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LOSS OF REGULARITY FOR SECOND ORDER HYPERBOLIC EQUATIONS WITH SINGULAR COEFFICIENTS

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Abstract

We consider the Cauchy problem of second order hyperbolic equation with time depending coefficients. Time depending singular coefficient can bring some loss of regularity of the solution; for instance "infinitely many oscillation", "infinite order degeneracy" and "accumulation of zeros" crucially influence on the regularity loss. In this paper we make clear the order of regularity loss from the interaction of the singular effects, and also discuss the optimality.

1. Introduction

We consider the loss of regularity of the solutions to the following Cauchy problem of second order hyperbolic equation:

$$(1)\qquad \begin{cases} (\partial_t^2 - a(t)^2\Delta)u(t,x) = 0, & (t,x) \in [0,T]\times\mathbb{R}^n, \\ u(T,x) = u_0(x),\ \ u_t(T,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

where $\Delta = \sum_{j=1}^n \partial_{x_j}^2$, $a(t) \geq 0$ and T is a small positive number.

Let $p(\xi)$ be a positive function for $\xi \in \mathbb{R}^n$. We introduce the following weighted energy $E(t;p(D))$ for the solution of (1) as follows:

$$E(t;p(D)) := \int_{\mathbb{R}^n} \mathcal{E}(t,\xi;p(\xi))\,d\xi := \int_{\mathbb{R}^n} p(\xi)^2(a(t)^2|\xi|^2|\hat{u}(t,\xi)|^2 + |\hat{u}_t(t,\xi)|^2)\,d\xi,$$

where $\hat{u}(t,\xi)$ denote the partial Fourier transformation with respect to the space variable x.

Let us suppose that $a(t)$ is strictly positive and Lipschitz continuous on $[0,T]$, then one can prove the following energy inequality:

$$(2)\qquad E(t;1) \leq C E(T;p(D))$$

with $p(\xi) \equiv 1$, it follows that (1) is L^2 well-posed, where C is a positive constant; we will denote by C and C_j $(j = 0, 1, \dots)$ some positive constants from below without any confusion.

On the other hand, if $a(t)$ has a singularity, which means non-Lipschitz continuity or having a zero, then L^2 well-posedness dose not hold in general. In the other words, the solution loses some regularity from the influences of singular behavior of $a(t)$. According to Colombini, De Giorgi and Spagnolo [1], if $a(t)$ is strictly positive and Hölder continuous of order $\alpha \in (0,1)$, then one can prove the inequality (2) with $p(\xi) = \exp(\langle\xi\rangle^{1/s})$ for $s < 1/(1-\alpha)$, but not for $s > 1/(1-\alpha)$ in general, where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$. Such a problem for non-strictly positive coefficient $a(t)$ is also considered by Colombini, Jannelli and Spagnolo [4], and the order of $p(\xi)$, which describes the loss of regularity, is higher than strictly positive case with the same Hölder continuity to $a(t)$. In any cases, such singularities of the coefficients causes infinite order regularity loss for the solution. Especially, if $a(t)$ is strictly positive and log-Lipschitz continuous, which means $|a(t_1) - a(t_2)| / |(t_1 - t_2)\ln|t_1 - t_2|| \le C$ for $t_1 \ne t_2$, then one can take $p(\xi) = \langle\xi\rangle^M$ for a positive constant M, it follows that (1) is C^∞ well-posed.

Hölder and log-Lipschitz continuity are appropriate to classify the relations between the singularity of non-Lipschitz continuous coefficient and the order of regularity loss of the solution. But such global regularity conditions for $a(t)$ are not appropriate if one is interested in the loss of regularity which is generated from one point singularity of $a(t)$. Actually, such a precise estimate for the loss of regularity will be required from an application to non-linear problem.

In the present paper we consider the cases that the coefficient $a(t)$ has only one or countably many singular points as t goes to 0.

2. Results

Let $a(t)$ be decomposed into a product of the two non-negative functions $\lambda(t)$ and $b(t)$ by

$$a(t) = \lambda(t)b(t),$$

where $\lambda(t)$, and $b(t)$ describe the *increasing behavior*, and the *oscillating behavior* of $a(t)$ respectively. Here we suppose that

$$\lambda(t) \in C^2([0,T]), \quad b(t) \in C^2((0,T]), \tag{3}$$

$$\lambda'(t) > 0, \quad \frac{|\lambda''(t)|}{\lambda(t)} \le C\left(\frac{\lambda(t)}{\Lambda(t)}\right)^2, \quad \frac{\lambda'(t)}{\lambda(t)} \le \frac{\lambda(t)}{\Lambda(t)} \quad \text{on} \quad (0,T], \tag{4}$$

$$\left|\ln\left(\frac{t\lambda(t)}{\Lambda(t)}\right)\right| \le C(\ln \Lambda(t)^{-1})^{\kappa_0}, \tag{5}$$

and

$$b_0 \le b(t) \le b_1,$$

where κ_0, b_0 and b_1 are non-negative constants and $\Lambda(t) = \int_0^t \lambda(s)\,ds$. Moreover, for a non-negative real number κ_1 we suppose the following conditions to $\lambda(t)$ and $b(t)$:

$$\sup_{t\in(0,T]}\left\{\frac{\Lambda(t)}{\lambda(t)(\ln\Lambda(t)^{-1})^{\kappa_1}}|b'(t)|\right\}+\sup_{t\in(0,T]}\left\{\left(\frac{\Lambda(t)}{\lambda(t)(\ln\Lambda(t)^{-1})^{\kappa_1}}\right)^2|b''(t)|\right\}<\infty. \tag{6}$$

Then our first theorem is described as follows:

Theorem 2.1. *Let* $b_0 > 0$ *and* $\kappa = \max\{\kappa_0, \kappa_1\}$. *Under the assumptions* (3)–(6) *the following estimate holds*:

$$E(t;1) \le C_0 E(T; \exp(C_1(\ln\langle D\rangle)^{\kappa})) \tag{7}$$

on $[0, T]$. *Moreover*, *the estimate* (7) *is optimal*, *that is*, *for any given positive real number* ε *there exists* $\lambda(t)$ *and* $b(t)$ *satisfying the assumptions of the theorem such that for any positive constants* C_0 *and* C_1 *the estimate*

$$E(t;1) \le C_0 E(T; \exp(C_1(\ln\langle D\rangle)^{\kappa-\varepsilon})) \tag{8}$$

does not hold in general.

The following corollary is concluded form the theorem:

Corollary 2.1. *Under the assumptions* (3)–(6) *with* $b_0 > 0$ *we have the followings*:

(i) *If* $\kappa \le 1$, *then* (1) *is* C^∞ *well-posed on* $[0, T]$. *On the other hands, if* $\kappa > 1$, *then* (1) *is not* C^∞ *well-posed in general*.

(ii) *If* $\kappa < 1$ *and* $\lambda(0) > 0$, *then the loss of regularity for the solution of* (1) *at* $t = 0$ *is arbitrarily small*. *Moreover*, *if* $\kappa = 0$, *then* (1) *is* L^2 *well-posed on* $[0, T]$, *that is*, *the solution of* (1) *loses any regularity at* $t = 0$.

Theorem 2.1 gives some optimal regularity estimates in the case that the coefficient $a(t)$ has only one (or no) zero. But we can consider the case that $a(t)$ has countably many number of zeros under some restriction to $a(t)$ around the zeros. Besides the conditions (3)–(6), we suppose the following conditions:

For a small positive real number d there exist sequences of positive real numbers $\{\tau_j^-\}_{j\ge1}$, $\{\tau_j^+\}_{j\ge1}$ and $\{t_j\}_{j\ge1}$ satisfying $\tau_{j+1}^+ < \tau_j^- \le t_j \le \tau_j^+$ for any j such that

$$\{t; b(t) \le d\} = \bigcup_{j\ge1}\left[\tau_j^-, \tau_j^+\right]$$

and

$$C_0 \le \frac{\lambda(t_j)}{\Lambda(t_j)} \frac{\Lambda(t_{j+1})}{\lambda(t_{j+1})} \le C_1. \tag{9}$$

Here we note that $b(\tau_j^{\pm}) = d$. Then we introduce the following conditions to $b(t)$ for $t \in I_j := [\tau_j^-, \tau_j^+]$ $(j = 1, 2, \dots)$:

There exist constants m_0 and d_0 independent of j, and sequences of real numbers $\{d_j\}_{j\ge1}$ and $\{m_j\}_{j\ge1}$ satisfying $1 \le m_j \le m_0$ and $0 \le d_j \le d_0$ such that

$$\frac{|b''(t)|}{b(t)} + \left(\frac{b'(t)}{b(t)}\right)^2 \le C \left(\frac{b(t)}{|B(t,t_j)|}\right)^2 \tag{10}$$

for any $t \in I_j$ and

$$C_0 \left(\left(\frac{\lambda(t_j)}{\Lambda(t_j)} |t - t_j| \right)^{m_j} + d_j \right) \le b(t) \le C_1 \left(\left(\frac{\lambda(t_j)}{\Lambda(t_j)} |t - t_j| \right)^{m_j} + d_j \right) \tag{11}$$

for any $t \in I_j \setminus \{t_j\}$, where $B(t_1, t_2) = \int_{t_1}^{t_2} b(s)\,ds$.

