathrm{LG}(K[\beta; R_1, R_2, \dots, R_{n-1}, R_n]) = \mathrm{LG}(K[\beta; R_n, R_{n-1}, \dots, R_2, R_1]). \tag{10}$$

In order to prove this, we need some lemmas. The following is easy to see.

Lemma 4.2. *Suppose that β is a pure 3-braid. If R_k is the 0 tangle, $1 \le k \le n$, then $K[\beta; R_1, \dots, R_n]$ is isotopic to*:

$$\begin{cases} \text{the trivial 2-component link} & \text{if } n = k = 1;\\ D(R_2)\#K[\beta; R_3, \dots, R_n] & \text{if } n \ge 2,\ k = 1;\\ K[\beta; R_1, \dots, R_{k-2}, R_{k-1} + R_{k+1}, R_{k+2}, \dots, R_n] & \text{if } 2 \le k \le n-1;\\ K[\beta; R_1, \dots, R_{n-2}]\#D(R_{n-1}) & \text{if } n \ge 2,\ k = n, \end{cases}$$

where $D(R)$ is a link as shown in Fig. 8, *and both $K[\beta; R_3, \dots, R_n]$ and $K[\beta; R_1, \dots, R_{n-2}]$ with $n = 2$ mean $K[\beta]$.*

Lemma 4.3. *Suppose that β is a strongly amphicheiral pure 3-braid. If R_k is the ∞ tangle, $1 \le k \le n$, then*

$$\begin{aligned} &\mathrm{LG}(K[\beta; R_1, R_2, \dots, R_n]) \\ &\qquad = \mathrm{LG}(K[\beta; R_1, R_2, \dots, R_{k-1}])\,\mathrm{LG}(K[\beta; R_{k+1}, R_{k+2}, \dots, R_n]), \end{aligned} \tag{11}$$

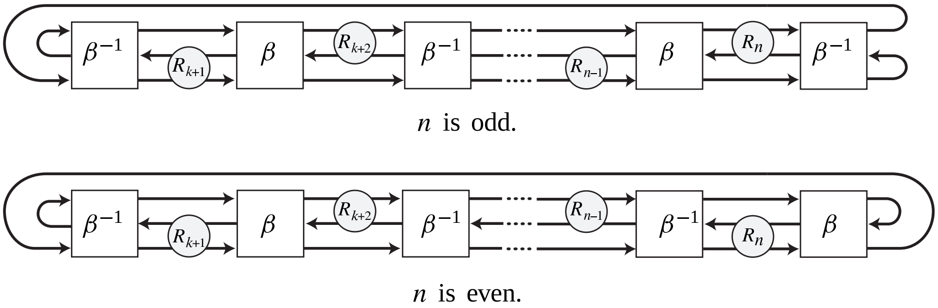


Fig. 9. $K'[\beta^{-1}; R_{k+1}, R_{k+2}, \dots, R_{n-1}, R_n]$.

where both $K[\beta; R_1, R_2, \dots, R_{k-1}]$ *with* $k = 1$ *and* $K[\beta; R_{k+1}, R_{k+2}, \dots, R_n]$ *with* $k = n$ *mean* $K[\beta]$.

Proof. Let K_k denote the link $K[\beta; R_1, R_2, \dots, R_n]$ with R_k the ∞ tangle. If k is even, then K_k is the connected sum

$$K[\beta; R_1, \dots, R_{k-1}] \# K[\beta; R_{k+1}, \dots, R_n], \tag{12}$$

and so we obtain (11) by Proposition 3.2 (iii). If k is odd, then K_k is the connected sum

$$K[\beta; R_1, \dots, R_{k-1}] \# K'[\beta^{-1}; R_{k+1}, \dots, R_n], \tag{13}$$

where $K'[\beta^{-1}; R_{k+1}, \dots, R_n]$ is the link as shown in Fig. 9. Rotating it through angle 180° about a horizontal axis, we obtain

$$K[\beta; \rho_X R_{k+1}, \rho_X R_{k+2}, \dots, \rho_X R_n]. \tag{14}$$

In fact, applying such a rotation $\beta = (\sigma_2^{p_1}\sigma_1^{p_2}\cdots)(\cdots\sigma_2^{-p_2}\sigma_1^{-p_1})$ becomes $\beta^{-1} = (\sigma_1^{p_1}\sigma_2^{p_2}\cdots)(\cdots\sigma_1^{-p_2}\sigma_2^{-p_1})$ and β^{-1} becomes β. Then the link (14) is a mutant of $K[\beta; R_{k+1}, R_{k+2}, \dots, R_n]$, and so, by Proposition 3.2 (vi), we obtain

