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NO 2-KNOT HAS TRIPLE POINT NUMBER TWO OR THREE

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Abstract

It is known that a 2-knot has the triple point number less than two if and only if it is of ribbon-type. We prove that there is no 2-knot of triple point number two or three. Hence the 2-twist-spun trefoil, which is known as a 2-knot of triple point number four, is one of the simplest non-ribbon 2-knots.

1. Introduction

In classical knot theory, the knot table is usually made according to the crossing number of a knot, which is the minimal number of crossings among all possible projections into the plane. There is no classical knot of crossing number one or two, and the trefoil of crossing number three is the simplest non-trivial knot in this sense. It is natural to consider a similar tabulation in surface-knot theory. A *surface-knot* means a (possibly disconnected or non-orientable) closed surface embedded in 4-space $\mathbb{R}^4$ smoothly. In particular, a surface-knot is called a 2-*knot* if it is a knotted 2-sphere in $\mathbb{R}^4$. One remarkable table is made by Yoshikawa [17] by introducing a certain kind of quantity, which he calls the "ch-index" of a surface-knot. In this paper, we use another criterion, the *triple point number* of a surface-knot, which has a natural analogy to the crossing number of a classical knot. Precisely, it is defined to be the minimal number of triple points among all possible projections of a surface-knot $K \subset \mathbb{R}^4$ into 3-space $\mathbb{R}^3$, and is denoted by $\mathrm{t}(K)$. The aim of this paper is to prove the following.

Theorem 1.1. *There is no* 2*-knot* K *with* $0 < \mathrm{t}(K) < 4$.

Let K be a surface-knot. We say that K is a *pseudo-ribbon* surface-knot if it satisfies $\mathrm{t}(K) = 0$ (cf. [7]), and a *ribbon* surface-knot if it is obtained from a split union of trivial 2-knots by surgeries along some 1-handles connecting them (cf. [6]). It is known that these families are coincident in the case of 2-knots (cf. [5, 16]). Hence Theorem 1.1 implies that if K is a non-ribbon 2-knot, then it holds that $\mathrm{t}(K) \geq 4$.

For the import of Theorem 1.1, it is reasonable to refer to some open problems on triple point numbers. In [10], it is proved that any surface-knot K satisfies $\mathrm{t}(K) \neq 1$. This result holds regardless of the genus, orientability, or connectivity of K. The only known example of $\mathrm{t}(K) = 2$ is given in [11], which is a 2-component surface-link with non-orientable components.

Question 1.2. Is there an *orientable* surface-knot K with $\mathrm{t}(K) = 2$?

In the case of $\mathrm{t}(K) = 3$, we have no examples even if K is non-orientable, or disconnected. More generally, we have no examples of odd triple point numbers.

Question 1.3. Is there a surface-knot K such that $\mathrm{t}(K) > 1$ is *odd*?

As an orientable surface-knot K whose triple point number $\mathrm{t}(K) > 0$ is concretely determined, we have the 2-twist-spun trefoil (and its connected sum with an arbitrary orientable pseudo-ribbon surface-knot) which satisfies $\mathrm{t}(K) = 4$ (cf. [13]). Hence it follows by Theorem 1.1 that the 2-twist-spun trefoil is one of the simplest non-ribbon 2-knots according to the triple point number. From the viewpoint of tabulation, the following is an important problem to be considered in future.

Question 1.4. Is there a 2-knot K with $\mathrm{t}(K) = 4$ *except* the connected sum of the 2-twist-spun trefoil with an arbitrary ribbon 2-knot?

It is proved in [14] that the 3-twist-spun trefoil K satisfies $\mathrm{t}(K) = 6$; however, nothing on $\mathrm{t}(K) = 5$ follows from this result.

This paper is organized as follows. In Section 2, we review the definition of a diagram of a surface-knot, which is a projection image in $\mathbb{R}^3$ with crossing information. In Sections 3 and 4, we prove $\mathrm{t}(K) \neq 3$ (Theorem 3.3) and $\mathrm{t}(K) \neq 2$ (Theorem 4.5) for any 2-knot K, respectively. This paper is motivated from Shima's result [15] that if a 2-knot K has a diagram with two triple points and *no* branch points, then K is a ribbon 2-knot. Hence, to prove $\mathrm{t}(K) \neq 2$, it is sufficient to consider a diagram with two triple points and *some* branch points.

2. Preliminaries

2.1. Double, triple, and branch points. Throughout this paper, we always assume that all surface-knots are oriented. Let us fix an orthogonal projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$. We can isotope a surface-knot $K \subset \mathbb{R}^4$ slightly so that the projection image $\pi(K) \subset \mathbb{R}^3$ has only double points and triple points as its multiple points, and has only branch points as its singular points missing multiple points (cf. [3]). See Fig. 1. We denote by M_2, M_3, and $S \subset \pi(K)$ the sets of double points, triple points, and branch points, respectively. Then M_3 and S appear as discrete sets, while M_2 appears as a disjoint union of open arcs and simple closed curves. Note that the boundary points of each arc of M_2 belong to $M_3 \cup S$. We say that such an open arc of M_2 is called an *edge*, and in particular, a *bb-edge*, *bt-edge*, or *tt-edge* if its boundary points are branch points both, a branch point and a triple point, or triple points both, respectively (cf. [12]). We will write double points, triple points, branch points, and edges in capital letters such as D, T, B, and E, respectively.