Then we have the following theorem:

Theorem 2.2. *Suppose the conditions* (3)–(6) *and* (9)–(11). *If* $b(T) > 0$, *then the estimate* (7) *holds on* $[0, T]$ *with*

$$\kappa := \max\left\{ \kappa_0, \kappa_1, \sup_j \left\{ \frac{\ln j}{\ln\ln \lambda(t_j)^{-1}} \right\} \right\}. \tag{12}$$

REMARK 2.1. One can replace the definition of κ from (12) to

$$\kappa := \max\left\{ \kappa_0, \kappa_1, \limsup_{j\to\infty} \left\{ \frac{\ln j}{\ln\ln \lambda(t_j)^{-1}} \right\} + \varepsilon \right\}$$

where ε is an arbitrary given positive real number.

Let us briefly introduce some related results to our theorems.

In the strictly hyperbolic case $\lambda(t) \equiv 1$, $a(t) = b(t)$ and $b_0 > 0$, by Colombini, Del Santo and Kinoshita [2] it is proved that the conditions

$$b(t) \in C^1((0, T]) \quad \text{and} \quad \sup_{t\in(0,T]} \{t|b'(t)|\} < \infty \tag{13}$$

are sufficient for the C^∞ well-posedness of (1). For instance, $a(t) = b(t) = 2 + \cos(\ln t^{-1})$ satisfies the condition (13). If $b(t) \in C^2((0, T])$, then more singular behavior for $b'(t)$ at $t = 0$ is allowed for the C^∞ well-posedness than (13). Indeed the condition is given as follows:

$$(14)\quad b(t) \in C^2((0,T]) \quad \text{and} \quad \sup_{t\in(0,T]}\left\{\frac{t}{\ln t^{-1}}|b'(t)|\right\} + \sup_{t\in(0,T]}\left\{\left(\frac{t}{\ln t^{-1}}\right)^2 |b''(t)|\right\} < \infty.$$

Moreover, the condition (14) is optimal (see Colombini, Del Santo and Reissig [3] and Hirosawa [6]). In Hirosawa [7] it is considered not only finite loss of regularity (C^∞ well-posedness) but also small loss of regularity from the point of view for the generalization of the condition (14), and this result corresponds to Theorem 2.1 of the conclusions (7) with $\lambda(t) = 1$. Thus, for the coefficient

$$(15)\qquad a(t) = 2 + \cos((\ln t^{-1})^{\kappa+1})$$

we see that if "$\kappa > 1$", "$\kappa = 1$", "$0 < \kappa < 1$" and "$\kappa = 0$", then the order of regularity loss of the solution to (1) is "infinite", "finite", "arbitrarily small" and "nothing" respectively.

REMARK 2.2. We only see from [2] that the regularity loss is at most finite for the coefficient (15) with $\kappa = 0$. On the other hand, in virtue of $a(t) \in C^2((0,T])$, actually the regularity loss is nothing from the conclusion of [7]. One cannot say that the conclusion of [7] (and also [3], [6]) contains the conclusion of [2], because of the difference of the assumptions to the differentiability of $a(t)$. Incidentally, it is proved in Hirosawa and Reissig [8] that one can weaken the assumption $a(t) \in C^2((0,T])$ of (14) to $a(t) \in C^{1+\varepsilon}((0,T])$ for any $\varepsilon > 0$.

In the weakly hyperbolic case $\lambda(0) = 0$ and $b_0 > 0$, by Tarama [11] it is proved that for $\lambda(t) = e^{-t^{-\alpha}}$ and a positive C^2 periodic function $b(t)$ the condition $\alpha \geq 1/2$ is necessary and sufficient for the C^∞ well-posedness of (1). By Yagdjian [12] the functions $\lambda(t)$ and $b(t)$ are generalized. Their conditions for the C^∞ well-posedness correspond to Theorem 2.1 with $\kappa \geq 1$. Recently, by Reissig [10] the estimate (7) is proved for $0 < \kappa < 1$. Thus actually the new point of Theorem 2.1 is the optimality of the estimate (7) for $\kappa \neq 1$, that is, the estimate (8) does not hold in general even if $\kappa \neq 1$. Indeed, the proof of such an optimality is not a simple analogy of the case $\kappa = 1$.

REMARK 2.3. The estimate (7) with $\kappa \leq 1$ implies that the Cauchy problem (1) is H^∞ (or C^∞) well-posed on $[0,T]$, in the other words, there exist a positive number M and a unique solution $u(t,x)$ such that $u(t,x) \in \bigcap_{j=0}^2 C^{2-j}([0,T];H^j)$ for any $(u_0,u_1) \in H^{M+2} \times H^{M+1}$, where H^s denotes the usual Sobolev space of order s. Then M describes the order of regularity loss. If $a(t)$ is strictly positive, then the estimate (7) with $\kappa < 1$ implies that the order of regularity loss is arbitrarily small. On the other hand, such a small loss of regularity does not follow from the estimate (7) in general, because $E(t;1)$ is a weighted energy. Indeed, if $a(t_0) = 0$ at $t = t_0$, then the estimate (7) shows only the boundedness of $\|u_t(t,\cdot)\|$.

If $a(t)$ has infinitely many number of zeros, we meet other difficulties for the proof of the estimate (7). By Yamazaki [13] and [14] the C^∞ and L^2 well-posedness of (1) for such coefficients are considered under some suitable assumptions to $a(t)$, which corresponds to the assumptions (9)–(10). (For the details will be mentioned below.)

Theorem 2.2 seems to be interpreted as only a unified description of the preceding results. But actually there are gaps between these results. The following examples will be appropriate to understand what is the meaning of our theorems, and also the connections with the preceding results.

EXAMPLE 2.1. Let $\lambda(t) = e^{-t^{-\alpha}}$ and $b(t)$ be defined by

$$b(t) = \chi_1(t)(1 + \left|\sin(\pi t^{-\gamma})\right|^2) + \chi_2(t)\left|\sin(\pi t^{-\alpha})\right|^2 + \chi_3(t),$$

where $\chi_j(t) = 1$ or $\chi_j(t) = 0$, $\chi_1(t) + \chi_2(t) + \chi_3(t) = 1$.
(i) In the case that $\chi_1(t) = 1$ and $\chi_2(t) = \chi_3(t) = 0$, (6) holds for $\kappa = \max\{0, (\gamma/\alpha) - 1\}$.
(ii) In the case that $\chi_2(t) = 1$ and $\chi_1(t) = \chi_3(t) = 0$, (6) holds for $\kappa = \max\{0, 0, 1\} = 1$.
(iii) Let $\chi_j(t)$ be defined by $\chi_j(t) = 1$ for $t \in \mathcal{I}_j$ and $\chi_j(t) = 0$ for $t \notin \mathcal{I}_j$, where

$$\mathcal{I}_1 := [0, T] \setminus (\mathcal{I}_2 \cup \mathcal{I}_3),$$

$$\mathcal{I}_2 := \bigcup_{j=1}^{\infty} \left(\left(t_j^{-\alpha} + \frac{1}{2} \right)^{-1/\alpha}, \left(t_j^{-\alpha} - \frac{1}{2} \right)^{-1/\alpha} \right],$$

$$\mathcal{I}_3 := \bigcup_{j=1}^{\infty} \left(\left((k_j + \delta_j)^{-1/\gamma}, \left(t_j^{-\alpha} + \frac{1}{2} \right)^{-1/\gamma} \right], \left(\left(t_j^{-\alpha} - \frac{1}{2} \right)^{-1/\alpha}, k_j^{-1/\gamma} \right] \right),$$

$$t_j := \left[j^{1/\beta} \right]^{-1/\alpha}, \quad k_j := \left[\left(t_j^{-\alpha} - \frac{1}{2} \right)^{-\gamma/\alpha} \right],$$

$$\delta_j := \begin{cases} \left(t_j^{-\alpha} + \frac{1}{2} \right)^{-\gamma/\alpha} - \left[\left(t_j^{-\alpha} - \frac{1}{2} \right)^{-\gamma/\alpha} \right] & \text{for} \quad \left(t_j^{-\alpha} + \frac{1}{2} \right)^{-\gamma/\alpha} \in \mathbb{Z}, \\ \left[\left(t_j^{-\alpha} + \frac{1}{2} \right)^{-\gamma/\alpha} \right] - \left[\left(t_j^{-\alpha} - \frac{1}{2} \right)^{-\gamma/\alpha} \right] + 1 & \text{for} \quad \left(t_j^{-\alpha} + \frac{1}{2} \right)^{-\gamma/\alpha} \notin \mathbb{Z} \end{cases}$$

with $\alpha > 0$, $1 \ge \beta > 0$, $\gamma \ge 0$ and $[\,\cdot\,]$ denotes Gauss' symbol. Then (6) holds for

$$\kappa = \max\left\{0, \max\left\{0, \frac{\gamma}{\alpha} - 1\right\}, \beta\right\} = \max\left\{\frac{\gamma}{\alpha} - 1, \beta\right\}.$$

