

Inoue, A.  
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## BORSUK-ULAM TYPE THEOREMS ON STIEFEL MANIFOLDS

AKIRA INOUE

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### Abstract

In this paper, we study the degree of equivariant maps between Stiefel manifolds by using cohomological index theory. As applications, we have some Borsuk-Ulam type theorems on Stiefel manifolds.

### 1. Introduction

We are concerned with the following classical version of the Borsuk-Ulam theorem:

(i) If  $n > k$  then there is no map  $f: S^n \rightarrow S^k$  such that  $f(-x) = -f(x)$  for all  $x$ .

This easily follows from the next proposition:

(ii) Let  $f: S^n \rightarrow S^n$  be a map of the sphere such that  $f(-x) = -f(x)$  for all  $x$ . Then  $\deg f \equiv 1 \pmod{2}$ .

Now let  $S^n$  denote the standard  $n$ -dimensional sphere with antipodal  $\mathbf{Z}_2$ -action, then the proposition (ii) implies that for any  $\mathbf{Z}_2$ -map  $f: S^n \rightarrow S^n$ , the degree of  $f$  is odd.

Many authors have been contributing to generalizing and extending the Borsuk-Ulam theorem in various ways. E. Fadell-S. Husseini and J. Jaworowski introduced an ideal-valued cohomological index theory, and generalized the Borsuk-Ulam theorem (see [2], [3] and [5]). Let  $V_k(\mathbf{R}^m)$  denote the space of orthonormal  $k$ -frames in  $\mathbf{R}^m$  and  $O(k)$  the orthogonal group. If we represent an element of  $V_k(\mathbf{R}^m)$  as a column vector  $[v_1 \cdots v_k]^T$ , and if  $O(k)$  is the orthogonal group of  $k \times k$  matrices, then  $V_k(\mathbf{R}^m)$  is a free  $O(k)$ -space under the action induced by matrix multiplication  $g[v_1 \cdots v_k]^T$ ,  $g \in O(k)$ . In [4], Yasuhiro Hara considered the degree of  $O(k)$ -maps  $f: V_k(\mathbf{R}^m) \rightarrow V_k(\mathbf{R}^m)$ .

In this paper, we will consider the degree of  $(\mathbf{Z}_2)^k$ -maps  $f: V_k(\mathbf{R}^m) \rightarrow V_k(\mathbf{R}^m)$  where  $(\mathbf{Z}_2)^k = \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$  ( $k$  times) is the subgroup of  $O(k)$  which is diagonally imbedded. We will show

**Theorem 3.3.** *Let  $f: V_k(\mathbf{R}^m) \rightarrow V_k(\mathbf{R}^m)$  be a  $(\mathbf{Z}_2)^k$ -map. Then the degree of  $f$  is odd.*

By a similar way,  $U(k)$  acts freely on the complex Stiefel manifold  $V_k(\mathbf{C}^m)$ . We restrict the  $U(k)$ -action on  $V_k(\mathbf{C}^m)$  to the subgroup  $(\mathbf{Z}_p)^k$  where  $p$  is a prime number. Then we will show

**Theorem 3.5.** *Let  $f: V_k(\mathbf{C}^m) \rightarrow V_k(\mathbf{C}^m)$  be a  $(\mathbf{Z}_p)^k$ -map. Then the degree of  $f$  is not congruent to zero modulo  $p$ .*

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## 2. Index theory

In this section we will recall the definition and basic properties of index theory which was first introduced by Fadell and Husseini and independently by Jaworowski.