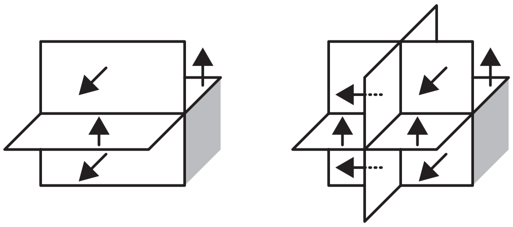

Fig. 1.

2.2. Alexander numbering. We fix an *Alexander numbering* for the complement $\mathbb{R}^3 \setminus \pi(K)$, which is a numbering of the set of connected regions of $\mathbb{R}^3 \setminus \pi(K)$ with integers such that (i) two regions separated by a sheet of $\pi(K)$ are numbered consecutively, and (ii) the orientation normal to the sheet points toward the region with larger number (see [8], for example). The Alexander numbering induces a map $\lambda\colon M_2 \cup M_3 \cup S \to \mathbb{Z}$ such that, for each point $X \in M_2$, M_3, or S, the integer $\lambda(X)$ is the minimal Alexander number among the four, eight, or three regions around X, respectively. In other words, $\lambda(X)$ is the Alexander number of the specific region R where all orientation normals to the bounded sheets point away from R. See Fig. 1 again, where the orientation normals to the sheets are depicted by small arrows, and the specific regions are shaded. For an edge E and a double point $D \in E$, since the Alexander number $\lambda(D)$ is independent of the choice of D, we use the extended notation $\lambda(E) = \lambda(D)$.

2.3. A diagram of a surface-knot. For a double point $D \in M_2$, let $\{D^{\mathrm{U}}, D^{\mathrm{L}}\} \subset K$ denote the preimage of D by $(\pi|_K)^{-1}$ such that $h(D^{\mathrm{U}}) > h(D^{\mathrm{L}})$, where $h\colon \mathbb{R}^4 \to \mathbb{R}$ is the height function orthogonal to π. Let $N^{\mathrm{W}} \subset K$ ($\mathrm{W} = \mathrm{U}, \mathrm{LS}$) be a sufficiently small regular neighborhood of the point D^{W} in K. Then we say that $\pi(N^{\mathrm{U}})$ and $\pi(N^{\mathrm{L}})$ are *upper* and *lower sheets* at D, respectively.

Similarly, for a triple point $T \in M_3$, let $\{T^{\mathrm{T}}, T^{\mathrm{M}}, T^{\mathrm{B}}\} \subset K$ denote the preimage of T by $(\pi|_K)^{-1}$ such that $h(T^{\mathrm{T}}) > h(T^{\mathrm{M}}) > h(T^{\mathrm{B}})$. Let $N^{\mathrm{W}} \subset K$ (W = T, M, B) be a sufficiently small regular neighborhood of T^{W} in K. Then $\pi(N^{\mathrm{T}})$, $\pi(N^{\mathrm{M}})$, and $\pi(N^{\mathrm{B}})$ are called *top*, *middle*, and *bottom sheets* at T, respectively.

A *diagram* of K is a projection image $\pi(K)$ with crossing information specified by breaking under-sheets at double points and middle and bottom sheets at triple points in a similar way to classical knot diagrams (see [3], for example). Hence, in a diagram, the lower sheet is divided into two pieces, and the middle and bottom sheets are divided into two and four pieces, respectively. See Fig. 2(i) and (ii). In this paper, we use the Greek letter Δ to stand for a diagram of a surface-knot.

2.4. Signs and orientations. The *sign* of a branch point B, denoted by $\varepsilon(B) \in \{\pm 1\}$, is defined according to crossing information along the edge incident to B. More

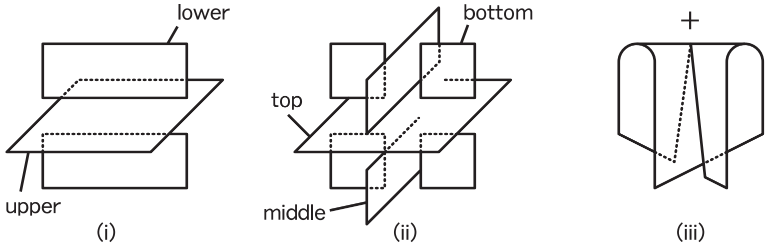

Fig. 2.

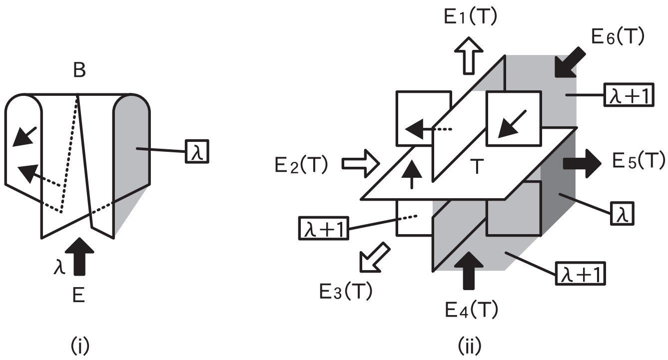

Fig. 3.

precisely, the branch point illustrated in Fig. 2 (iii) has the sign $\varepsilon(B) = +1$, and its mirror image with opposite crossing information has $\varepsilon(B) = -1$. This definition does not depend on the particular choice of an orientation of the sheet near a branch point (cf. [2]).