The conclusions (i) and (ii)–(iii) are proved form Theorem 2.1, and Theorem 2.2 respectively. The order of regularity loss is determined by the parameters α, β and γ, which describe the *vanishing order*, *distribution of zeros*, and *accumulation of the* *oscillation as* $t \to 0$ respectively. Then the corresponding problems and results which considered in the previous papers are the followings:

- for (i) with $\alpha \geq 1/2$ and $\gamma = 1$, (7) holds for $\kappa = 1$ ([11]);
- for (ii) with $\beta = 1$ and $\alpha = \gamma$, (7) holds for $\kappa = 1$ ([13]);
- for (i) with $\alpha < 1$ and $\gamma = 1$, (7) holds for $\kappa = 1/\alpha - 1$ ([10]);

and nothing more concerning Example 2.1. Thus we could consider only some restricted singular effects of the coefficient which are described by the parameters α, β and γ in the previous results. On the other hand, the assertion of our theorems is that the loss of regularity is brought from the three different singular effects of the coefficient:

(s-α) vanishing order of $\lambda(t)$ as $t = 0$;
(s-β) distribution of zeros, which is described by the order of $\{t_j\}$;
(s-γ) accumulation of the oscillation as $t = 0$, which is described by the order of $|b'(t)|$ and $|b''(t)|$.

Then, these orders are denoted by α, β and γ respectively for Example 2.1 (iii), and the order of regularity loss of the solution is determined by the interactions of these parameters such as (12).

3. Proof of the theorems

3.1. Zones. Let us briefly introduce our strategy of the proof. Our goal is to have a good representation of the solution to conclude the estimate (7). After partial Fourier transformation with respect to x, (1) is rewritten as the following Cauchy problem:

$$\begin{cases} (\partial_t^2 + a(t)^2|\xi|^2)v(t,\xi) = 0, & (t,\xi) \in [0,T] \times \mathbb{R}^n, \\ v(T,\xi) = v_0(\xi), \;\; v_t(T,\xi) = v_1(\xi), & \xi \in \mathbb{R}^n, \end{cases} \tag{16}$$

where $v(t,\xi) = \widehat{u}(t,\xi)$, $v_0(\xi) = \widehat{u}_0(\xi)$ and $v_1(\xi) = \widehat{u}_1(\xi)$. The second order scalar equation of (16) can be rewritten as the following first order system:

$$V_t(t,\xi) = (A(t,\xi) + Q(t,\xi))V(t,\xi).$$

If $A(t,\xi)$ is diagonal and $ReA(t,\xi) = 0$, $\int_0^T |Q(t,\xi)|dt \leq C_0(\ln\langle\xi\rangle)^\kappa$, $|V(t,\xi)|^2 \geq C_1\mathcal{E}(t,\xi;1)$ and $|V(T,\xi)|^2 \leq C_2\mathcal{E}(T,\xi;1)$, then our proof is concluded. Thus the main part of the proof is how to extract such a vector valued function $V(t,\xi)$.

REMARK 3.1. We transform our problem (1) to (16) in the first step. This step performs only the case that the coefficient depends only on t, but this step is not essential. Indeed, the method to use some properties of pseudo-differential operator, which was introduced in [8], hints a possibility to be generalized our problem to the case of x dependent coefficient.

Let M be a positive real number, to be chosen later. For an arbitrary given large number R we define τ_R, the sequences of positive real numbers $\{t_j^-\}$ and $\{t_j^+\}$ satisfying $t_j^- \le t_j \le t_j^+$ implicitly by

$$R\Lambda(\tau_R) = M(\ln \Lambda(\tau_R)^{-1})^\kappa \tag{17}$$

and

$$M = R\lambda(t_j^-)B(t_j^-, t_j) = R\lambda(t_j^+)B(t_j, t_j^+). \tag{18}$$

Moreover, we denote by $N = N(R)$ the positive integer satisfying $t_N \le \tau_R \le t_{N-1}$. Then we define the sets of intervals $Z_{\Psi,0} = Z_{\Psi,0}(R, M)$, $Z_{H,0} = Z_{H,0}(R, M)$, $Z_{\Psi,1} = Z_{\Psi,1}(R, M)$ and $Z_{H,1} = Z_{H,1}(R, M)$ by

$$Z_{\Psi,0} := \{t \in [0, T]; t < \tau_R\},$$

$$Z_{H,0} := \left\{ t \in [0, T]; [\tau_R, T] \cap \bigcup_{j\ge1} [\tau_{j+1}^+, \tau_j^-) \right\},$$

$$Z_{\Psi,1} := \left\{ t \in [0, T]; [\tau_R, T] \cap \bigcup_{j\ge1} [t_j^-, t_j^+) \right\}$$

and

$$Z_{H,1} := \left\{ t \in [0, T]; [\tau_R, T] \cap \bigcup_{j\ge1} ([\tau_j^-, t_j^-) \cup [t_j^+, \tau_j^+)) \right\}.$$

In particular, if $t_j^- < \tau_j^-$, and $t_j^+ > \tau_j^+$, then we regard that $[\tau_j^-, t_j^-)$, and $[t_j^+, \tau_j^+)$ are empty respectively. We shall call the sets $Z_{\Psi,0}$ and $Z_{\Psi,1}$ the pseudo-differential zones, and the sets $Z_{H,0}$ and $Z_{H,1}$ the hyperbolic zones respectively.

Let us define $\widetilde{a}(t) = \widetilde{a}(t; R)$ in the respective zones as follows:

$$\widetilde{a}(t; R) := \begin{cases} \tau_R^{-1}\Lambda(\tau_R) + b_1\lambda(t)\Lambda(t)^{1/2}\Lambda(\tau_R)^{-1/2} & \text{for} \quad t \in Z_{\Psi,0}, \\ a(t_j^-) & \text{for} \quad t \in [t_j^-, t_j), \\ a(t_j^+) & \text{for} \quad t \in [t_j, t_j^+), \\ a(t) & \text{for} \quad t \in Z_{H,0} \cup Z_{H,1}. \end{cases}$$

Here we see from the definition of $\widetilde{a}(t)$ that

$$a(t) \le \begin{cases} \widetilde{a}(\tau_R)\exp(C(\ln \Lambda(\tau_R)^{-1})^\kappa) & \text{for} \quad t \in Z_{\Psi,0}, \\ C\widetilde{a}(t) & \text{for} \quad t \in Z_{\Psi,1}. \end{cases} \tag{19}$$

Indeed, the estimate of (19) in $Z_{\Psi,0}$ is straightforward from (5). We introduce the following lemma in order to show the estimate in $Z_{\Psi,1}$:

Lemma 3.1. *The following estimate holds for any j:*

$$\max_{s_0,s_1\in I_j}\left\{\frac{\lambda(s_1)}{\lambda(s_0)}\right\} < C.$$

Proof. Note that $\lambda(t)/\Lambda(t)$ is monotone decreasing by (4). By (4), (9), (11) and mean value theorem for $\tau_j^- \le s_0 < s_1 \le \tau_j^+$ with $j \ge 2$ there exists $s_2 \in (s_0, s_1)$ such that

$$\begin{aligned}\ln\frac{\lambda(s_1)}{\lambda(s_0)} &= \frac{\lambda'(s_2)}{\lambda(s_2)}(s_1 - s_0) \le C\frac{\lambda(t_{j-1})}{\Lambda(t_{j-1})}(\tau_j^+ - \tau_j^-)\\ &\le C\left(\frac{\lambda(t_j)}{\Lambda(t_j)}(\tau_j^+ - t_j) + \frac{\lambda(t_j)}{\Lambda(t_j)}(t_j - \tau_j^-)\right) \le Cd^{1/m_0}.\end{aligned}$$

□

Thus by (11) and Lemma 3.1 we have

$$a(t) \le C\lambda(t_j)\left(\left(\frac{\lambda(t_j)}{\Lambda(t_j)}(t - t_j)\right)^{m_j} + d_j\right) \le C\lambda(t_j^-)b(t_j^-)$$

for $t \in [t_j^-, t_j)$. If $t \in [t_j, t_j^+)$, then (19) is proved by the same way.