Let  $G$  be a compact Lie group and  $X$  a  $G$ -CW complex. We denote the universal principal  $G$ -bundle by  $EG \rightarrow BG$ . Then  $G$  acts freely on  $EG \times X$  by  $g(e, x) = (ge, gx)$ . We denote the quotient space of this action by  $EG \times_G X$ . Note that the orbit map  $p: EG \times X \rightarrow EG \times_G X$  is a fiber bundle of the fiber  $G$ . The Borel cohomology of  $X$  with coefficients in a field  $\mathbf{K}$  is defined by  $H_G^*(X; \mathbf{K}) = H^*(EG \times_G X; \mathbf{K})$ , where  $H^*(\ )$  is singular cohomology theory. Let  $c_X: X \rightarrow *$  be a constant map to one-point space. The  $G$ -index of  $X$ , denoted by  $\text{Ind}^G(X; \mathbf{K})$ , is an ideal in  $H^*(BG; \mathbf{K})$ .  $\text{Ind}^G(X; \mathbf{K})$  is defined to be the kernel of the homomorphism  $\bar{c}_X^* = (\text{id} \times_G c_X)^*: H^*(BG; \mathbf{K}) = H_G^*(*; \mathbf{K}) \rightarrow H_G^*(X; \mathbf{K})$ . If  $X$  is a free  $G$ -space, then  $\text{Ind}^G(X)$  coincides with the kernel of the homomorphism  $H^*(BG) \rightarrow H^*(X/G)$  induced from a classifying map  $X/G \rightarrow BG$  for the free  $G$ -action on  $X$ . Furthermore for an integer  $k$  we set

$$\text{Ind}_k^G(X; \mathbf{K}) = \text{Ind}^G(X; \mathbf{K}) \cap H^k(BG; \mathbf{K}) = \ker(\bar{c}_X^*: H^k(BG; \mathbf{K}) \rightarrow H_G^k(X; \mathbf{K})).$$

The following proposition is a basic property of the  $G$ -index.

**Proposition 2.1** ([2], [5]). *If there exists a  $G$ -map  $f: X \rightarrow Y$ , then for any  $k \in \mathbf{Z}$*

$$\text{Ind}_k^G(X) \supset \text{Ind}_k^G(Y).$$

We now consider a basic computation which is important to an application which we give later on.

$V_k(\mathbf{R}^m)$  denotes the space of orthonormal  $k$ -frames in  $\mathbf{R}^m$  and  $O(k)$  denotes the orthogonal group. Then  $O(k)$  acts freely on  $V_k(\mathbf{R}^m)$  by the usual action  $gv, g \in O(k)$  and  $v$  is a column vector representing  $k$ -frame. We restrict this action to the subgroup  $(\mathbf{Z}_2)^k$  of diagonal matrices with entries  $\pm 1$ . Then  $V_k(\mathbf{R}^m)$  is also a free  $(\mathbf{Z}_2)^k$ -space.

Recall that  $B(\mathbf{Z}_2)^k = B\mathbf{Z}_2 \times \cdots \times B\mathbf{Z}_2$  ( $k$  times) and

$$H^*(B(\mathbf{Z}_2)^k; \mathbf{Z}_2) = H^*(B\mathbf{Z}_2) \otimes \cdots \otimes H^*(B\mathbf{Z}_2) = \mathbf{Z}_2[t_1, \dots, t_k],$$

where  $\dim t_i = 1$ . Fadell proved the following in [3].

**Proposition 2.2.** *The monomial  $t_1^{m-1}t_2^{m-2} \cdots t_k^{m-k}$  does not belong to  $\text{Ind}^{(\mathbf{Z}_2)^k}(V_k(\mathbf{R}^m); \mathbf{Z}_2)$ .*

*In particular, since  $\dim V_k(\mathbf{R}^m) = mk - k(k + 1)/2$ , we can assert*

$$\text{Ind}_{\dim V_k(\mathbf{R}^m)}^{(\mathbf{Z}_2)^k}(V_k(\mathbf{R}^m); \mathbf{Z}_2) \neq H^{\dim V_k(\mathbf{R}^m)}(B(\mathbf{Z}_2)^k; \mathbf{Z}_2).$$

We have an analogous proposition for complex Stiefel manifolds.  $V_k(\mathbf{C}^m)$  denotes the space of orthonormal  $k$ -frames in  $\mathbf{C}^m$  and  $U(k)$  denotes the unitary group. Then  $U(k)$  acts freely on  $V_k(\mathbf{C}^m)$  by the usual action  $gv, g \in U(k)$  and  $v$  is a column vector representing  $k$ -frame. We restrict this action to the subgroup  $(\mathbf{Z}_p)^k$  of diagonal matrices with entries  $p$ -th root of one and consider  $\text{Ind}^{(\mathbf{Z}_p)^k}(V_k(\mathbf{C}^m); \mathbf{Z}_p)$ , where  $p$  is a prime number.