Near a double point $D \in M_2$, we take orientation normals $\vec{n}_{\mathrm{U}}$ and $\vec{n}_{\mathrm{L}}$ to the upper and lower sheets, respectively. We define a vector $\vec{v}$ at D such that the ordered triple $(\vec{n}_{\mathrm{U}}, \vec{n}_{\mathrm{L}}, \vec{v})$ matches the fixed right-handed orientation of $\mathbb{R}^3$. For an edge $E \subset M_2$, the set of vectors at the double points on E defines an orientation of E. If the boundary points of E are X and $Y \in M_3 \cup S$ such that the orientation of E points from X toward Y, we use the notation $E = \overrightarrow{XY}$. If X or Y is a branch point B, then it holds that $\lambda(B) = \lambda(E)$. Moreover, we have $\varepsilon(B) = +1$ if $E = \overrightarrow{XB}$, and $\varepsilon(B) = -1$ if $E = \overrightarrow{BY}$. See Fig. 3 (i), where the case of $\varepsilon(B) = +1$ is depicted.

Near a triple point $T \in M_3$, we take orientation normals $\vec{n}_{\mathrm{T}}$, $\vec{n}_{\mathrm{M}}$, and $\vec{n}_{\mathrm{B}}$ to the top, middle, and bottom sheets, respectively. We define the *sign* of T, denoted by $\varepsilon(T) \in \{\pm 1\}$, such that $\varepsilon(T) = +1$ if and only if the ordered triple $(\vec{n}_{\mathrm{T}}, \vec{n}_{\mathrm{M}}, \vec{n}_{\mathrm{B}})$ matches the fixed right-handed orientation of $\mathbb{R}^3$.

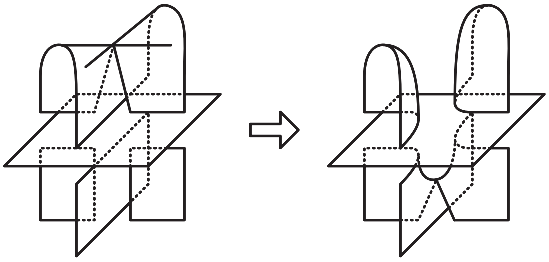

Fig. 4.

2.5. Edges at a triple point. There are six edges incident to T, which are distinguished by the orientations of top, middle, and bottom sheets; the edges are denoted by $E_1(T), E_2(T), \dots, E_6(T)$ such that
(i) $E_1(T) \cup E_4(T)$, $E_2(T) \cup E_5(T)$, and $E_3(T) \cup E_6(T)$ form straight paths across T, which are transverse to the top, middle, and bottom sheets, respectively, and
(ii) the orientation normal to the sheet points from $E_{k+3}(T)$ toward $E_k(T)$ for $k = 1, 2, 3$.
Then the Alexander number of each edge $E_k(T)$ satisfies

$$\lambda\bigl(E_k(T)\bigr) = \begin{cases} \lambda(T) + 1 & \text{for } k = 1, 2, 3, \\ \lambda(T) & \text{for } k = 4, 5, 6. \end{cases}$$

Moreover, if $\varepsilon(T) = +1$, then the orientation of $E_k(T)$ points away from T for $k = 1, 3, 5$ and toward T for $k = 2, 4, 6$. Similarly, if $\varepsilon(T) = -1$, then the orientation of $E_k(T)$ is opposite (cf. [1]). See Fig. 3 (ii), where a positive triple point is depicted, and the Alexander numbers of edges with black and white big arrows are $\lambda(T)$ and $\lambda(T) + 1$, respectively.

2.6. A minimal diagram. Let Δ be a diagram of a surface-knot K. We denote by $t(\Delta)$ the number of triple points of Δ, that is, $t(\Delta) = |M_3|$. The *triple point number* of K, denoted by $\mathrm{t}(K)$, is the minimal number of $t(\Delta)$'s for all possible diagrams of K. We say that Δ is a *minimal* diagram if $t(\Delta) = \mathrm{t}(K)$ holds. It is known that if Δ has a triple point $T \in M_3$ such that at least one of the four edges $E_1(T)$, $E_3(T)$, $E_4(T)$, or $E_6(T)$ is a bt-edge, then Δ is not a minimal diagram (see [11], for example). Fig. 4 shows a deformation of eliminating a triple point along a bt-edge $E_1(T)$ or $E_4(T)$. This deformation is realized by a finite sequence of Roseman moves [9], which are sufficient to connect any two diagrams of a surface-knot.