$$\mathrm{LG}(K'[\beta^{-1}; R_{k+1}, R_{k+2}, \dots, R_n]) = \mathrm{LG}(K[\beta; R_{k+1}, R_{k+2}, \dots, R_n]), \tag{15}$$

completing the proof. □

Proof of Theorem 4.1. We prove by induction on n. The case $n = 1$ is trivial. Let m be a fixed integer with $m \geq 2$. Assuming that (10) holds for $n < m$, we will prove (10) with $n = m$.

Let M_{R_k} be a link that contains the tangle R_k. Since R_k is LG decomposable, the

LG invariant of M_{R_k} is of the form

$$\mathrm{LG}(M_{R_k}) = f_k\,\mathrm{LG}(M_0) + g_k\,\mathrm{LG}(M_2) + h_k\,\mathrm{LG}(M_\infty), \tag{16}$$

where $f_k,\ g_k,\ h_k \in \mathbb{Z}[t_0^{\pm 1}, t_1^{\pm 1}]$.

First, consider the case R_k is the 0 tangle. If $k = 1$, then

$$\begin{aligned}
&\mathrm{LG}(K[\beta; 0, R_2 \dots, R_m]) && \\
&= \mathrm{LG}(D(R_2)\# K[\beta; R_3, \dots, R_m]) && \text{(by Lemma 4.2)} \\
&= \mathrm{LG}(D(R_2))\,\mathrm{LG}(K[\beta; R_3, \dots, R_m]) && \text{(by Proposition 3.2 (iii))} \\
&= \mathrm{LG}(D(R_2))\,\mathrm{LG}(K[\beta; R_m, \dots, R_3]) && \text{(by inductive hypothesis)} \\
&= \mathrm{LG}(K[\beta; R_m, \dots, R_2, 0]). && \text{(by Lemma 4.2)}
\end{aligned} \tag{17}$$

The case $k = m$ is similar. If $2 \leq k \leq m - 1$, then

$$\begin{aligned}
&\mathrm{LG}(K[\beta; R_1, \dots, R_{k-1}, 0, R_{k+1}, \dots, R_m]) && \\
&= \mathrm{LG}(K[\beta; R_1, \dots, R_{k-1} + R_{k+1}, \dots, R_m]) && \text{(by Lemma 4.2)} \\
&= \mathrm{LG}(K[\beta; R_1, \dots, R_{k+1} + R_{k-1}, \dots, R_m]) && \text{(by Proposition 3.2 (vi))} \\
&= \mathrm{LG}(K[\beta; R_m, \dots, R_{k+1} + R_{k-1}, \dots, R_1]) && \text{(by inductive hypothesis)} \\
&= \mathrm{LG}(K[\beta; R_m, \dots, R_{k+1}, 0, R_{k-1}, \dots, R_1]). && \text{(by Lemma 4.2)}
\end{aligned} \tag{18}$$

Thus (10) with $n = m$ holds.

Also if R_k is the ∞ tangle, then by Lemma 4.3 and inductive hypothesis, (10) with $n = m$ holds. Therefore by (16), we may consider the case each tangle R_k is the 2 tangle. However such links are both $K[\beta; \underbrace{2, \dots, 2}_{n}]$, and thus we have (10) with $n = m$, completing the proof. $\square$

REMARK 4.4. Similarly, we can prove: if β is a pure 3-braid, then the HOMFLY polynomials of two links

$$K[\beta; R_1, R_2, \dots, R_{n-1}, R_n], \quad K[\beta; R_n, R_{n-1}, \dots, R_2, R_1] \tag{19}$$

are equal, which is a generalization of [15, Proposition 1]. Also, we can prove: if β is a strongly amphicheiral pure 3-braid, then the Kauffman polynomials of two links (19) coincide, which is a generalization of [15, Proposition 3].