In case  $p = 2$  we show that  $t_1^{2(m-1)+1}t_2^{2(m-2)+1} \cdots t_k^{2(m-k)+1}$  is not in  $\text{Ind}^{(\mathbf{Z}_2)^k}(V_k(\mathbf{C}^m); \mathbf{Z}_2)$  by induction on  $k$ . The computation will be based on the fibration

$$(1) \quad S^{2(m-k)+1} \rightarrow V_k(\mathbf{C}^m) \xrightarrow{\pi} V_{k-1}(\mathbf{C}^m),$$

where  $\pi$  is the projection on the first  $k - 1$  coordinates. Consider the sequence

$$(2) \quad \mathbf{Z}_2 \rightarrow (\mathbf{Z}_2)^k \rightarrow (\mathbf{Z}_2)^{k-1},$$

where  $\mathbf{Z}_2$  injects on the last coordinate and  $(\mathbf{Z}_2)^k$  projects on the first  $k - 1$  coordinates. Dividing out the action of (2) on (1), we obtain

$$\mathbf{R}P^{2(m-k)+1} \rightarrow V_k(\mathbf{C}^m)/(\mathbf{Z}_2)^k \rightarrow V_{k-1}(\mathbf{C}^m)/(\mathbf{Z}_2)^{k-1}.$$

We then have an induced diagram of fibrations

$$\begin{array}{ccc} \mathbf{R}P^{2(m-k)+1} & \xrightarrow{\alpha_{m-k+1,1}} & B\mathbf{Z}_2 \\ i_m \downarrow & & i_\infty \downarrow \\ V_k(\mathbf{C}^m)/(\mathbf{Z}_2)^k & \xrightarrow{\alpha_{m,k}} & B(\mathbf{Z}_2)^k \\ p_m \downarrow & & p_\infty \downarrow \\ V_{k-1}(\mathbf{C}^m)/(\mathbf{Z}_2)^{k-1} & \xrightarrow{\alpha_{m,k-1}} & B(\mathbf{Z}_2)^{k-1} \end{array}$$

where the  $\alpha_{i,j}$  are classifying maps. Recall that our coefficients are  $\mathbf{Z}_2$ , and since  $i_\infty^*$  and  $\alpha_{m-k+1,1}^*$  are surjective,  $i_m^*: H^*(V_k(\mathbf{C}^m)/(\mathbf{Z}_2)^k) \rightarrow H^*(\mathbf{R}P^{2(m-k)+1})$  is surjective.

Thus, the Leray-Hirsch theorem applies and we have a diagram

$$\begin{array}{ccc}
 H^* (V_{k-1}(\mathbf{C}^m)/(\mathbf{Z}_2)^{k-1}) \otimes H^*(\mathbf{R}P^{2(m-k)+1}) & \xrightarrow{\varphi_m} & H^* (V_k(\mathbf{C}^m)/(\mathbf{Z}_2)^k) \\
 \alpha_{m,k-1}^* \otimes \alpha_{m-k+1,1}^* \uparrow & & \alpha_{m,k}^* \uparrow \\
 H^* (B(\mathbf{Z}_2)^{k-1}) \otimes H^*(\mathbf{R}P^\infty) & \xrightarrow{\varphi_\infty} & H^* (B(\mathbf{Z}_2)^k)
 \end{array}$$

with  $\varphi_m$  and  $\varphi_\infty$  isomorphisms. Then

$$\begin{aligned}
 & \alpha_{m,k}^* \left[ t_1^{2(m-1)+1} t_2^{2(m-2)+1} \dots t_k^{2(m-k)+1} \right] \\
 = & \alpha_{m,k}^* \circ \varphi_\infty \left[ t_1^{2(m-1)+1} t_2^{2(m-2)+1} \dots t_{k-1}^{2(m-k+1)+1} \otimes t_k^{2(m-k)+1} \right] \\
 = & \varphi_m \left[ \alpha_{m,k-1}^* \left( t_1^{2(m-1)+1} t_2^{2(m-2)+1} \dots t_{k-1}^{2(m-k+1)+1} \right) \otimes \alpha_{m-k+1,1}^* \left( t_k^{2(m-k)+1} \right) \right].
 \end{aligned}$$