2.7. Numbers of triple points. Assume that Δ is a minimal diagram. Then the triple points of Δ are divided into four classes according to whether $E_2(T)$ or $E_5(T)$, or both are bt-edges. (Recall that $E_2(T)$ and $E_5(T)$ are transverse to the middle sheet.) We say that the *type* of a triple point $T \in M_3$ is

$\langle 0\rangle$ if both of $E_2(T)$ and $E_5(T)$ are tt-edges,
$\langle 2\rangle$ if $E_2(T)$ is a bt-edge and $E_5(T)$ is a tt-edge,
$\langle 5\rangle$ if $E_2(T)$ is a tt-edge and $E_5(T)$ is a bt-edge, and
$\langle 25\rangle$ if both of $E_2(T)$ and $E_5(T)$ are bt-edges.
For each $\varepsilon \in \{\pm 1\}$, $w \in \{0, 2, 5, 25\}$, and $\lambda \in \mathbb{Z}$, we denote by $t_w^{\varepsilon}(\lambda)$ the number of triple points of Δ with the sign ε, type $\langle w\rangle$, and Alexander number λ. Moreover, we put $t_w(\lambda) = t_w^{+1}(\lambda) - t_w^{-1}(\lambda)$, which is equal to the sum of signs for all triple points of type $\langle w\rangle$ with Alexander number λ. Then it is proved in [12] that

$$\begin{aligned} &t_0(\lambda) + 2t_2(\lambda) + t_5(\lambda) + 2t_{25}(\lambda) \\ &= t_0(\lambda + 1) + t_2(\lambda + 1) + 2t_5(\lambda + 1) + 2t_{25}(\lambda + 1) \end{aligned} \tag{1}$$

for any $\lambda \in \mathbb{Z}$.

2.8. Double point curves. Let Δ be a (not necessary minimal) diagram of a surface-knot K. By connecting diagonal edges $E_k(T)$ and $E_{k+3}(T)$ for $k = 1, 2, 3$ at each triple point T of Δ, the set $M_2 \cup M_3 \cup S$ is regarded as a union of oriented curves (circle and arc components) immersed in $\mathbb{R}^3$. More precisely, if there is a sequence of tt-edges

$$E_1 = \overrightarrow{T_0T_1}, E_2 = \overrightarrow{T_1T_2}, \dots, E_{n-1} = \overrightarrow{T_{n-2}T_{n-1}}, E_n = \overrightarrow{T_{n-1}T_n},$$

where $T_0, T_1, \dots, T_n = T_0 \in M_3$, such that E_i and E_{i+1} are diagonal at T_i for $i = 1, 2, \dots, n$ $(E_{n+1} = E_1)$, then they form a circle component. Similarly, if there is a sequence of bt- and tt-edges

$$E_1 = \overrightarrow{B_0T_1}, E_2 = \overrightarrow{T_1T_2}, \dots, E_{n-1} = \overrightarrow{T_{n-2}T_{n-1}}, E_n = \overrightarrow{T_{n-1}B_n},$$

where $T_1, T_2, \dots, T_{n-1} \in M_3$ and $B_0, B_n \in S$, such that E_i and E_{i+1} are diagonal at T_i for $i = 1, 2, \dots, n-1$, then they form an arc component. We call such oriented curves the *double point curves*.

2.9. Decker curves. Let C be a double point curve of a diagram Δ of a surface-knot K. For each edge E contained in C, let $(\pi|_K)^{-1}(E) = \{E^{\mathrm{U}}, E^{\mathrm{L}}\}$ be a pair of open arcs on K such that $E^{\mathrm{W}} = \bigcup_{D\in E} D^{\mathrm{W}}$ for W = U and L. Then the curve $C^{\mathrm{W}} = \mathrm{Cl}\big(\bigcup_{E\subset C} E^{\mathrm{W}}\big)$ on K is called the *upper decker curve* of C for W = U, and the *lower decker curve* for W = L, where Cl stands for the closure. If C is a circle component, then the corresponding decker curve C^{W} (W = U, L) is a circle immersed in K. On the other hand, if C is an arc component, then C^{W} is an immersed arc such that the union $C^{\mathrm{U}} \cup C^{\mathrm{L}}$ forms a circle by connecting their boundary points (cf. [3]).

Throughout this paper, we use the notation defined in this section.

3. Diagrams with three triple points

It is proved in [10] that any surface-knot K satisfies $\mathrm{t}(K) \neq 1$. Hence, to prove Theorem 1.1, it is sufficient to study the cases $\mathrm{t}(K) = 2$ and 3. The proof of $\mathrm{t}(K) \neq 3$ for any 2-knot K is divided into Lemma 3.1 and Proposition 3.2. We first consider the types of triple points in a minimal diagram Δ with $t(\Delta) = 3$.

Lemma 3.1. *Assume that there is a surface-knot K with $\mathrm{t}(K) = 3$. Let Δ be a minimal diagram of K whose triple points are T_1, T_2, and T_3. Then after suitable changes of indexes, T_1 and T_2 are of type $\langle 0\rangle$, and T_3 is of type $\langle 25\rangle$. Moreover, it holds that $\lambda(T_1) = \lambda(T_2) = \lambda(T_3)$ and $\varepsilon(T_1) = \varepsilon(T_2) = -\varepsilon(T_3)$.*

Proof. We put $\lambda_i = \lambda(T_i)$ and $\varepsilon_i = \varepsilon(T_i)$ for $i = 1, 2, 3$. We may assume that $\lambda_1 \le \lambda_2 \le \lambda_3$. Since there is no triple point of Δ whose Alexander number is less than $\lambda_1 - 1$, or greater than λ_3, we obtain

$$t_0(\lambda_1) + t_2(\lambda_1) + 2t_5(\lambda_1) + 2t_{25}(\lambda_1) = 0, \quad \text{and} \tag{2}$$

$$t_0(\lambda_3) + 2t_2(\lambda_3) + t_5(\lambda_3) + 2t_{25}(\lambda_3) = 0 \tag{3}$$

by putting $\lambda = \lambda_1 - 1$ and λ_3 in the equation (1), respectively. If $\lambda_1 < \lambda_2$, then it follows by definition that

$$\big\{t_0(\lambda_1), t_2(\lambda_1), t_5(\lambda_1), t_{25}(\lambda_1)\big\} = \{\varepsilon_1, 0, 0, 0\},$$

which contradicts to the equation (2). Here, we use the notation { } for a multi-set, so that the above equality means that one of $t_0(\lambda_1), \ldots, t_{25}(\lambda_1)$ is equal to ε_1, and the others are zeros.