5. The LG invariants of the link $\boldsymbol{\Sigma[\beta; R_1, R_2, \dots, R_{2n}; J, p]}$

For a 3-braid β and tangles $R_1,\ R_2, \dots, R_{2n-1},\ R_{2n}$, we define a class of oriented links $L[\beta; R_1, R_2, \dots, R_{2n-1}, R_{2n}]$ as shown in Fig. 10. We denote this link by

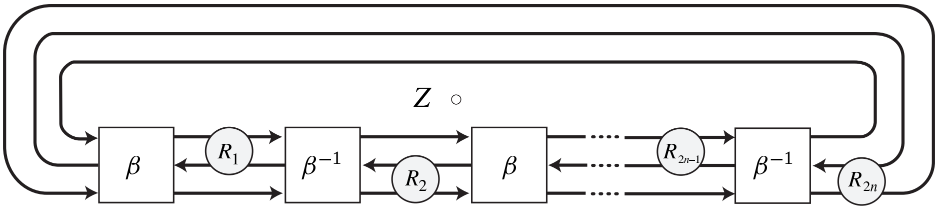


Fig. 10. $L[\beta; R_1, R_2, \dots, R_{2n-1}, R_{2n}]$.

L for short. Let V be a solid torus $S^3 - \operatorname{Int} N(Z)$, where Z is an axis perpendicular to the projection plane of the diagram and $\operatorname{Int} N(Z)$ is the interior of the regular neighborhood $N(Z)$ of Z. We suppose that Z is oriented so that the linking number of Z and L is +1 and that the preferred meridian of V is oriented parallel to Z. Let J be an oriented knot and $\varphi\colon V \to N(J)$ a homeomorphism sending (longitude) to (longitude)+p(meridian) of $N(J)$. Denote the satellite link $\varphi(L)$ with companion J and pattern $L[\beta; R_1, R_2, \dots, R_{2n-1}, R_{2n}]$ by $\Sigma[\beta; R_1, R_2, \dots, R_{2n}; J, p]$; cf. [24, p. 111]. In particular, if J is a trivial knot, then $\varphi(L)$ is obtained from L by twisting V p times.

Let L and L' be oriented links. If they are isotopic, then we write $L \approx L'$, and if they are mutants one another, then we write $L \overset{\mathrm{m}}{\approx} L'$. If β is a strongly amphicheiral pure 3-braid, then for any integer i it is easy to see:

(20) $$L[\beta; R_{1+i}, R_{2+i}, \dots, R_{2n+i}] \overset{\mathrm{m}}{\approx} L[\beta; R_1, R_2, \dots, R_{2n}],$$

and so

(21) $$\Sigma[\beta; R_{1+i}, R_{2+i}, \dots, R_{2n+i}; J, p] \overset{\mathrm{m}}{\approx} \Sigma[\beta; R_1, R_2, \dots, R_{2n}; J, p],$$

where the subscripts of R_k are understood to be reduced modulo $2n$.

Theorem 5.1. *If β is a strongly amphicheiral pure 3-braid and $R_1, R_2, \dots, R_{2n-1}, R_{2n}$ are LG decomposable tangles, then for any integer p and a knot J it holds that*

(22) $$\mathrm{LG}(\Sigma[\beta; R_1, R_2, \dots, R_{2n-1}, R_{2n}; J, p]) = \mathrm{LG}(\Sigma[\beta; R_{2n}, R_{2n-1}, \dots, R_2, R_1; J, p]).$$

In particular, it holds that

(23) $$\mathrm{LG}(L[\beta; R_1, R_2, \dots, R_{2n-1}, R_{2n}]) = \mathrm{LG}(L[\beta; R_{2n}, R_{2n-1}, \dots, R_2, R_1]).$$

Proof. We prove by induction on n. The case $n = 1$ follows from (21) with $n = i = 1$. Let m be a fixed integer with $m \geq 2$ and assume that (22) holds for $n < m$.

First we consider the case R_k is the 0 tangle. Then

$$\begin{aligned}(24)\qquad &\Sigma[\beta; R_1, \dots, R_{k-1}, 0, R_{k+1}, \dots, R_{2m}; J, p] \\ &\qquad \approx \Sigma[\beta; R_1, \dots, R_{k-1} + R_{k+1}, \dots, R_{2m}; J, p];\\ (25)\qquad &\Sigma[\beta; R_{2m}, \dots, R_{k+1}, 0, R_{k-1}, \dots, R_1; J, p] \\ &\qquad \approx \Sigma[\beta; R_{2m}, \dots, R_{k+1} + R_{k-1}, \dots, R_1; J, p].\end{aligned}$$

By the inductive hypothesis, the link (25) and

$$(26)\qquad \Sigma[\beta; R_1, \dots, R_{k-2}, R_{k+1} + R_{k-1}, R_{k+2}, \dots, R_{2m}; J, p]$$

share the same LG invariant. On the other hand, the link (24) and (26) are mutants one another, and so have the same LG invariant by Proposition 3.2 (vi). Therefore, the links (24) and (25) share the same LG invariant.