But  $\alpha_{m-k+1,1}^* (t_k^{2(m-k)+1}) \neq 0$  and assuming by induction that

$$\alpha_{m,k-1}^* \left( t_1^{2(m-1)+1} t_2^{2(m-2)+1} \dots t_{k-1}^{2(m-k+1)+1} \right) \neq 0,$$

we have

$$\alpha_{m,k}^* \left[ t_1^{2(m-1)+1} t_2^{2(m-2)+1} \dots t_k^{2(m-k)+1} \right] \neq 0.$$

Thus  $t_1^{2(m-1)+1} t_2^{2(m-2)+1} \dots t_k^{2(m-k)+1}$  is not in  $\ker \alpha_{m,k}^*$ .

When  $p$  is an odd prime,  $H^*(B(\mathbf{Z}_p)^k; \mathbf{Z}_p) = \mathbf{Z}_p[x_1, x_2, \dots, x_k] \otimes E(y_1, y_2, \dots, y_k)$ , where  $\mathbf{Z}_p[x_1, x_2, \dots, x_k]$  denotes the  $\mathbf{Z}_p$ -polynomial algebra on 2-dimensional generators  $x_i$  and  $E(y_1, y_2, \dots, y_k)$  denotes the  $\mathbf{Z}_p$ -exterior algebra on 1-dimensional generators  $y_i$ . The ring is graded-commutative, i.e.  $xy = (-1)^{\deg(x)\deg(y)}yx$ . We next show that  $x_1^{m-1}y_1x_2^{m-2}y_2 \dots x_k^{m-k}y_k$  is not in  $\text{Ind}^{(\mathbf{Z}_p)^k}(V_k(\mathbf{C}^m); \mathbf{Z}_p)$  by induction on  $k$ . Consider the sequence

$$(3) \quad \mathbf{Z}_p \rightarrow (\mathbf{Z}_p)^k \rightarrow (\mathbf{Z}_p)^{k-1},$$

where  $\mathbf{Z}_p$  injects on the last coordinate and  $(\mathbf{Z}_p)^k$  projects on the first  $k - 1$  coordinates. Dividing out the action of (3) on (1), we obtain

$$S^{2(m-k)+1}/\mathbf{Z}_p \rightarrow V_k(\mathbf{C}^m)/(\mathbf{Z}_p)^k \rightarrow V_{k-1}(\mathbf{C}^m)/(\mathbf{Z}_p)^{k-1}.$$

We then have an induced diagram of fibrations

$$\begin{array}{ccc}
 L_p^{2(m-k)+1} & \xrightarrow{\alpha_{m-k+1,1}} & BZ_p \\
 i_m \downarrow & & i_\infty \downarrow \\
 V_k(\mathbf{C}^m)/(\mathbf{Z}_p)^k & \xrightarrow{\alpha_{m,k}} & B(\mathbf{Z}_p)^k \\
 p_m \downarrow & & p_\infty \downarrow \\
 V_{k-1}(\mathbf{C}^m)/(\mathbf{Z}_p)^{k-1} & \xrightarrow{\alpha_{m,k-1}} & B(\mathbf{Z}_p)^{k-1}
 \end{array}$$

where the orbit space  $L_p^{2(m-k)+1} = S^{2(m-k)+1}/\mathbf{Z}_p$  is the lens space and the  $\alpha_{i,j}$  are classifying maps. Recall that our coefficients are  $\mathbf{Z}_p$ , and since  $i_\infty^*$  and  $\alpha_{m-k+1,1}^*$  are surjective,  $i_m^* : H^*(V_k(\mathbf{C}^m)/(\mathbf{Z}_p)^k) \rightarrow H^*(L_p^{2(m-k)+1})$  is surjective. Thus, the Leray-Hirsch theorem applies and we have a diagram