Similarly, if $\lambda_2 < \lambda_3$, then it holds that

$$\big\{t_0(\lambda_3), t_2(\lambda_3), t_5(\lambda_3), t_{25}(\lambda_3)\big\} = \{\varepsilon_3, 0, 0, 0\},$$

which contradicts to the equation (3). Hence, we have $\lambda_1 = \lambda_2 = \lambda_3$. We put $t_w = t_w(\lambda_i)$ regardless of i ($w = 0$, 2, 5, 25), which is the algebraic number of triple points of type $\langle w\rangle$. It is sufficient to consider the following three cases.

- $\underline{\{t_0, t_2, t_5, t_{25}\} = \{\varepsilon_1 + \varepsilon_2 + \varepsilon_3, 0, 0, 0\}}$. Since $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \neq 0$, we have a contradiction to (2) clearly. Hence this case does not happen.
- $\underline{\{t_0, t_2, t_5, t_{25}\} = \{\varepsilon_1 + \varepsilon_2, \varepsilon_3, 0, 0\}}$. If $\varepsilon_1 = -\varepsilon_2$, then it reduces to the the previous case. If $\varepsilon_1 = \varepsilon_2$, then we have $\varepsilon_1 + \varepsilon_2 = \pm 2$. Since $t_2 = t_5$ by (2) and (3), we obtain $t_0 = \varepsilon_1 + \varepsilon_2 = \pm 2$, $t_2 = t_5 = 0$, and $t_{25} = \varepsilon_3 = \mp 1$. This is the desired solution.
- $\underline{\{t_0, t_2, t_5, t_{25}\} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, 0\}}$. If $t_0 = 0$, we have $t_0 + t_2 + 2t_5 + 2t_{25} \equiv t_2 \equiv 1 \pmod 2$, which contradicts to (2). If $t_2 = 0$ or $t_5 = 0$, it contradicts to $t_2 = t_5$. If $t_{25} = 0$, we have $t_0 + 3t_2 = 0$ by (2) and (3). However, this contradicts to $|t_0| = 1$ and $|3t_2| = 3$.

Thus we have the conclusion. □

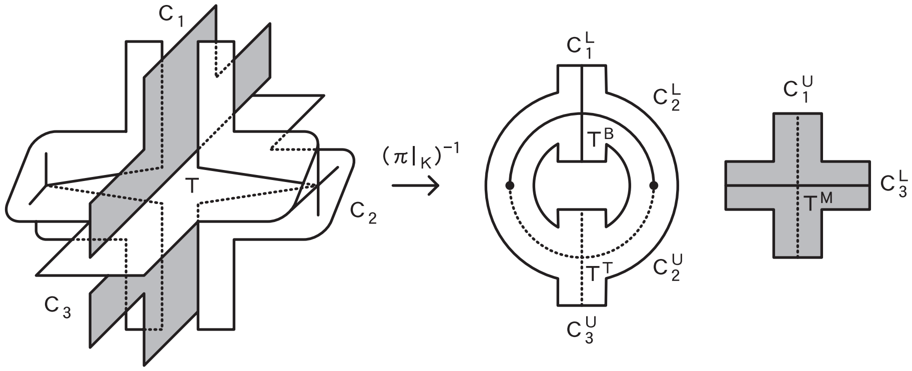

Fig. 5.

The following proposition is proved by counting the number of intersections between decker curves on K. In [4] Hasegawa generalizes this proposition without the condition "no triple points of type $\langle 2\rangle$ or $\langle 5\rangle$."

Proposition 3.2. *Let Δ be a minimal diagram of a surface-knot K. If Δ has at least one triple point T of type $\langle 25\rangle$ and no triple points of type $\langle 2\rangle$ or $\langle 5\rangle$, then the genus of K is positive.*

Proof. Let C_k be the double point curve containing $E_k(T) \cup E_{k+3}(T)$ for $k = 1, 2, 3$. Since Δ has no triple points of type $\langle 2\rangle$ or $\langle 5\rangle$, C_1 and C_3 are circle components. On the other hand, since T is of type $\langle 25\rangle$, C_2 is an arc component consisting of $E_2(T)$ and $E_5(T)$. Consider the decker curves corresponding to C_k as shown in Fig. 5, where we draw upper and lower decker curves by dotted and solid lines, respectively. Then the circle $C_2^{\mathrm{U}} \cup C_2^{\mathrm{L}}$ intersects the circles C_1^{L} and C_3^{U} at T^{B} and T^{T}, respectively. Since C_1^{L} and C_3^{U} are upper and lower decker circles, respectively, it holds that $C_1^{\mathrm{L}} \neq C_3^{\mathrm{U}}$ by definition. Hence there is a pair of circles on K, such as $\{C_2^{\mathrm{U}} \cup C_2^{\mathrm{L}}, C_1^{\mathrm{L}}\}$, with a single intersection. This is possible only if the genus of K is positive. □

Theorem 3.3. *There is no* 2*-knot* K *with* $\mathrm{t}(K) = 3$.