3.2. First step of diagonalization procedure. Let us carry out some diagonalization procedure to have a representation of the solution. Let us fix $\xi_0 \in \mathbb{R}^n$ satisfying $|\xi_0| = R$. For the solution $v(t, \xi_0)$ of (16) we define the vector valued function $V_0(t) = V_0(t; R)$ by

$$V_0(t; R) := \begin{pmatrix} R\widetilde{a}(t; R)v(t, \xi_0) \\ -iv_t(t, \xi_0) \end{pmatrix},$$

where $i = -\sqrt{-1}$. Then $V_0(t)$ is a solution to the following first order system:

$$(\partial_t - A_0(t) - B_0(t))V_0(t) = 0, \tag{20}$$

where

$$A_0(t) = A_0(t; R) := iR\widetilde{a}(t; R)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$B_0(t) = B_0(t; R) := \begin{pmatrix} \dfrac{\widetilde{a}'(t; R)}{\widetilde{a}(t; R)} & 0 \\ \dfrac{iR(a(t)^2 - \widetilde{a}(t; R)^2)}{\widetilde{a}(t; R)} & 0 \end{pmatrix}.$$

In the first step of diagonalization procedure which will be done below, we transform the equation (20) taking the hyperbolicity into account. We define the matrix M_1 by

$$M_1 := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then (20) is rewritten as follows:

$$(\partial_t - A_1(t) - B_{11}(t) - B_{12}(t))M_1^{-1}V_0(t) = 0, \tag{21}$$

where

$$A_1(t) = A_1(t; R) := iR\widetilde{a}(t; R)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$B_{11}(t) = B_{11}(t; R) := \frac{\widetilde{a}'(t; R)}{2\widetilde{a}(t; R)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and

$$B_{12}(t) = B_{12}(t; R) := \frac{iR(a(t)^2 - \widetilde{a}(t; R)^2)}{2\widetilde{a}(t; R)}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

respectively. Now (21) is a sufficiently good formula for the estimate in the pseudo-differential zones $Z_{\Psi,0}$ and $Z_{\Psi,1}$.

3.3. Estimate in $Z_{\Psi,0}$. We define the 2×2 matrix valued functions $\Theta_1(t, \tau)$ by

$$\Theta_1(t, \tau) = \Theta_1(t, \tau; R) := \begin{pmatrix} \exp\left(-iR\int_t^\tau \widetilde{a}(s; R)\,ds\right) & 0 \\ 0 & \exp\left(iR\int_t^\tau \widetilde{a}(s; R)\,ds\right) \end{pmatrix}.$$

Noting $\Theta_1(t, \tau)^{-1} = \Theta_1(\tau, t)$ and $\Theta_1(\tau, t)(\partial_t - A_1(t))\Theta_1(t, \tau) = \partial_t$ the equation (21) is rewritten as follows:

$$(\partial_t - \tilde{B}_1(t, \tau))W_1(t, \tau) = 0, \tag{22}$$

where

$$\tilde{B}_1(t, \tau) = \tilde{B}_1(t, \tau; R) := \Theta_1(\tau, t; R)(B_{11}(t; R) + B_{12}(t; R))\Theta_1(t, \tau; R)$$

and

$$W_1(t, \tau) = W_1(t, \tau; R) := \Theta_1(\tau, t; R)M_1^{-1}V_0(t; R).$$

Then the formal solution of (22) is represented by

$$W_1(t,\tau) = W_1(\tau,\tau)\sum_{k=0}^{\infty}\int_{\tau}^{s_0}\tilde{B}_1(s_1,\tau)\cdots\int_{\tau}^{s_k}\tilde{B}_1(s_{k+1},\tau)\,ds_{k+1}\cdots ds_1, \tag{23}$$

where $s_0 = t$. it follows that

$$\begin{aligned}|W_1(t,\tau)| &\le |W_1(\tau,\tau)|\sum_{k=0}^{\infty}\frac{1}{k!}\left(\int_t^{\tau}\left\|\tilde{B}_1(s,\tau)\right\|ds\right)^k\\ &\le |W_1(\tau,\tau)|\exp\left(\int_t^{\tau}(\|B_{11}(s)\| + \|B_{12}(s)\|)\,ds\right),\end{aligned}$$

where $\|A\|$ denotes the matrix norm of 2×2 matrix, that is, for $A = \{a_{jk}\}_{j,k=1}^2$ it is defined by $\|A\| := \max_{j,k}\{|a_{jk}|\}$. Consequently, we have

$$\begin{aligned}|W_1(t,\tau_R)| &\le |W_1(\tau_R,\tau_R)|\exp\left(\int_0^{\tau_R}\frac{|\widetilde{a}'(s)|}{2\widetilde{a}(s)}\,ds + \int_0^{\tau_R}\frac{R\left|a(s)^2-\widetilde{a}(s)^2\right|}{2\widetilde{a}(s)}\,ds\right)\\ &\le |W_1(\tau_R,\tau_R)|\exp\left(\int_0^{\tau_R}\frac{\widetilde{a}'(s)}{2\widetilde{a}(s)}\,ds + \frac{R}{2}\int_0^{\tau_R}\widetilde{a}(s)\,ds\right)\\ &= |W_1(\tau_R,\tau_R)|\exp\left(\frac{1}{2}\ln\left(1+\frac{b_1\tau_R\lambda(\tau_R)}{\Lambda(\tau_R)}\right)+\frac{R\Lambda(\tau_R)}{2}+b_1R\Lambda(\tau_R)\right)\\ &\le |W_1(\tau_R,\tau_R)|\exp(C(\ln\Lambda(\tau_R)^{-1})^{\kappa})\end{aligned}$$

by (5) and (17), it follows that

$$|W_1(t,\tau_R)| \le |W_1(\tau_R,\tau_R)|\exp(C(\ln R)^{\kappa}) \tag{24}$$

in $Z_{\Psi,0}$.

3.4. Estimate in $\mathbf{Z_{\Psi,1}}$. We note that $B_{11}(r)\equiv 0$ in $Z_{\Psi,1}$. By (18) we have

$$\begin{aligned}\int_{t_j^-}^{t_j}\|B_{12}(s)\|\,ds &\le \frac{R}{2}\int_{t_j^-}^{t_j}\frac{\left|a(s)^2-\widetilde{a}(s)^2\right|}{\widetilde{a}(s)}\,ds\\ &\le CRa(t_j^-)(t_j-t_j^-) \le CR\lambda(t_j^-)B(t_j^-,t_j) = CM,\end{aligned}$$

where we note that (11) implies the following estimates in $t\in I_j\setminus\{t_j\}$:

$$C_0|t-t_j|^{-1} \le \frac{b(t)}{|B(t,t_j)|} \le C_1|t-t_j|^{-1} \tag{25}$$

and

$$\frac{\lambda(t)}{\Lambda(t)} \le C\frac{b(t)}{|B(t,t_j)|}. \tag{26}$$

Analogously, we have

$$\int_{t_j}^{t_j^+} \|B_{12}(s)\|\, ds \le CM.$$

Consequently, there exists a positive constant ρ independent of j such that

$$|W_1(t,\tau)| \le \sqrt{\rho}\,|W_1(\tau,\tau)| \quad \text{for} \quad t_j^- \le t < \tau < t_j, \tag{27}$$

and

$$\left|W_1(t,t_j^+)\right| \le \sqrt{\rho}\left|W_1(t_j^+,t_j^+)\right| \quad \text{for} \quad t_j \le t < t_j^+. \tag{28}$$

3.5. Second step of diagonalization procedure. The representation (23), which is obtained after the first step of diagonalization procedure, performs well for the estimates in the pseudo-differential zones. However, such a representation is insufficient for the estimate in the hyperbolic zones. In the next step we transform the equation (21) in the hyperbolic zones $Z_{H,0} \cup Z_{H,1}$ taking account of the assumption $a(t) \in C^2((0,t])$; we shall call this step the second step of diagonalization procedure.

Let us define $M_2(t) = M_2(t;R)$ by

$$M_2(t;R) := \begin{pmatrix} 1 & -p(t;R) \\ p(t;R) & 1 \end{pmatrix},$$

where

$$p(t) = p(t;R) := \frac{-ia'(t)}{4Ra(t)^2}.$$

Now we suppose that $M_2(t)$ is invertible; which will be confirmed later. Noting $\widetilde{a}(t) = a(t)$, and the identities:

$$M_2(t)^{-1}\partial_t M_2(t) = \partial_t + \frac{1}{1+p(t)^2}\begin{pmatrix} p(t)p'(t) & -p'(t) \\ p'(t) & p(t)p'(t) \end{pmatrix},$$
$$M_2(t)^{-1}A_1(t)M_2(t) = \frac{iRa(t)}{1+p(t)^2}\begin{pmatrix} 1-p(t)^2 & -2p(t) \\ -2p(t) & -1+p(t)^2 \end{pmatrix}$$

and

$$M_2(t)^{-1}B_{11}(t)M_2(t) = \frac{a'(t)}{2a(t)(1+p(t)^2)}\begin{pmatrix} 1+2p(t)+p(t)^2 & 1-p(t)^2 \\ 1-p(t)^2 & 1-2p(t)+p(t)^2 \end{pmatrix},$$

we have

$$M_2^{-1}(t)(\partial_t - A_1(t) - B_{11}(t))M_2(t) = \partial_t - A_2(t) - B_2(t),$$

where

$$A_2(t) = A_2(t;R) := \begin{pmatrix} iRa(t) + \dfrac{a'(t)}{2a(t)} & 0 \\ 0 & -iRa(t) + \dfrac{a'(t)}{2a(t)} \end{pmatrix},$$

$$B_2(t) = B_2(t;R) := \begin{pmatrix} b_{1+}(t) & b_{2+}(t) \\ b_{2-}(t) & b_{1-}(t) \end{pmatrix},$$

$$b_{1\pm}(t) = b_{1\pm}(t;R) = \frac{R^2a(t)^4}{16R^2a(t)^4 - a'(t)^2}\left(\mp\frac{2ia'(t)^2}{Ra(t)^3} + \frac{a'(t)a''(t)}{R^2a(t)^4} - \frac{2a'(t)^3}{R^2a(t)^5}\right)$$

and

$$b_{2\pm}(t) = b_{2\pm}(t;R) = \frac{R^2a(t)^4}{16R^2a(t)^4 - a'(t)^2}\left(\mp\left(\frac{4ia''(t)}{Ra(t)^2}\right) \pm \left(\frac{8ia'(t)^2}{Ra(t)^3}\right) + \frac{a'(t)^3}{2R^2a(t)^5}\right).$$