$$\begin{array}{ccc}
 H^*(V_{k-1}(\mathbf{C}^m)/(\mathbf{Z}_p)^{k-1}) \otimes H^*(L_p^{2(m-k)+1}) & \xrightarrow{\varphi_m} & H^*(V_k(\mathbf{C}^m)/(\mathbf{Z}_p)^k) \\
 \alpha_{m,k-1}^* \otimes \alpha_{m-k+1,1}^* \uparrow & & \alpha_{m,k}^* \uparrow \\
 H^*(B(\mathbf{Z}_p)^{k-1}) \otimes H^*(BZ_p) & \xrightarrow{\varphi_\infty} & H^*(B(\mathbf{Z}_p)^k)
 \end{array}$$

with  $\varphi_k$  and  $\varphi_\infty$  isomorphisms. Then

$$\begin{aligned}
 & \alpha_{m,k}^* [x_1^{m-1}y_1x_2^{m-2}y_2 \cdots x_k^{m-k}y_k] \\
 &= \alpha_{m,k}^* \circ \varphi_\infty [x_1^{m-1}y_1x_2^{m-2}y_2 \cdots x_{k-1}^{m-k+1}y_{k-1} \otimes x_k^{m-k}y_k] \\
 &= \varphi_m [\alpha_{m,k-1}^* (x_1^{m-1}y_1x_2^{m-2}y_2 \cdots x_{k-1}^{m-k+1}y_{k-1}) \otimes \alpha_{m-k+1,1}^* (x_k^{m-k}y_k)].
 \end{aligned}$$

But  $\alpha_{m-k+1,1}^*(x_k^{m-k}y_k) \neq 0$  and assuming by induction that

$$\alpha_{m,k-1}^* (x_1^{m-1}y_1x_2^{m-2}y_2 \cdots x_{k-1}^{m-k+1}y_{k-1}) \neq 0,$$

we have

$$\alpha_{m,k}^* [x_1^{m-1}y_1x_2^{m-2}y_2 \cdots x_k^{m-k}y_k] \neq 0.$$

Therefore  $x_1^{m-1}y_1x_2^{m-2}y_2 \cdots x_k^{m-k}y_k$  is not in  $\ker \alpha_{m,k}^*$ . Thus we have the following result.

**Proposition 2.3.** (1) *The monomial  $t_1^{2(m-1)+1}t_2^{2(m-2)+1} \cdots t_k^{2(m-k)+1}$  does not belong to  $\text{Ind}^{(\mathbf{Z}_2)^k}(V_k(\mathbf{C}^m); \mathbf{Z}_2)$ .*

*In particular, since  $\dim V_k(\mathbf{C}^m) = 2mk - k^2$ , we can assert*

$$\text{Ind}_{\dim V_k(\mathbf{C}^m)}^{(\mathbf{Z}_2)^k}(V_k(\mathbf{C}^m); \mathbf{Z}_2) \neq H^{\dim V_k(\mathbf{C}^m)}(B(\mathbf{Z}_2)^k; \mathbf{Z}_2).$$

(2) When  $p$  is an odd prime, the monomial  $x_1^{m-1}y_1x_2^{m-2}y_2 \cdots x_k^{m-k}y_k$  does not belong to  $\text{Ind}_{\dim V_k(\mathbf{C}^m)}^{(\mathbf{Z}_p)^k}(V_k(\mathbf{C}^m); \mathbf{Z}_p)$ .

In particular, since  $\dim V_k(\mathbf{C}^m) = 2mk - k^2$ ,  $\dim x_i = 2$  and  $\dim y_i = 1$ , we can assert

$$\text{Ind}_{\dim V_k(\mathbf{C}^m)}^{(\mathbf{Z}_p)^k}(V_k(\mathbf{C}^m); \mathbf{Z}_p) \neq H^{\dim V_k(\mathbf{C}^m)}(B(\mathbf{Z}_p)^k; \mathbf{Z}_p).$$

### 3. Borsuk-Ulam type theorems on Stiefel manifolds

Let  $G$  be a compact Lie group and  $X$  be a free  $G$ -CW complex. We denote by  $X/G$  the orbit space of  $X$ . Note that the orbit map  $p: X \rightarrow X/G$  is a fiber bundle with fiber  $G$ . Following [4], we define the transfer  $p_!: H^n(X; \Gamma) \rightarrow H^{n-\dim G}(X/G; \Gamma)$  where  $\Gamma$  is a commutative group. Then we have the following.