Next we consider the case R_k is the ∞ tangle. It is easy to see:

$$\begin{aligned}(27)\qquad &\Sigma[\beta; R_1, \dots, R_{k-1}, \infty, R_{k+1}, \dots, R_{2m}; J, p] \\ &\qquad \approx K[\beta; R_{k+1}, \dots, R_{2m}, R_1, \dots, R_{k-1}]\#J;\\ (28)\qquad &\Sigma[\beta; R_{2m}, \dots, R_{k+1}, \infty, R_{k-1}, \dots, R_1; J, p] \\ &\qquad \approx K[\beta; R_{k-1}, \dots, R_1, R_{2m}, \dots, R_{k+1}]\#J.\end{aligned}$$

Thus by Proposition 3.2 (iii) and Theorem 4.1 these two links share the same LG invariant.

Therefore, since R_k is LG decomposable, we may consider the case each tangle R_k is the 2 tangle. However, such links are both $\Sigma[\beta; \underbrace{2, \dots, 2}_{2m}; J, p]$, and thus we obtain (22) with $n = m$, completing the proof. □

REMARK 5.2. Similarly, we can prove: if β is a pure strongly amphicheiral 3-braid, then the HOMFLY and Kauffman polynomials of the two links

$$\Sigma[\beta; R_1, R_2, \dots, R_{2n-1}, R_{2n}; J, p], \quad \Sigma[\beta; R_{2n}, R_{2n-1}, \dots, R_2, R_1; J, p]$$

coincide, respectively; see [15, Propositions 4 and 5].

6. 2-bridge knots and links

In this section, we show the following theorems using Theorem 4.1.

Theorem 6.1. *For any positive integer N, there exist 2^N, mutually distinct, amphicheiral, fibered 2-bridge knots, which are skein equivalent and have the same Kauffman polynomial and LG invariant.*

Theorem 6.2. *For any positive integer N, there exist 2^N, mutually distinct, amphicheiral, fibered 2-bridge links, which are skein equivalent and have the same Kauffman and 2-variable Alexander polynomials and LG invariant.*

In order to prove these theorems, we prepare some notations for 3-braids and their properties, which are given in [15]. Let α be a 3-braid and $a_1, a_2, \dots, a_{n-1}, a_n$ integers. We define a 3-braid $\alpha(a_1, a_2, \dots, a_{n-1}, a_n)$ as follows:

$$\alpha(a_1, a_2, \dots, a_{n-1}, a_n) = \begin{cases} \alpha\sigma_2^{a_1}\alpha^{-1}\sigma_1^{a_2}\alpha\cdots\alpha^{-1}\sigma_1^{a_{n-1}}\alpha\sigma_2^{a_n}\alpha^{-1} & \text{if } n \text{ is odd;} \\ \alpha\sigma_2^{a_1}\alpha^{-1}\sigma_1^{a_2}\alpha\cdots\alpha\sigma_2^{a_{n-1}}\alpha^{-1}\sigma_1^{a_n}\alpha & \text{if } n \text{ is even.} \end{cases} \tag{29}$$

Thus if β is a pure 3-braid and n is even, then

$$K[\beta; a_1, a_2, \dots, a_n] = K[\beta(a_1, a_2, \dots, a_{n-1}, a_n)]; \tag{30}$$

see Figs. 6 and 7.

Furthermore, for a 3-braid α and integers $p_1, p_2, \dots, p_{n-1}, p_n$, we define 3-braids as follows:

$$\alpha\langle p_1\rangle = \alpha(p_1, -p_1) = \alpha\sigma_2^{p_1}\alpha^{-1}\sigma_1^{-p_1}\alpha; \tag{31}$$

$$\alpha\langle p_1, p_2, \dots, p_{n-1}, p_n\rangle = \alpha\langle p_1, p_2, \dots, p_{n-1}\rangle\langle p_n\rangle. \tag{32}$$