Thus (20) is rewritten as follows:

$$(\partial_t - A_2(t) - B_2(t))M_2^{-1}(t)M_1^{-1}V_0(t) = 0. \tag{29}$$

3.6. Estimate in $Z_{H,0}$. By (4), (6), (17) and noting $b(t) \geq d$ in $Z_{H,0}$ we have the followings:

$$\begin{aligned} |p(t;R)| &= \frac{|a'(t)|}{4Ra(t)^2} \leq \frac{\lambda'(t)}{4dR\lambda(t)^2} + \frac{|b'(t)|}{4d^2R\lambda(t)} \\ &\leq \frac{C}{R\Lambda(t)}(\ln\Lambda(t)^{-1})^{\kappa} \leq \frac{C}{R\Lambda(\tau_R)}(\ln\Lambda(\tau_R)^{-1})^{\kappa} = CM^{-1}. \end{aligned}$$

Therefore, $M_2(t)$ is uniformly invertible in $Z_{H,0}$ by choosing the constant M sufficiently large.

We define the 2×2 matrix valued functions $\Theta_2(t,\tau)$ by

$$\Theta_2(t,\tau) = \Theta_2(t,\tau;R) := \sqrt{\frac{a(t)}{a(\tau)}}\Theta_1(t,\tau;R).$$

Noting $\Theta_2(t,\tau)^{-1} = \Theta_2(\tau,t)$ and $\Theta_2(\tau,t)(\partial_t - A_2(t))\Theta_2(t,\tau) = \partial_t$, (29) is rewritten as follows:

$$(\partial_t - \tilde{B}_2(t,\tau))W_2(t,\tau) = 0, \tag{30}$$

where

$$\tilde{B}_2(t,\tau) = \tilde{B}_2(t,\tau;R) := \Theta_2(\tau,t;R)B_2(t;R)\Theta_2(t,\tau;R)$$

and

$$W_2(t,\tau) = W_2(t,\tau;R) := \Theta_2(\tau,t;R)M_2^{-1}(t;R)M_1^{-1}V_0(t;R).$$

Then the solution of (30) is represented by

$$W_2(t,\tau) = W_2(\tau,\tau)\sum_{k=0}^{\infty}\int_{\tau}^{s_0}\tilde{B}_2(s_1,\tau)\cdots\int_{\tau}^{s_k}\tilde{B}_2(s_{k+1},\tau)\,ds_{k+1}\cdots ds_1,$$

it follows that

$$|W_2(t,\tau)| \le |W_2(\tau,\tau)|\exp\left(\int_t^{\tau}\|B_2(s)\|\,ds\right).$$

By (4) and (6) we see that

$$|a'(t)| \le C\frac{\lambda(t)^2}{\Lambda(t)}(\ln\Lambda(t)^{-1})^{\kappa}$$

and

$$|a''(t)| \le C\frac{\lambda(t)^3}{\Lambda(t)^2}(\ln\Lambda(t)^{-1})^{2\kappa}.$$

Therefore, we have

$$\begin{aligned}|b_{1\pm}(t)| + |b_{2\pm}(t)| &\le C\left(\frac{\lambda(t)}{R\Lambda(t)^2}(\ln\Lambda(t)^{-1})^{2\kappa} + \frac{\lambda(t)}{R^2\Lambda(t)^3}(\ln\Lambda(t)^{-1})^{3\kappa}\right)\\ &\le C\frac{\lambda(t)}{R\Lambda(t)^2}(\ln\Lambda(t)^{-1})^{2\kappa},\end{aligned}$$

it follows that

$$\begin{aligned}\int_t^{\tau}\|B_2(s)\|\,ds &\le C\int_t^{\tau}\frac{\lambda(s)}{R\Lambda(s)^2}(\ln\Lambda(s)^{-1})^{2\kappa}ds\\ &\le C\left(\frac{(\ln\Lambda(t)^{-1})^{2\kappa}}{R\Lambda(t)} - \frac{(\ln\Lambda(\tau)^{-1})^{2\kappa}}{R\Lambda(\tau)} - \frac{2\kappa}{R}\int_t^{\tau}\frac{\lambda(s)}{\Lambda(s)^2}(\ln\Lambda(s)^{-1})^{2\kappa-1}\,ds\right)\\ &\le C\left(\frac{(\ln\Lambda(t)^{-1})^{2\kappa}}{R\Lambda(t)} - \frac{(\ln\Lambda(\tau)^{-1})^{2\kappa}}{R\Lambda(\tau)}\right).\end{aligned}$$

Thus, we obtain

$$|W_2(t,\tau)| \le |W_2(\tau,\tau)|\exp\left(C\left(\frac{(\ln\Lambda(t)^{-1})^{2\kappa}}{R\Lambda(t)} - \frac{(\ln\Lambda(\tau)^{-1})^{2\kappa}}{R\Lambda(\tau)}\right)\right) \tag{31}$$

for any $\tau_j^+ \le t < \tau \le \tau_{j-1}^-$.

3.7. Estimate in $Z_{H,1}$. We restrict ourselves that $t \in \left[\tau_j^-, t_j^-\right]$; otherwise, we have the same estimates below in the analogy of the present case.

By (10), (26) and noting the definition of $Z_{H,1}$ we have

$$\begin{aligned}|p(t;R)| &= \frac{|a'(t)|}{4Ra(t)^2} \le \frac{|\lambda'(t)b(t)| + |\lambda(t)b'(t)|}{4Rb(t)^2\lambda(t)^2} \\ &\le C\left(\frac{1}{Rb(t)\Lambda(t)} + \frac{1}{RB(t,t_j)\lambda(t)}\right) \le \frac{C}{RB(t,t_j)\lambda(t)} \le \frac{C}{RB\left(t_j^-,t_j\right)\lambda\left(t_j^-\right)} \\ &= CM^{-1}.\end{aligned}$$

Thus $M_2(t)$ is uniformly invertible on $\left[\tau_j^-, t_j^-\right]$ with respect to j for large M.

By (4), (10) and (26) we have

$$|a'(t)| \le Ca(t)\frac{b(t)}{B(t,t_j)} \quad \text{and} \quad |a''(t)| \le Ca(t)\left(\frac{b(t)}{B(t,t_j)}\right)^2.$$

Hence we obtain

$$\begin{aligned}|b_{1\pm}(t)| + |b_{2\pm}(t)| &\le C\left(\frac{b(t)}{R\lambda(t)B(t,t_j)^2} + \frac{b(t)}{R^2\lambda(t)^2B(t,t_j)^3}\right) \\ &\le \frac{Cb(t)}{R\lambda(t)B(t,t_j)^2}\left(1 + \frac{1}{R\lambda(t)B(t,t_j)}\right) \le \frac{Cb(t)}{R\lambda(t)B(t,t_j)^2}.\end{aligned}$$

Noting (4) we see that

$$\begin{aligned}\frac{d}{dt}(\lambda(t)B(t,t_j)^{1/2}) &= \lambda(t)B(t_j,t)^{1/2}\left(\frac{\lambda'(t)}{\lambda(t)} - \frac{b(t)}{2B(t,t_j)}\right) \\ &\le \frac{\lambda(t)b(t)}{2B(t_j,t)^{1/2}}(Cd^{1/m_j} - 1) \le 0\end{aligned}$$

for sufficiently small $d > 0$, it follows that $\lambda(t)B(t_j,t)^{1/2}$ is monotone decreasing on $\left[\tau_j^-, t_j^-\right]$. Thus we obtain

$$\begin{aligned}\int_t^\tau \frac{b(s)}{R\lambda(s)B(s,t_j)^2}\,ds &\le \int_{\tau_j^-}^{t_j^-} \frac{b(s)}{R\lambda(s)B(s,t_j)^2}\,ds \le \frac{1}{R\lambda\left(t_j^-\right)B\left(t_j^-,t_j\right)^{1/2}}\int_{\tau_j^-}^{t_j^-}\frac{b(s)}{B(s,t_j)^{3/2}}\,ds \\ &\le \frac{2}{R\lambda\left(t_j^-\right)B\left(t_j^-,t_j\right)} = \frac{2}{M}\end{aligned}$$

for any $\tau_j^- \le t < \tau \le t_j^-$. Consequently, there exists a positive constant ρ such that

$$|W_2(t,\tau)| \le \sqrt{\rho}\,|W_2(\tau,\tau)|. \tag{32}$$

We easily see that the estimate (32) also holds for any $t_j^+ \le t < \tau \le \tau_j^+$.