Proof. Assume that there is a 2-knot K with $\mathrm{t}(K) = 3$. Then any minimal diagram Δ of K has three triple points of type $\langle 0\rangle$, $\langle 0\rangle$, and $\langle 25\rangle$ by Lemma 3.1, which contradicts to Proposition 3.2. □

4. Diagrams with two triple points

In this section, we study the case $\mathrm{t}(K) = 2$. We first consider the types, Alexander numbers, and signs of triple points in a minimal diagram Δ with $t(\Delta) = 2$.

Lemma 4.1. *Assume that there is a surface-knot K with $\mathrm{t}(K) = 2$. Let Δ be a minimal diagram of K whose triple points are T_1 and T_2. Then T_1 and T_2 are of the same type with $\lambda(T_1) = \lambda(T_2)$ and $\varepsilon(T_1) = -\varepsilon(T_2)$.*

Proof. We put $\lambda_i = \lambda(t_i)$ and $\varepsilon_i = \varepsilon(t_i)$ for $i = 1$, 2. We may assume that $\lambda_1 \leq \lambda_2$. By putting $\lambda = \lambda_1 - 1$ and λ_2 in the equation (1), we obtain

$$t_0(\lambda_1) + t_2(\lambda_1) + 2t_5(\lambda_1) + 2t_{25}(\lambda_1) = 0, \text{ and} \tag{4}$$

$$t_0(\lambda_2) + 2t_2(\lambda_2) + t_5(\lambda_2) + 2t_{25}(\lambda_2) = 0. \tag{5}$$

If $\lambda_1 < \lambda_2$, then it follows by definition that

$$\{t_0(\lambda_1), t_2(\lambda_1), t_5(\lambda_1), t_{25}(\lambda_1)\} = \{\varepsilon_1, 0, 0, 0\},$$

which contradicts to the equation (4). Hence we have $\lambda_1 = \lambda_2$. We put $t_w = t_w(\lambda_i)$ regardless of i ($w = 0$, 2, 5, 25). It is sufficient to consider the following two cases.

- $\underline{\{t_0, t_2, t_5, t_{25}\} = \{\varepsilon_1 + \varepsilon_2, 0, 0, 0\}}$. If $\varepsilon_1 = \varepsilon_2$, then we have $\varepsilon_1 + \varepsilon_2 = \pm 2$, which contradicts to (4). If $\varepsilon_1 = -\varepsilon_2$, then this is the desired solution.
- $\underline{\{t_0, t_2, t_5, t_{25}\} = \{\varepsilon_1, \varepsilon_2, 0, 0\}}$. By (4) and (5), we have $t_2 = t_5$. If $t_2 = t_5 = 0$, then we have $t_0 + 2t_{25} = 0$. This contradicts to $|t_0| = 1$ and $|2t_{25}| = 2$. If $t_2 = t_5 \neq 0$, then we have $t_0 = t_{25} = 0$, and $t_2 = t_5 = \pm 1$. This contradicts to (4) clearly.

Hence we obtain the conclusion. □

For a diagram Δ of a surface-knot K, we denote by $b(\Delta)$ the number of branch points of Δ. The following lemma is proved by a Roseman move [9].

Lemma 4.2 (cf. [2]). *Let Δ be a diagram of a surface-knot K. Assume that Δ has two branch points B_1 and $B_2 \in S$ with $\lambda(B_1) = \lambda(B_2)$ and $\varepsilon(B_1) = -\varepsilon(B_2)$. If there is an embedded arc L in Δ connecting B_1 and B_2 which misses $M_2 \cup M_3 \cup S$ except the boundary, then K has a diagram Δ' with $t(\Delta') = t(\Delta)$ and $b(\Delta') = b(\Delta) - 2$.*

Proof. By assumption, the arc L has a neighborhood as shown in Fig. 6 (i). Let Δ' be a diagram obtained from Δ by replacing the neighborhood with Fig. 6 (ii). Since the deformation from Δ to Δ' is a Roseman move, Δ' is a diagram of K with $t(\Delta') = t(\Delta)$ and $b(\Delta') = b(\Delta) - 2$. □

We remark that, in the assumption of Lemma 4.2, if the branch points do not satisfy the condition $\lambda(B_1) = \lambda(B_2)$, then L has a neighborhood as shown in Fig. 6 (iii).

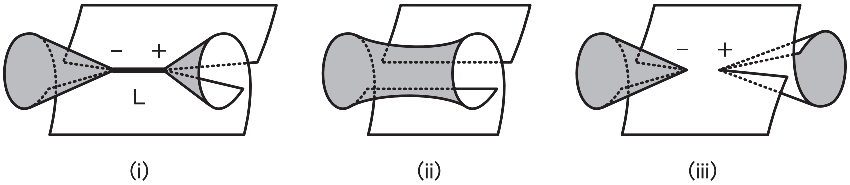

Fig. 6.