Then it is easy to see:

$$\alpha\langle p_1, p_2, \dots, p_{n-1}, p_n\rangle = \alpha\langle p_1, p_2, \dots, p_{i-1}\rangle\langle p_i, \dots, p_n\rangle, \tag{33}$$

where $2 \le i \le n$ [15, Lemma 1]. Let $q_1, q_2, \dots, q_m$, $m = 3^n - 1$, be the integers obtained by $p_i = \epsilon_j q_j$, where $j \equiv 0 \pmod{3^{i-1}}$, $j \not\equiv 0 \pmod{3^i}$, and

$$\epsilon_j = \begin{cases} 1 & \text{if } \dfrac{j}{3}^{i-1} \equiv 1 \pmod 3; \\ -1 & \text{if } \dfrac{j}{3}^{i-1} \equiv 2 \pmod 3. \end{cases} \tag{34}$$

Then $\alpha\langle p_1, p_2, \dots, p_n\rangle = \alpha(q_1, q_2, \dots, q_{m-1}, q_m)$. Furthermore, we have

$$\alpha\langle -p_1, -p_2, \dots, -p_n\rangle = \alpha(q_m, q_{m-1}, \dots, q_2, q_1). \tag{35}$$

Lemma 6.3. *Suppose that β is a strongly amphicheiral pure 3-braid and $b_1, b_2, \dots, b_n$ are integers. Then*

$$\mathrm{LG}(K[\beta\langle 2b_1, 2b_2, \dots, 2b_n\rangle]) = \mathrm{LG}(K[\beta\langle -2b_1, -2b_2, \dots, -2b_n\rangle]). \tag{36}$$

Proof. Choose integers c_1, $c_2, \dots, c_m$, $m = 3^n - 1$, so that $\beta\langle 2b_1, 2b_2, \dots, 2b_n\rangle = \beta(2c_1, 2c_2, \dots, 2c_{m-1}, 2c_m)$ as above. Then we have

$$K[\beta\langle 2b_1, 2b_2, \dots, 2b_n\rangle] = K[\beta(2c_1, 2c_2, \dots, 2c_{m-1}, 2c_m)] \\ = K[\beta; 2c_1, 2c_2, \dots, 2c_{m-1}, 2c_m]. \tag{37}$$

By using (35), we have

$$K[\beta\langle -2b_1, -2b_2, \dots, -2b_n\rangle] = K[\beta(2c_m, c_{m-1}, \dots, 2c_2, 2c_1)] \\ = K[\beta; 2c_m, 2c_{m-1}, \dots, 2c_2, 2c_1]. \tag{38}$$

Thus by using Theorem 4.1, we obtain (36). □

Proof of Theorem 6.1. Let $\beta = (\sigma_2^{2\delta_1}\sigma_1^{2\delta_2}\cdots)(\cdots\sigma_2^{-2\delta_2}\sigma_1^{-2\delta_1})$ be a nontrivial pure amphicheiral 3-braid, $\delta_i = \pm 1$. We show that the 2^N knots in the set

$$\mathcal{K}_{\beta,N} = \{\, K[\beta\langle 2\epsilon_1, 2\epsilon_2, \dots, 2\epsilon_N\rangle] \mid \epsilon_i = \pm 1 \,\} \tag{39}$$

are the desired ones. In the proof of Theorem 1 of [15] it is proved that these knots are mutually distinct, amphicheiral, fibered 2-bridge knots, and are skein equivalent and have the same Kauffman polynomial. So we have to prove that they share the same LG invariant. In fact, we have

$$\begin{aligned} &\mathrm{LG}(K[\beta\langle 2\epsilon_1, 2\epsilon_2, \dots, 2\epsilon_{i-1}, 2\epsilon_i, 2\epsilon_{i+1}, \dots, 2\epsilon_N\rangle]) \\ &= \mathrm{LG}(K[\beta\langle 2\epsilon_1, 2\epsilon_2, \dots, 2\epsilon_{i-1}\rangle\langle 2\epsilon_i, 2\epsilon_{i+1}, \dots, 2\epsilon_N\rangle]) \\ &= \mathrm{LG}(K[\beta\langle 2\epsilon_1, 2\epsilon_2, \dots, 2\epsilon_{i-1}\rangle\langle -2\epsilon_i, -2\epsilon_{i+1}, \dots, -2\epsilon_N\rangle]) \\ &= \mathrm{LG}(K[\beta\langle 2\epsilon_1, 2\epsilon_2, \dots, 2\epsilon_{i-1}, -2\epsilon_i, -2\epsilon_{i+1}, \dots, -2\epsilon_N\rangle]) \\ &= \mathrm{LG}(K[\beta\langle 2\epsilon_1, 2\epsilon_2, \dots, 2\epsilon_{i-1}, -2\epsilon_i\rangle\langle -2\epsilon_{i+1}, \dots, -2\epsilon_N\rangle]) \\ &= \mathrm{LG}(K[\beta\langle 2\epsilon_1, 2\epsilon_2, \dots, 2\epsilon_{i-1}, -2\epsilon_i\rangle\langle 2\epsilon_{i+1}, \dots, 2\epsilon_N\rangle]) \\ &= \mathrm{LG}(K[\beta\langle 2\epsilon_1, 2\epsilon_2, \dots, 2\epsilon_{i-1}, -2\epsilon_i, 2\epsilon_{i+1}, \dots, 2\epsilon_N\rangle]), \end{aligned} \tag{40}$$