**Lemma 3.1** ([4]). *Let  $X, Y$  be  $G$ -CW complexes and  $f: X \rightarrow Y$  a  $G$ -map. Let  $p_X: EG \times X \rightarrow EG \times_G X$  and  $p_Y: EG \times Y \rightarrow EG \times_G Y$  denote the orbit maps. Then the commutativity holds in the diagram:*

$$\begin{array}{ccc} H^i(Y; \Gamma) & \xrightarrow{f^*} & H^i(X; \Gamma) \\ (p_Y)_! \downarrow & & \downarrow (p_X)_! \\ H_G^{i-\dim G}(Y; \Gamma) & \xrightarrow{\bar{f}^*} & H_G^{i-\dim G}(X; \Gamma) \end{array}$$

where  $\bar{f} = \text{id} \times_G f: EG \times_G X \rightarrow EG \times_G Y$  is the induced map from a  $G$ -map  $\text{id} \times f: EG \times X \rightarrow EG \times Y$ .

Let  $M$  be a smooth closed connected oriented  $G$ -manifold of dimension  $n$ . Suppose that the  $G$ -action on  $M$  is free. Note that the orbit space  $M/G$  is also a manifold of dimension  $n - \dim G$  in this case. Let  $p: M \rightarrow M/G$  be the orbit map. Suppose that  $M/G$  is orientable over  $\mathbf{K}$ . Then the transfer  $p_!$  of the  $p$  is described as  $p_! = D_{M/G}^{-1} \circ p_* \circ D_M$  where  $D$  is the Poincaré duality isomorphism. Then  $p_!: H^n(M; \mathbf{K}) \rightarrow H^{n-\dim G}(M/G; \mathbf{K})$  is an isomorphism.

The following theorem has been essentially proved in [4].

**Theorem 3.2** ([4]). *Let  $G$  be a compact Lie group and let  $M$  and  $N$  be smooth closed connected  $G$ -free manifolds of dimension  $n$  which are orientable over  $\mathbf{K}$ . Assume that the orbit space  $M/G$  and  $N/G$  are also orientable. Then we have the following.*

(1) *Suppose  $\text{Ind}_{n-\dim G}^G(M; \mathbf{K})$  is not equal to  $H^{n-\dim G}(BG; \mathbf{K})$ . Then for any  $G$ -map  $f: M \rightarrow N$ ,  $f^*: H^n(N; \mathbf{K}) \rightarrow H^n(M; \mathbf{K})$  is non-trivial.*

(2) Suppose that  $\text{Ind}_{n-\dim G}^G(N; \mathbf{K})$  is not equal to  $\text{Ind}_{n-\dim G}^G(M; \mathbf{K})$ . Then for any  $G$ -map  $f: M \rightarrow N$ ,  $f^*: H^n(N; \mathbf{K}) \rightarrow H^n(M; \mathbf{K})$  is not injective.

Proof. (1) Assume that there exists a  $G$ -map  $f: M \rightarrow N$  such that  $f^*: H^n(N; \mathbf{K}) \rightarrow H^n(M; \mathbf{K})$  is trivial. By Lemma 3.1,  $(p_M)_! \circ f^* = \bar{f}^* \circ (p_N)_!$ .

Therefore  $\bar{f}^*: H_G^{n-\dim G}(N; \mathbf{K}) \rightarrow H_G^{n-\dim G}(M; \mathbf{K})$  is trivial, because  $(p_M)_!$  and  $(p_N)_!$  are isomorphism and  $f^*$  is the trivial homomorphism. Since  $c_M = c_N \circ f$ ,

$$\text{Ind}_{n-\dim G}^G(M; \mathbf{K}) = (\bar{c}_M^*)^{-1}(0) = (\bar{c}_N^*)^{-1}\left((\bar{f}^*)^{-1}(0)\right) = H^{n-\dim G}(M; \mathbf{K}).$$