3.8. Estimate in the whole zones. Let us introduce the following lemmas:

Lemma 3.2. *There exists a positive constant $\mu \geq 1$ independent of j such that*

$$\frac{a(t_j^{\pm})}{a(t_j^{\mp})} \leq \mu.$$

Proof. By (18) and (25) we have

$$\frac{a\bigl(t_j^{\pm}\bigr)}{a\bigl(t_j^{\mp}\bigr)} = \frac{b\bigl(t_j^{\pm}\bigr)}{\bigl|B\bigl(t_j^{\pm}, t_j\bigr)\bigr|} \frac{\bigl|B\bigl(t_j, t_j^{\mp}\bigr)\bigr|}{b\bigl(t_j^{\mp}\bigr)} \leq C \frac{\bigl|t_j - t_j^{\mp}\bigr|}{\bigl|t_j^{\pm} - t_j\bigr|}.$$

Therefore, the lemma is proved if

$$t_j - t_j^- \leq C\bigl(t_j^+ - t_j\bigr) \tag{33}$$

and

$$t_j^+ - t_j \leq C\bigl(t_j - t_j^-\bigr). \tag{34}$$

We only prove (33); otherwise the proof is easier. By (18) and Lemma 3.1 we have

$$B\bigl(t_j^-, t_j\bigr) \leq CB\bigl(t_j, t_j^+\bigr),$$

it follows from (11) that

$$\frac{\bigl(t_j - t_j^-\bigr)\Bigl(\bigl((\lambda(t_j)/\Lambda(t_j))\bigl(t_j - t_j^-\bigr)\bigr)^{m_j} + d_j\Bigr)}{\bigl(t_j^+ - t_j\bigr)\Bigl(\bigl((\lambda(t_j)/\Lambda(t_j))\bigl(t_j^+ - t_j\bigr)\bigr)^{m_j} + d_j\Bigr)} \leq C. \tag{35}$$

Let us consider the case that $d_j > 0$ and $t_j - t_j^- \geq t_j^+ - t_j$; otherwise the estimate (33) is trivial. Denoting for $L_j \geq 1$ and $l_j > 0$ that

$$\left(\frac{\lambda(t_j)}{\Lambda(t_j)}(t_j^+ - t_j)\right)^{m_j} = l_j d_j \quad \text{and} \quad \left(\frac{\lambda(t_j)}{\Lambda(t_j)}(t_j - t_j^-)\right)^{m_j} = L_j l_j d_j,$$

by (35) we have

$$C \geq L_j^{1/m_j} \frac{L_j l_j + 1}{l_j + 1} \geq L_j^{1/m_0},$$

it follows that L_j is bounded. Therefore, we obtain (33). □

Lemma 3.3. *The following equality and inequalities are established for large M:*

$$|V_0(t; R)| = \sqrt{2}\,|W_1(t, \tau; R)| \tag{36}$$

and

$$\frac{1}{\sqrt{2}}\sqrt{\frac{a(\tau)}{a(t)}}\,|V_0(t;R)| \le |W_2(t,\tau;R)| \le \sqrt{\frac{a(\tau)}{a(t)}}\,|V_0(t;R)|\,. \tag{37}$$

Proof. (36) is straightforward from the definition of $V_0(t;R)$ and $W_1(t,\tau;R)$. We note the equalities:

$$\begin{aligned}
|W_2(t,\tau)| &= \sqrt{\frac{a(\tau)}{a(t)}}\left|\Theta_1(\tau,t)M_2^{-1}(t)M_1^{-1}V_0(t)\right| \\
&= \frac{1}{2(1+p(t)^2)}\sqrt{\frac{a(\tau)}{a(t)}}\left|\Theta_1(\tau,t)\begin{pmatrix}1 & p\\ -p & 1\end{pmatrix}\begin{pmatrix}\widetilde{a}(t)Rv(t,\xi_0)-iv_t(t,\xi_0)\\ \widetilde{a}(t)Rv(t,\xi_0)+iv_t(t,\xi_0)\end{pmatrix}\right| \\
&= \frac{1}{2(1+p(t)^2)}\sqrt{\frac{a(\tau)}{a(t)}}\left|\begin{pmatrix}(1+p)\widetilde{a}(t)Rv(t,\xi_0)-i(1-p)v_t(t,\xi_0)\\ (1-p)\widetilde{a}(t)Rv(t,\xi_0)+i(1+p)v_t(t,\xi_0)\end{pmatrix}\right|.
\end{aligned}$$

Recalling the estimate of $p(t)$ in Section 3.6, $|p(t)|$ can be taken arbitrarily small with large M. Thus (37) is proved. □

By (27), (28), (31), (32), Lemma 3.2 and Lemma 3.3 we have

$$\begin{aligned}
\left|V_0\left(t_j^-;R\right)\right| &= \sqrt{2}\left|W_1\left(t_j^-,t_j;R\right)\right| \le \sqrt{2\rho}\lim_{\tau\to t_j-0}|W_1(\tau,\tau)| \\
&= \sqrt{\rho}\left(R^2a\left(t_j^-\right)^2|v(t_j,\xi_0)|^2+|v_t(t_j,\xi_0)|\right)^{1/2} \le \sqrt{2\rho}\mu\left|W_1\left(t_j,t_j^+;R\right)\right| \\
&\le \rho\mu\left|V_0\left(t_j^+;R\right)\right|
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
&\left|V_0\left(t_{j+1}^+;R\right)\right| \\
&\le \sqrt{2}\rho\sqrt{\frac{a\left(t_{j+1}^+\right)}{a\left(t_j^-\right)}}\exp\left(C\left(\frac{\left(\ln\Lambda\left(t_{j+1}^+\right)^{-1}\right)^{2\kappa}}{R\Lambda\left(t_{j+1}^+\right)}-\frac{\left(\ln\Lambda\left(t_j^-\right)^{-1}\right)^{2\kappa}}{R\Lambda\left(t_j^-\right)}\right)\right)\left|V_0\left(t_j^-;R\right)\right|.
\end{aligned} \tag{39}$$

Suppose that $t\in\left[t_{j+1},t_{j+1}^+\right]$ and $t\notin Z_{\Psi,0}$, that is, $j\le N$. Then by (27), (38), (39) Lemma 3.2 and Lemma 3.3 we have the following estimates:

$$\begin{aligned}
|V_0(t;R)| &= \sqrt{2}\left|W_1\left(t,t_{j+1}^+\right)\right| \le \sqrt{2}\rho\left|W_1\left(t_{j+1}^+,t_{j+1}^+\right)\right| = \rho\left|V_0\left(t_{j+1}^+;R\right)\right| \\
&\le \sqrt{2}\rho^2\sqrt{\frac{a\left(t_{j+1}^+\right)}{a\left(t_j^-\right)}}\exp\left(C\left(\frac{\left(\ln\Lambda\left(t_{j+1}^+\right)^{-1}\right)^{2\kappa}}{R\Lambda\left(t_{j+1}^+\right)}-\frac{\left(\ln\Lambda\left(t_j^-\right)^{-1}\right)^{2\kappa}}{R\Lambda\left(t_j^-\right)}\right)\right)\left|V_0\left(t_j^-;R\right)\right| \\
&\le \sqrt{2}\rho^3\mu\sqrt{\frac{a\left(t_{j+1}^+\right)}{a\left(t_j^-\right)}}\exp\left(C\left(\frac{\left(\ln\Lambda\left(t_{j+1}^+\right)^{-1}\right)^{2\kappa}}{R\Lambda\left(t_{j+1}^+\right)}-\frac{\left(\ln\Lambda\left(t_j^-\right)^{-1}\right)^{2\kappa}}{R\Lambda\left(t_j^-\right)}\right)\right)\left|V_0\left(t_j^+;R\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&\vdots\\
&\le \rho\sqrt{\rho}\left(\sqrt{2}\rho^2\mu\right)^j \sqrt{\frac{a\left(t_{j+1}^+\right)a\left(t_j^+\right)\cdots a\left(t_2^+\right)a\left(t_1^+\right)}{a\left(t_j^-\right)a\left(t_{j-1}^-\right)\cdots a\left(t_1^-\right)a\left(\tau_1^+\right)}}\\
&\quad\times \exp\left(C\sum_{k=1}^{j}\left(\frac{\left(\ln\Lambda\left(t_{k+1}^+\right)^{-1}\right)^{2\kappa}}{R\Lambda\left(t_{k+1}^+\right)} - \frac{\left(\ln\Lambda\left(t_k^-\right)^{-1}\right)^{2\kappa}}{R\Lambda\left(t_k^-\right)}\right)\right)|V_0(\tau_1^+;R)|\\
&\le C\left(\sqrt{2}\rho^2\mu\sqrt{\mu}\right)^j\sqrt{\frac{b_1\lambda\left(t_{j+1}^+\right)}{d\lambda\left(\tau_1^+\right)}}\exp\left(\frac{C\left(\ln\Lambda\left(t_{j+1}^+\right)^{-1}\right)^{2\kappa}}{R\Lambda\left(t_{j+1}^+\right)}\right)|V_0(T;R)|\\
&\le C\left(\sqrt{2}\rho^2\mu\sqrt{\mu}\right)^N\exp\left(\frac{C(\ln\Lambda(\tau_R)^{-1})^{2\kappa}}{R\Lambda(\tau_R)}\right)|V_0(T;R)|\\
&\le \exp(C(N+(\ln R)^\kappa))|V_0(T;R)|,
\end{aligned}
$$