where we use (33) and Lemma 6.3. This completes the proof. □

Proof of Theorem 6.2. Let β be as in the proof of Theorem 6.1. Then the links in the set

$$\mathcal{L}_{\beta,N} = \{\, K[\beta\langle 2\epsilon_1, 2\epsilon_2, \dots, 2\epsilon_N\rangle(2)] \mid \epsilon_i = \pm 1 \,\} \tag{41}$$

are the desired ones. Note that the link $K[\beta\langle 2\epsilon_1, 2\epsilon_2, \dots, 2\epsilon_N\rangle(2)]$ is also expressed as $K[\beta\langle 2\epsilon_1, 2\epsilon_2, \dots, 2\epsilon_N\rangle; 2]$. In the proof of Theorem 2 in [15], it is proved that these links are mutually distinct, amphicheiral, fibered 2-bridge links, and are skein equivalent and have the same Kauffman and 2-variable Alexander polynomials. Also we may prove that they share the same LG invariant in a similar way to the proof of Theorem 6.1, and so we omit it. □

In [15, Sect. 4], all the pairs of 2-bridge knots and links through 20 crossings sharing the same Kauffman polynomial are given; cf. [18, 19]. Using the formula [7, Proposition 5.1], the first author has given a computer program calculating the LG invariant of 2-bridge knots and links. Making use of this, he has seen that the following 11 pairs of 2-bridge knots and a single pair of 2-bridge links also share the same LG invariants: for 2-bridge knots $K[\sigma_2^2\sigma_1^{-2}; p, q]$ and $K[\sigma_2^2\sigma_1^{-2}; q, p]$, where $(p, q) = (2, -2)$, $(4, 2)$, $(4, -2)$, $(6, 2)$, $(6, -2)$, $(4, -4)$, $(8, 2)$, $(8, -2)$, $(6, 4)$, $(6, -4)$; $K[\sigma_2^2\sigma_1^2\sigma_2^{-2}\sigma_1^{-2}; 2, -2]$ and $K[\sigma_2^2\sigma_1^2\sigma_2^{-2}\sigma_1^{-2}; -2, 2]$; and for 2-bridge links $K[\sigma_2^2\sigma_1^{-2}; 2, 2, -2]$ and $K[\sigma_2^2\sigma_1^{-2}; -2, 2, 2]$. They also share the same HOMFLY polynomials.

Furthermore, the knots and links with the same LG invariant that are constructed in this paper also share the same HOMFLY and Kauffman polynomials; see Remarks 4.4 and 5.2. Compare Theorem 8.1. So we may ask:

PROBLEM 6.4. Does there exist a pair of knots or links with the same LG invariant that have distinct HOMFLY or Kauffman polynomials?

7. Chiral knots

In this section, we show certain knots of the form $L[\beta; R_1, R_2, \dots, R_{2n}]$ have the same LG invariants as their mirror images. Making use of this, we construct a chiral knot, whose chirality is undetected by the LG invariant.

Proposition 7.1. *If β is a strongly amphicheiral pure 3-braid, then the link*

$$L[\beta; R_1, R_2, \dots, R_n, \mu R_1, \mu R_2, \dots, \mu R_n] \tag{42}$$

and its mirror image have the same LG invariant.