(2) Assume that there exists a  $G$ -map  $f: M \rightarrow N$  such that  $f^*: H^n(N; \mathbf{K}) \rightarrow H^n(M; \mathbf{K})$  is injective. Then  $\bar{f}^*: H_G^{n-\dim G}(N; \mathbf{K}) \rightarrow H_G^{n-\dim G}(M; \mathbf{K})$  is injective, using Lemma 3.1 again. Hence

$$\begin{aligned} \text{Ind}_{n-\dim G}^G(N; \mathbf{K}) &= \ker \bar{c}_N^* = (\bar{c}_N^*)^{-1}(0) = (\bar{c}_M^*)^{-1}\left((\bar{f}^*)^{-1}(0)\right) = (\bar{c}_M^*)^{-1}(0) \\ &= \text{Ind}_{n-\dim G}^G(M; \mathbf{K}) \end{aligned} \quad \square$$

As a consequence of Proposition 2.2 and Theorem 3.2 (1) we get the following theorem.

**Theorem 3.3.** *Let  $f: V_k(\mathbf{R}^m) \rightarrow V_k(\mathbf{R}^m)$  be a  $(\mathbf{Z}_2)^k$ -map. Then the degree of  $f$  is odd.*

Proof. Set  $n = \dim V_k(\mathbf{R}^m)$ . By Proposition 2.2,  $\text{Ind}_n^{(\mathbf{Z}_2)^k}(V_k(\mathbf{R}^m); \mathbf{Z}_2)$  is not equal to  $H^n(B(\mathbf{Z}_2)^k; \mathbf{Z}_2)$ . Hence  $f^*: H^n(N; \mathbf{Z}_2) \rightarrow H^n(M; \mathbf{Z}_2)$  is non-trivial from assertion (1) of Theorem 3.2. □

This theorem implies the following.

**Corollary 3.4.** *If there exists a  $(\mathbf{Z}_2)^k$ -map  $f: V_k(\mathbf{R}^m) \rightarrow V_k(\mathbf{R}^n)$ , then  $m \leq n$ .*

Proof. Let  $f: V_k(\mathbf{R}^m) \rightarrow V_k(\mathbf{R}^n)$  be a  $(\mathbf{Z}_2)^k$ -map. Assume that  $m > n$ . The canonical inclusion  $i: V_k(\mathbf{R}^n) \rightarrow V_k(\mathbf{R}^m)$  is a  $(\mathbf{Z}_2)^k$ -map. Since  $i \circ f: V_k(\mathbf{R}^m) \rightarrow V_k(\mathbf{R}^m)$  is a  $(\mathbf{Z}_2)^k$ -map, the degree of  $i \circ f$  is not even. Otherwise, because  $(i \circ f)^* = f^* \circ i^*$  and  $H^{\dim V_k(\mathbf{R}^m)}(V_k(\mathbf{R}^n); \mathbf{Z}_2) = 0$ ,  $(i \circ f)^*: H^{\dim V_k(\mathbf{R}^m)}(V_k(\mathbf{R}^m)) \rightarrow H^{\dim V_k(\mathbf{R}^m)}(V_k(\mathbf{R}^m))$  is trivial. This is a contradiction. □

Next if  $l < k$ , then we regard  $(\mathbf{Z}_p)^l$  as any subgroup of  $(\mathbf{Z}_p)^k$ . We get a commutative diagram

$$\begin{array}{ccc} E(\mathbf{Z}_2)^k \times_{(\mathbf{Z}_2)^l} V_k(\mathbf{R}^m) & \xrightarrow{\bar{c}} & B(\mathbf{Z}_2)^l \\ \pi \downarrow & & \rho \downarrow \\ E(\mathbf{Z}_2)^k \times_{(\mathbf{Z}_2)^k} V_k(\mathbf{R}^m) & \xrightarrow{\bar{c}} & B(\mathbf{Z}_2)^k. \end{array}$$

Then we have

$$\begin{array}{ccc} H_{(\mathbf{Z}_2)^l}^*(V_k(\mathbf{R}^m)) & \xleftarrow{\bar{c}^*} & H^*(B(\mathbf{Z}_2)^l) \\ \pi^* \uparrow & & \rho^* \uparrow \\ H_{(\mathbf{Z}_2)^k}^*(V_k(\mathbf{R}^m)) & \xleftarrow{\bar{c}^*} & H^*(B(\mathbf{Z}_2)^k). \end{array}$$