where we note that $T \in Z_{H,0}$ for small d. Analogously, we have the same estimate if $t \in \left[t_j^+, t_{j-1}\right]$ and $t \notin Z_{\Psi,0}$. Thus we obtain

$$|V_0(t;R)| \le \exp(C(N + (\ln R)^\kappa))|V_0(T;R)|$$

for any $t \notin Z_{\Psi,0}$. On the other hand, by (24) we have

$$|V_0(t;R)| = \sqrt{2}\,|W_1(t,\tau_R)| \le C\exp(C(\ln R)^\kappa)|W_1(\tau_R,\tau_R)| \le C\exp(C(\ln R)^\kappa)|V_0(\tau_R;R)|$$

for $t \in Z_{\Psi,0}$. Noting the inequalities

$$
\begin{aligned}
N &\le \exp\left((\ln\ln\lambda(t_N)^{-1})\sup_j\left\{\frac{\ln j}{\ln\ln\lambda(t_j)^{-1}}\right\}\right)\\
&\le \exp\left((\ln\ln\Lambda(t_N)^{-1})\sup_j\left\{\frac{\ln j}{\ln\ln\lambda(t_j)^{-1}}\right\}\right)\\
&\le \exp\left((\ln\ln R)\sup_j\left\{\frac{\ln j}{\ln\ln\lambda(t_j)^{-1}}\right\}\right) = (\ln R)^{\sup_j\{\ln j/\ln\ln\lambda(t_j)^{-1}\}}
\end{aligned}
$$

we obtain

$$|V_0(t;R)| \le \exp(C(\ln R)^\kappa)|V_0(T;R)|$$

for any $t \in [0,T]$ and any larger number R. Thus by (19) we obtain (7).

REMARK 3.2. We have never considered for non-large R, but such a case does not bring any problem for the loss of regularity.

3.9. Optimality. Finally, we shall prove the optimality of the estimate (7) on the assumption (6). Precisely, we shall give an example of $a(t) = \lambda(t)b(t)$ and initial data satisfying all the assumptions of Theorem 2.1, but the estimate (8) does not hold for any positive constants C_0, C_1 and ε.

Main idea of the proof is based on [11] by Tarama. He proved that for the coefficient $a(t) = e^{-t^{-\alpha}} p(t^{-1})$, where p is a positive periodic function, there exist initial data such that (1) is not C^∞ well-posed since $\alpha < 1/2$. In the other words, if $\kappa = 1/\alpha - 1$, then there exist initial data such that the estimate (7) with $\kappa = 1$ dose not hold for any positive constants C_0 and C_1. Here we remark that we only see that the estimate (7) with $\kappa = 1$ dose not hold even if $\kappa < 1$ by [11]. On the other hand, our theorem asserts that the optimality of (7) is true without any restriction to κ.

Let us consider the Cauchy problem

$$
\begin{cases} (\partial_t^2 + \lambda(t)^2 p(t^{-1})^2|\xi|^2)v(t,\xi) = 0, & (t,\xi) \in [0,T] \times \mathbb{R}^n, \\ v(T,\xi) = v_0(\xi),\, v_t(T,\xi) = v_1(\xi), & \xi \in \mathbb{R}^n, \end{cases} \tag{40}
$$

where $\lambda(t) = e^{-t^{-\alpha}}$, $\alpha < 1$ and p is a positive 1-periodic function. By setting $s := 1/t$ and $w(s) = w(s;\xi) := sv(t,\xi)$ we have

$$
\left(\frac{d^2}{ds^2} + s^{-4}\lambda(s^{-1})^2 p(s)^2|\xi|^2\right) w(s) = 0. \tag{41}
$$

Let us understand now the ordinary differential equation (41) as a small perturbation of the simpler equation

$$
\left(\frac{d^2}{ds^2} + \lambda_0 p(s)^2\right) w(s) = 0, \tag{42}
$$

where λ_0 is a positive constant; indeed, such a equation of Hill's type is studied well and the properties will help to solve our problem.

The second order scalar equation (42) is rewritten as the following first order system:

$$
\frac{d}{ds}X(s;s_0) = \begin{pmatrix} 0 & -\lambda_0 p(s)^2 \\ 1 & 0 \end{pmatrix} X(s;s_0),
$$

where $X(s,s_0)$ is 2×2-matrix valued function. Then one can describe a result of Floquet theory as follows:

Lemma 3.4 (Floquet theory). *Let $p(t)$ be a continuous, 1-periodic and non-constant function, and*

$$
X(s_0,s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

There exists a positive real number λ_0 *such that* $X(s_0+1, s_0)$ *has the eigenvalues* μ *and* μ^{-1} *satisfying* $|\mu| > 1$. *Here we remark that the eigenvalues of* $X(s_0+1, s_0)$ *are independent of* s_0 *since* $p(s)$ *is* 1*-periodic*.

For a proof of this lemma refer to [9] Chapter 1, for instance.

Let us prepare some properties to apply Lemma 3.4 for our problem.

Lemma 3.5. *For any given real number* $\delta \in [0, 1)$ *there exists a positive constant* C_δ *such that*

$$\left|s^{-4}\lambda(s^{-1})^2 - (s-\sigma)^{-4}\lambda((s-\sigma)^{-1})^2\right| \le C_\delta \sigma s^{\alpha-5}\lambda(s^{-1})^2$$

for any $s \ge 1$ *and* $0 \le \sigma \le \delta s^{1-\alpha}$.

Proof. By mean value theorem there exist constants $\theta_1, \theta_2, \theta_3 \in (0, 1)$ such that

$$\begin{aligned}
&\left|s^{-4}\lambda(s^{-1})^2 - (s-\sigma)^{-4}\lambda((s-\sigma)^{-1})^2\right| \\
&\le 4\sigma(s-\theta_1\sigma)^{-5}e^{-2s^\alpha} + 2\alpha\sigma(s-\sigma)^{-4}(s-\theta_2\sigma)^{\alpha-1}e^{-2(s-\theta_2\sigma)^\alpha} \\
&\le 4\sigma(s-\sigma)^{-5}e^{-2s^\alpha} + 2\alpha\sigma(s-\sigma)^{\alpha-5}e^{-2(s-\sigma)^\alpha} \\
&\le C\sigma s^{\alpha-5}e^{2s^\alpha - 2(s-\sigma)^\alpha}\lambda(s^{-1})^2 \\
&\le C\sigma s^{\alpha-5}e^{2\alpha\sigma(s-\theta_3\sigma)^{\alpha-1}}\lambda(s^{-1})^2 \\
&\le C\sigma s^{\alpha-5}e^{C_2\sigma s^{\alpha-1}}\lambda(s^{-1})^2 \\
&\le C_\delta \sigma s^{\alpha-5}\lambda(s^{-1})^2,
\end{aligned}$$

where the constant C_δ depends only on δ. Thus the lemma is proved. □

Let λ_0 be a positive real number and define $s_\xi = s_\xi(\lambda_0)$ be the solution to

$$s_\xi^{-4}\lambda\left(s_\xi^{-1}\right)^2|\xi|^2 = \lambda_0.$$

Here we remark that for large s_ξ we have

$$\frac{1}{2}\ln|\xi| \le s_\xi^\alpha \le \ln|\xi|.$$

Let ε be a small positive real number satisfying $1/\alpha - 1 - \varepsilon > 0$, and $0 \le \sigma \le (\ln|\xi|)^{1/\alpha-1-\varepsilon}$. Noting $\sigma \le \delta s_\xi^{1-\alpha-\varepsilon\alpha}$ for $\delta \in [0, 1)$ with large $|\xi|$ we have from Lemma 3.5 that

$$\left|\lambda_0 - (s_\xi - \sigma)^{-4}e^{-2(s_\xi-\sigma)^\alpha}|\xi|^2\right| \le C(\ln|\xi|)^{-\varepsilon} \tag{43}$$

for large $|\xi|$.