Proof. Let us consider a link $L[\beta; R_1, R_2, \dots, R_{2n}]$ as shown in Fig. 10 with $\beta = (\sigma_2^{p_1}\sigma_1^{p_2}\cdots)(\cdots\sigma_2^{-p_2}\sigma_1^{-p_1})$. Flip it over about a vertical axis in the projection plane. Then we obtain the link as shown in Fig. 11, where $\alpha = (\sigma_2^{-p_1}\sigma_1^{-p_2}\cdots)(\cdots\sigma_2^{p_2}\sigma_1^{p_1})$ and $R'_k = \rho_Y R_k$. Then it may be presented as $-L[\alpha; R'_{2n-1}, \dots, R'_2, R'_1, R'_{2n}]$. Since α is a strongly amphicheiral pure 3-braid, this and $-L[\alpha; R'_{2n}, R'_{2n-1}, \dots, R'_2, R'_1]$ are mutants by (20), whose mirror image is $-L[\beta; \mu R'_{2n}, \mu R'_{2n-1}, \dots, \mu R'_2, \mu R'_1]$. Therefore, in particular, the mirror image of the link (42) and
$-L[\beta; R'_n, \dots, R'_1, \mu R'_n, \dots, \mu R'_1]$, are mutants, whose LG invariant is equal to that of $L[\beta; R_n, \dots, R_1, \mu R_n, \dots, \mu R_1]$ by using Proposition 3.2 (v) and (vi). By (20) and (23) in Theorem 5.1, it is equal to the LG invariant of the link (42). This com-

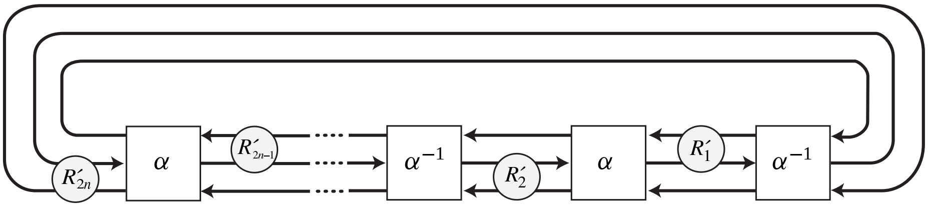


Fig. 11. The link obtained by flipping $L[\beta; R_1, R_2, \dots, R_{2n-1}, R_{2n}]$.

pletes the proof. □

EXAMPLE 7.2. Let us consider the knot

$$L\Big[\sigma_2^{-2}\sigma_1^2; \frac{1}{2}, \frac{1}{4}, -\frac{1}{2}, -\frac{1}{4}\Big]. \tag{43}$$

By Proposition 7.1, this knot and its mirror image share the same LG invariant. Thus we cannot detect the chirality of this knot using the LG invariant. Furthermore, they also share the same HOMFLY and Kauffman polynomials according to the Remark 5.2.

We then consider the 3-manifolds obtained by performing +1 and −1 surgeries on this knot. If the knot (43) were amphicheiral, then they must be homeomorphic. However, applying the computer program SnapPea of Jeffrey R. Weeks, we obtain the hyperbolic volumes of these manifolds: they are 32.52083188 and 32.52148150, respectively, which are given without rounding, out to eight decimal places. Thus we may conclude that the knot (43) is chiral; cf. [24, Exercise 9H8]. By using the result of Ruberman [25], this also shows that the knot (43) and its mirror image are not mutants one another. Furthermore, the knot (43) is hyperbolic, and so in particular prime, since its volume is 33.62982454.

REMARK 7.3. In general, we can easily construct a pair of amphicheiral and chiral knots with the same LG invariant, when they are (i) nonprime or (ii) mutants one another: (i) Let J and K be distinct prime knots that share the same LG invariant such that $\mathrm{LG}(J; t_0, t_1) \neq \mathrm{LG}(J; t_0^{-1}, t_1^{-1}) = \mathrm{LG}(J!; t_0, t_1)$. Then $J\#J!$ and $J\#K!$ are such a pair. In fact, $\mathrm{LG}(J\#K!) = \mathrm{LG}(J)\mathrm{LG}(J!) = \mathrm{LG}(J\#J!)$, and $J\#K!$ is chiral. If $J\#K!$ were amphicheiral, then J and K should equivalent by the unique factorization theorem and considering the LG invariants, which is false; cf. [11, Sect. 2].

(ii) The knot given in [11, Fig. 7] is chiral, which has a mutant knot that is amphicheiral; see [17], [21, p.428].