**Theorem 3.5.** *If  $\dim V_k(\mathbf{R}^m) = \dim V_l(\mathbf{R}^n)$ , then for any  $(\mathbf{Z}_2)^l$ -map  $f: V_k(\mathbf{R}^m) \rightarrow V_l(\mathbf{R}^n)$  the degree of  $f$  is even.*

*Proof.* We set  $d = \dim V_k(\mathbf{R}^m) = \dim V_l(\mathbf{R}^n)$ . Then  $\pi^*: H_{(\mathbf{Z}_2)^k}^d(V_k(\mathbf{R}^m); \mathbf{Z}_2) \rightarrow H_{(\mathbf{Z}_2)^l}^d(V_k(\mathbf{R}^m); \mathbf{Z}_2)$  is trivial. Since  $\rho^*: H^*(B(\mathbf{Z}_2)^k; \mathbf{Z}_2) \rightarrow H^*(B(\mathbf{Z}_2)^l; \mathbf{Z}_2)$  is surjective,  $\bar{c}^*: H^d(B(\mathbf{Z}_2)^l; \mathbf{Z}_2) \rightarrow H_{(\mathbf{Z}_2)^l}^d(V_k(\mathbf{R}^m); \mathbf{Z}_2)$  is also trivial. Therefore we have  $\text{Ind}_d^{(\mathbf{Z}_2)^l}(V_k(\mathbf{R}^m); \mathbf{Z}_2) = H^d(B(\mathbf{Z}_2)^l; \mathbf{Z}_2)$ .

Otherwise  $\text{Ind}_d^{(\mathbf{Z}_2)^l}(V_l(\mathbf{R}^n); \mathbf{Z}_2) \neq H^d(B(\mathbf{Z}_2)^l; \mathbf{Z}_2)$  from Proposition 2.2. Therefore it follows from Theorem 3.2 (2) that for any  $(\mathbf{Z}_2)^l$ -map  $f: V_k(\mathbf{R}^m) \rightarrow V_l(\mathbf{R}^n)$  the degree of  $f$  is even.  $\square$

Still continuing our complex analogue of the propositions above, we get the following.

**Theorem 3.6.** *Let  $f: V_k(\mathbf{C}^m) \rightarrow V_k(\mathbf{C}^m)$  be a  $(\mathbf{Z}_p)^k$ -map. Then the degree of  $f$  is not congruent to zero modulo  $p$ .*

From this theorem, the following corollary is proved in the same way as Corollary 3.4.

**Corollary 3.7.** *If there exists a  $(\mathbf{Z}_p)^k$ -map  $f: V_k(\mathbf{C}^m) \rightarrow V_k(\mathbf{C}^n)$ , then  $m \leq n$ .*

Next if  $l < k$ , then we regard  $(\mathbf{Z}_p)^l$  as any subgroup of  $(\mathbf{Z}_p)^k$ . Hence  $V_k(\mathbf{C}^m)$  is a free  $(\mathbf{Z}_p)^l$ -manifold. Then we get the following in the same way as Theorem 3.5.



**Theorem 3.8.** *If  $\dim V_k(\mathbf{C}^m) = \dim V_l(\mathbf{C}^n)$ , then for any  $(\mathbf{Z}_p)^l$ -map  $f: V_k(\mathbf{C}^m) \rightarrow V_l(\mathbf{C}^n)$  the degree of  $f$  is congruent to zero modulo  $p$ .*

REMARK. If  $k$  is even, then  $\dim V_k(\mathbf{C}^m)$  is even. Hence there does not exist a free  $\mathbf{Z}_p$ -action on  $S^{\dim V_k(\mathbf{C}^m)}$ .

**Corollary 3.9.** *If  $\dim V_k(\mathbf{C}^m) = \dim V_l(\mathbf{C}^n)$ , then for any  $(S^1)^l$ -map  $f: V_k(\mathbf{C}^m) \rightarrow V_l(\mathbf{C}^n)$  the degree of  $f$  is zero.*

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Department of Mathematics  
 Graduate School of Science  
 Osaka University  
 Toyonaka 560-0043, Japan