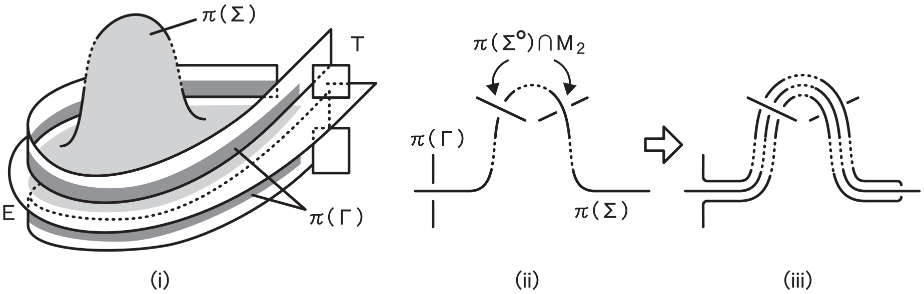

Fig. 7.

In this case, the branch points can not be canceled without introducing a new triple point locally.

Proposition 4.3. *Let Δ be a diagram of a surface-knot K. Assume that Δ has a triple point T with $E_2(T) = E_3(T)$, which is denoted by E simply. If E^{U} bounds a 2-disk Σ in K such that $\pi(\Sigma^\circ) \cap (M_3 \cup S) = \emptyset$, where Σ° is the interior of Σ, then K has a diagram Δ' with $t(\Delta') = t(\Delta) - 1$. Also, if $E_1(T) = E_2(T)$, $E_4(T) = E_5(T)$, or $E_5(T) = E_6(T)$, we have a similar result.*

Proof. We prove the case $E = E_2(T) = E_3(T)$; other cases are similarly proved. Let Γ be a sufficiently thin neighborhood of E^{L} in K. See Fig. 7 (i). First assume that $\pi(\Sigma^\circ) \cap M_2 \neq \emptyset$. Since $\pi(\Sigma^\circ) \cap (M_3 \cup S) = \emptyset$, we can shrink Γ parallel to Σ in $\mathbb{R}^4$ without introducing new triple points, so that we have $\pi(\Sigma^\circ) \cap M_2 = \emptyset$. Fig. 7 (ii) $\to$ (iii) shows this deformation schematically. [We remark that this process produces new double points near a double point on $\pi(\Sigma^\circ)$, but never produce triple points.] Hence, we may assume that $\pi(\Sigma^\circ) \cap M_2 = \emptyset$. Then it is not difficult to see that the triple point T can be eliminated by using the deformation as in Fig. 6 (ii) $\to$ (i). $\square$

The following theorem is due to Shima [15], which is our main motivation of this paper. Note that if a diagram Δ satisfies $b(\Delta) = 0$, then the underlying surface in $\mathbb{R}^3$ (without crossing information) is an immersion. In [14] Shima and the author studied the minimal number of triple points for all possible "immersed" diagrams of

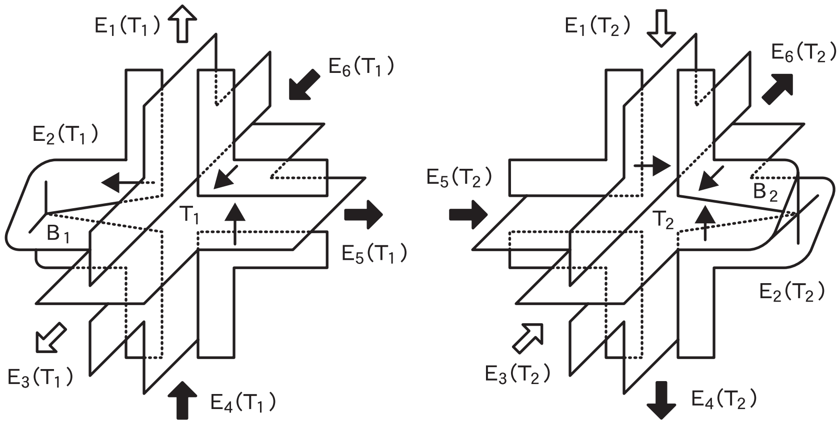

Fig. 8.

a surface-knot.

Theorem 4.4. *If a* 2*-knot* K *has a diagram* Δ *with* $t(\Delta) = 2$ *and* $b(\Delta) = 0$, *then it holds that* $\mathrm{t}(K) = 0$.

We are ready to prove the following.

Theorem 4.5. *There is no* 2*-knot* K *with* $\mathrm{t}(K) = 2$.

Proof. Assume that there is a 2-knot K with $\mathrm{t}(K) = 2$. Let Δ be a minimal diagram of K with the triple points T_1 and T_2. If Δ has a bb-edge, then we replace it with a simple closed curve by canceling the branch points as in Lemma 4.2. Hence, we may assume that Δ has no bb-edges; in other words, any branch point connects to a triple point by an edge.

By Lemma 4.1, there are four cases according to the types of T_1 and T_2. If T_1 and T_2 are of type $\langle 25 \rangle$ both, then we have a contradiction to Proposition 3.2. If both of T_1 and T_2 are of type $\langle 0 \rangle$, then it holds that $b(\Delta) = 0$ by assumption that Δ has no bb-edge. It follows by Theorem 4.4 that $\mathrm{t}(K) = 0$, which contradicts to the assumption that Δ is a minimal diagram. If both of T_1 and T_2 are of type $\langle 5 \rangle$, then this case reduces to that of type $\langle 2 \rangle$ by changing the orientation of Δ.