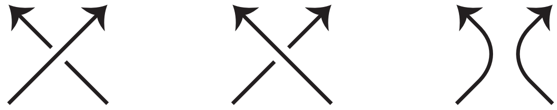


Fig. 12. Skein triple.

8. The LG invariant and the Conway polynomial

The *Conway polynomial* $\nabla(L; z) \in \mathbb{Z}[z]$ [3] is an invariant of an oriented link L, which is defined by the following formulas:

$$\nabla(U; z) = 1; \tag{44}$$

$$\nabla(L_+; z) - \nabla(L_-; z) = z\nabla(L_0; z), \tag{45}$$

where U is a trivial knot and L_+, L_-, L_0 are three links that are identical except near one point where they are as in Fig. 12.

The first author has proved the following [9].

Theorem 8.1. *The Conway polynomial* $\nabla(L; z)$ *is obtained from the LG invariant* $\mathrm{LG}(L; t_0, t_1)$ *by putting* t_0 *and* t_1 *as follows*:

$$t_0 + t_1 = z, \qquad t_0 t_1 = -1. \tag{46}$$

This result had been led by the following proposition:

Proposition 8.2. *Let L be an oriented link. Suppose that the LG invariant* $\mathrm{LG}(L; t_0, t_1)$ *of L is calculated recursively by using the relations* (1), (2) *in* Proposition 3.1 *and* Proposition 3.2 (i). *Then the Conway polynomial* $\nabla(L; z)$ *is obtained from it by putting* t_0 *and* t_1 *as* (46).

Proof. Let L_0, L_∞, $L_{1/m}$, L_n be oriented links as in Section 2. Then it follows from (45) that

$$\nabla(L_{-1/2}) = z\nabla(L_0) + \nabla(L_\infty); \tag{47}$$

$$\nabla(L_{-2}) = \nabla(L_0) - z\nabla(L_\infty); \tag{48}$$

$$\nabla(L_2) = \nabla(L_0) + z\nabla(L_\infty), \tag{49}$$

which imply

$$\nabla(L_{-1/2}) - \nabla(L_{-2}) = (-1 + z)\nabla(L_0) + (1 + z)\nabla(L_\infty); \tag{50}$$

$$\nabla(L_{-1/2}) - \nabla(L_2) = (-1 + z)\nabla(L_0) + (1 - z)\nabla(L_\infty). \tag{51}$$

On the other hand, let $\gamma(L;z)$ be the polynomial obtained from the LG invariant $\mathrm{LG}(L;t_0,t_1)$ by putting t_0 and t_1 as (46). If the LG invariant $\mathrm{LG}(L;t_0,t_1)$ of a link L is calculated by using (1), (2) and Proposition 3.2 (i), then the polynomial $\gamma(L)$ is also calculated by using:

$$\gamma(L_{-1/2}) - \gamma(L_{-2}) = (-1+z)\gamma(L_0) + (1+z)\gamma(L_\infty); \tag{52}$$

$$\gamma(L_{-1/2}) - \gamma(L_{2}) = (-1+z)\gamma(L_0) + (1-z)\gamma(L_\infty); \tag{53}$$

$$\gamma(U) = 1, \tag{54}$$

where (52) and (53) are obtained from (1) and (2), respectively. Comparing these formulas with (50) and (51), the two polynomials $\gamma(L)$ and $\nabla(L)$ should coincide. This completes the proof. □

Since the first author [7, Theorem 4.1] proved that the LG invariant of an algebraic link (see [3]) may be calculated recursively by using the relations (1), (2) and Proposition 3.2 (i). In particular, he gave a formula for calculating the LG invariant of a 2-bridge link [7, Proposition 5.1]. Thus Proposition 8.2 implies: The Conway polynomial of an algebraic link may be recovered from the LG invariant by putting t_0 and t_1 as (46). Also, for prime knots with up to 10 crossings, he checked that the Conway polynomials are recovered from the LG invariant by putting t_0 and t_1 as (46), and also he showed that a closed 3-braid has such a property. Finally, he succeeded in generalizing as in Theorem 8.1.

Corollary 8.3. *There exist arbitrarily many* 2*-bridge knots with the same Kauffman polynomial but distinct LG invariants.*

Proof. The second author has proved [16] that there exist arbitrarily many 2-bridge knots with the same Kauffman polynomial but distinct Conway polynomials. Since the Conway polynomial of a 2-bridge knot is recovered from the LG invariant, we obtain the result. □

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