By similar proof for the estimate (43) we have

$$\max_{\tau\in[-1,0]} \left|(s_\xi - n - 1 + \tau)^{-4} e^{-2(s_\xi - n-1+\tau)^\alpha} - (s_\xi - n + \tau)^{-4} e^{-2(s_\xi - n+\tau)^\alpha}\right| |\xi|^2 \leq C(\ln|\xi|)^{1-1/\alpha} \tag{44}$$

for any n satisfying $n-1 \leq (\ln|\xi|)^{1/\alpha-1-\varepsilon} \leq n$.

Let us prove now the following proposition:

Proposition 3.1. *Let n_0 be a large integer satisfying $n_0 - 1 \leq (\ln|\xi|)^{1/\alpha-1-\varepsilon} \leq n_0$ for a given large $|\xi|$. Then there exist initial data $(w(s_\xi), w_s(s_\xi))$, positive constants $C_1 = C_1(\kappa)$ and $C_2 = C_2(\kappa)$ such that the solution $w(s)$ of (41) satisfies the estimate*

$$\left|\frac{d}{ds} w(s_\xi - n_0 - 1)\right| + \left|w(s_\xi - n_0 - 1)\right| \geq C_1 \exp(C_2(\ln|\xi|))^{(1/\alpha)-1-\varepsilon}.$$

Proof. Let n be a non-negative integer. We consider the following first order system:

$$\begin{cases} \dfrac{d}{d\tau} X_n(\tau,\tau_0) = \begin{pmatrix} 0 & -(s_\xi - n + \tau)^{-4}\lambda(s_\xi - n + \tau)^2 p(s_\xi + \tau)^2 |\xi|^2 \\ 1 & 0 \end{pmatrix} X_n(\tau,\tau_0), \\ X_n(\tau_0,\tau_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{cases}$$

Then we have the following representation:

$$\begin{pmatrix} \dfrac{d}{ds} w(s_\xi - n_0 - 1) \\ w(s_\xi - n_0 - 1) \end{pmatrix} = X_{n_0}(-1,0) X_{n_0-1}(-1,0) \cdots X_0(-1,0) \begin{pmatrix} \dfrac{d}{ds} w(s_\xi) \\ w(s_\xi) \end{pmatrix}.$$

We set

$$X(s_\xi - 1, s_\xi) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad X_n(-1,0) = \begin{pmatrix} a_{11}(n) & a_{12}(n) \\ a_{21}(n) & a_{22}(n) \end{pmatrix},$$

and denote their eigenvalues by $\mu^\pm$, and $\mu_n^\pm$ respectively. Here we note the following lemma:

Lemma 3.6. *For any $n - 1 \leq (\ln|\xi|)^{1/\alpha-1-\varepsilon} \leq n$ with large $|\xi|$ we have the followings*:

$$\min_n\{|\mu_n|\} > 1 \tag{45}$$

and

$$\max\left\{\left|a_{ij}(n+1)-a_{ij}(n)\right|+|\mu_{n+1}-\mu_n|\right\}\le C(\ln|\xi|)^{1-1/\alpha}. \tag{46}$$

Proof. Noting $a_{11}+a_{22}=\mu+\mu^{-1}$ we have $|a_{11}-\mu|+|a_{22}-\mu|\ge|\mu-\mu^{-1}|$, from which follows

$$\max\{|a_{11}-\mu|,|a_{22}-\mu|\}\ge\frac{1}{2}\left|\mu-\mu^{-1}\right|.$$

Let us assume that

$$|a_{11}-\mu|\ge\frac{1}{2}\left|\mu-\mu^{-1}\right|;$$

the other case can be treated similarly. Then we also have

$$\left|a_{22}-\mu^{-1}\right|\ge\frac{1}{2}\left|\mu-\mu^{-1}\right|.$$

The estimate (43) implies that $\max_{s,\tau\in[-1,0]}\|X_n(s,\tau)\|$ is bounded, and

$$\max_{\tau\in[-1,0]}\|X_n(\tau,0)-X(s_\xi+\tau,s_\xi)\|\le C(\ln|\xi|)^{-\varepsilon}$$

for large $|\xi|$. Thus we have the estimate (45). By (44) we also have the estimate (46). □

Let us set

$$B_n=\begin{pmatrix}\dfrac{a_{12}(n)}{\mu_n-a_{11}(n)} & 1\\ 1 & \dfrac{a_{21}(n)}{\mu_n^{-1}-a_{22}(n)}\end{pmatrix}.$$

Then we have

$$X_n(-1,0)B_n=B_n\begin{pmatrix}\mu_n & 0\\ 0 & \mu_n^{-1}\end{pmatrix},$$

it follows that

$$X_{n_0}(-1,0)X_{n_0-1}(-1,0)\cdots X_0(-1,0)=B_{n_0}Y_{n_0}B_0^{-1},$$

where

$$Y_{n_0}=\begin{pmatrix}\mu_{n_0} & 0\\ 0 & \mu_{n_0}^{-1}\end{pmatrix}(I+G_{n_0})\begin{pmatrix}\mu_{n_0-1} & 0\\ 0 & \mu_{n_0-1}^{-1}\end{pmatrix}(I+G_{n_0-1})\cdots(I+G_1)\begin{pmatrix}\mu_0 & 0\\ 0 & \mu_0^{-1}\end{pmatrix}$$

and

$$G_{n+1} = B_{n+1}^{-1} B_n - I.$$

Let us denote by $y(1,1)$ the $(1,1)$ element of Y_{n_0}. By Lemma 3.6 we have $\|B_n\| + \|B_n^{-1}\| \le C$ and $\|G_{n+1}\| \le C(\ln|\xi|)^{1-1/\alpha}$. Thus we obtain

$$\left| y(1,1) - \prod_{n=1}^{n_0} \mu_n \right| \le C n_0 (\ln|\xi|)^{1-1/\alpha} \prod_{n=0}^{n_0} |\mu_n| \le C(\ln|\xi|)^{-\varepsilon} \prod_{n=0}^{n_0} |\mu_n|.$$

Therefore, for large $|\xi|$ we obtain

$$\left| y(1,1) - \prod_{n=1}^{n_0} \mu_n \right| \le \frac{1}{2} \prod_{n=0}^{n_0} |\mu_n|,$$

it follows from (45) that

$$|y(1,1)| \ge \frac{1}{2} \prod_{n=0}^{n_0} |\mu_n| \ge \frac{1}{2} \left(\frac{1}{2}(|\mu| + 1) \right)^{n_0} \ge C \exp(C(\ln|\xi|)^{1/\alpha - 1 - \varepsilon}).$$

This estimate implies the estimate of Proposition 3.1. □

We set $t_\xi = s_\xi^{-1}$ and $t_{\xi,n} = (s_\xi - n - 1)^{-1}$. Noting the inequality

$$t_{\xi,n}^{-2} = \frac{1}{\sqrt{\lambda_0}} \frac{t_\xi^2}{t_{\xi,n}^2} e^{t_{\xi,n}^\alpha - t_\xi^\alpha} \lambda(t_{\xi,n})|\xi| \le \frac{1}{\sqrt{\lambda_0} \min_t\{p(t^{-1})\}} a(t_{\xi,n})|\xi|$$

we have

$$\begin{aligned} \left| \frac{d}{ds} w(s_{\xi_n} - n - 1) \right| + \left| w(s_{\xi_n} - n - 1) \right| &= \left| t_{\xi,n} \frac{d}{dt} v(t_{\xi,n}) + v(t_{\xi,n}) \right| + \left| t_{\xi,n}^{-1} v(t_{\xi,n}) \right| \\ &\le 2 t_{\xi,n} \left(\left| \frac{d}{dt} v(t_{\xi,n}) \right| + t_{\xi,n}^{-2} \left| v(t_{\xi,n}) \right| \right) \\ &\le C \left(\left| \frac{d}{dt} v(t_{\xi,n}) \right| + \lambda(t_{\xi,n})|\xi| \left| v(t_{\xi,n}) \right| \right) \end{aligned}$$

for any large $|\xi|$. Therefore, by Proposition 3.1 and Lemma 3.6 we have the following:

Corollary 3.1. *Let $\alpha < 1$. For any given ξ and for any small positive real number ε there exist t_0, t_1 satisfying $0 < t_1 < t_0 < T$, positive constants C_0, C_1, and initial data $(v_0(\xi), v_1(\xi))$ such that the solution to (40) satisfies*

$$|v_t(t_1)| + \lambda(t_1) p(t_1^{-1}) |\xi| |v(t_1)| \ge C_0 e^{C_1 (\ln|\xi|)^{1/\alpha - 1 - \varepsilon}} |v(t_0)|.$$

This corollary concludes the proof of Theorem 2.1.

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