We consider the case that both of T_1 and T_2 are of type $\langle 2 \rangle$. We may assume that $\varepsilon(T_1) = +1$ and $\varepsilon(T_2) = -1$ by Lemma 4.1, and put $\lambda = \lambda(T_1) = \lambda(T_2)$. Fig. 8 shows the neighborhoods of T_1 and T_2, where we indicate orientations of the edges by white and black big arrows whose Alexander numbers are $\lambda + 1$ and λ, respectively. It holds that $\lambda(B_1) = \lambda(B_2) = \lambda + 1$. Since there is no triple point other than T_1 and T_2, it follows

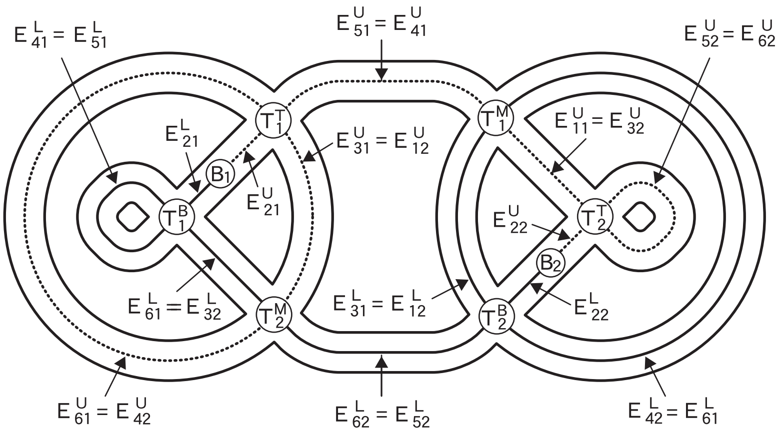

Fig. 9.

that

$$\{E_1(T_1), E_3(T_1)\} = \{E_1(T_2), E_3(T_2)\}, \text{ and} \tag{6}$$

$$\{E_5(T_1), E_4(T_2), E_6(T_2)\} = \{E_4(T_1), E_6(T_1), E_5(T_2)\}. \tag{7}$$

First we consider the case $E_1(T_1) = E_1(T_2)$. There is an embedded curve connecting B_1 and B_2 satisfying the assumption in Lemma 4.2. More precisely, we may take a parallel curve L along the sequence of edges

$$E_2(T_1) = \overrightarrow{B_1T_1}, \quad E_1(T_1) = E_1(T_2) = \overrightarrow{T_1T_2}, \quad \text{and} \quad E_2(T_2) = \overrightarrow{T_2B_2}.$$

By applying Lemma 4.2 to Δ, this case reduces to that of type $\langle 0 \rangle$. The cases $E_k(T_1) = E_k(T_2)$ for $k = 3, 4, 6$ are similarly proved. $\square$

Thus we may assume that $E_k(T_1) \neq E_k(T_2)$ for $i = 1, 3, 4, 6$. It follows by (6) that $E_1(T_1) = E_3(T_2)$ and $E_3(T_1) = E_1(T_2)$. Then there are three cases by (7).

- $E_5(T_1) = E_4(T_1)$, $E_4(T_2) = E_6(T_1)$, and $E_6(T_2) = E_5(T_2)$. We can apply Proposition 4.3 to one of the looped edges $E_5(T_1) = E_4(T_1)$ and $E_6(T_2) = E_5(T_2)$. To see this, it is sufficient to check that the preimage of the neighborhood is connected as shown in Fig. 9, where we write $E_{ki}^{\mathrm{W}} = E_k^{\mathrm{W}}(T_i)$; in fact, since K is a 2-sphere, each of $E_5^{\mathrm{L}}(T_1) = E_4^{\mathrm{L}}(T_1)$ and $E_6^{\mathrm{U}}(T_2) = E_5^{\mathrm{U}}(T_2)$ bounds a 2-disk, which does not contain triple points and branch points. Note Δ has no triple and branch points except $\{T_1, T_2\}$ and $\{B_1, B_2\}$. Hence, this contradicts to the assumption that Δ is a minimal diagram.
- $E_5(T_1) = E_6(T_1)$, $E_4(T_2) = E_5(T_2)$, and $E_6(T_2) = E_4(T_1)$. This case is the mirror image of the previous one. Hence we have a similar contradiction to the assumption that Δ is a minimal diagram.
- $E_5(T_1) = E_5(T_2)$, $E_4(T_2) = E_6(T_1)$, $E_6(T_2) = E_4(T_1)$. We have three double point

curves C_1, C_2, and C_3 consisting of

$$\begin{cases} C_1: E_1(T_1) = E_3(T_2) \text{ and } E_6(T_2) = E_4(T_1), \\ C_2: E_2(T_1),\ \ E_5(T_1) = E_5(T_2),\ \text{ and } E_2(T_2), \\ C_3: E_3(T_1) = E_1(T_2) \text{ and } E_4(T_2) = E_6(T_1). \end{cases}$$

Then there is a pair of circles on K with a single intersection; for example, the circle $C_2^{\mathrm{U}} \cup C_2^{\mathrm{L}}$ intersects C_1^{U}, C_1^{L}, C_3^{U}, and C_3^{L} at T_2^{T}, T_1^{B}, T_1^{T}, and T_2^{B}, respectively. This contradicts to the assumption that K is a 2-knot.
Hence we have the conclusion.

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