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# TOGLIATTI SYSTEMS

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**Abstract**

We find some examples in $\mathbf{P}^5(\mathbf{C})$ of surfaces satisfying Laplace equations. In particular, we study rational surfaces in $\mathbf{P}^5(\mathbf{C})$ whose hyperplane sections have genus one that satisfy a Laplace equation. Then we study monomial Togliatti systems of cubics for variety of dimension three, i.e. we find all the monomial examples of three-folds satisfying Laplace equations.

## Introduction

A $k$-dimensional variety $V$ whose $d$-osculating space at a general point has dimension $\binom{k+d}{d}-1-\delta$ is said to satisfy $\delta$ independent Laplace equations of order $d$. We have studied rational varieties satisfying Laplace equations in two papers ([4], [7]), i.e. we have considered rational varieties whose $d$-th osculating space does not have the expected dimension at the generic point. We have linked the dimension of the osculating spaces for a projection of a Veronese variety to the position of the linear space $\rho$ from which we project. We found both a lower bound and an upper bound for the dimension of $\rho$. Finally in [4, 7] we have examined a famous example, by Togliatti, of a smooth rational surface with elliptic hyperplane sections in $\mathbf{P}^5(\mathbf{C})$, satisfying a single Laplace equation. In [4] the approach is completely different and we use the so called "Voie ouest" introduced by Ellia and Hirschowitz in [3]. We make use of the Borel-fixed point method.

Now we use a different point of view.

In [9] Togliatti studies the not ruled surfaces, of degree less or equal than six of $\mathbf{P}^r(\mathbf{C})$, with $r \geq 5$, which represent one Laplace equation of order two. Following his arguments, we consider another step: i.e. the research of the surfaces of higher degree of $\mathbf{P}^5(\mathbf{C})$, which represent one Laplace equation of order two. In particular we study the surfaces of degree more than six with elliptic hyperplane sections in $\mathbf{P}^5(\mathbf{C})$, which satisfy one Laplace equation of order two.

In the second section we study the three-folds, in particular monomial Togliatti systems of cubics for variety of dimension three (for the definition of Togliatti system see later and for more details see [7]).

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In Section 1 we set some notations and recall some basic facts about the osculating spaces to the Veronese varieties and about the Togliatti systems. In Section 2 we study linearly 2-normal rational surfaces of $\mathbf{P}^5(\mathbf{C})$ (for the definition of linearly 2-normal surface see later).

For instance, a smooth example is given considering the Togliatti triangle ([4], [7]). This is the unique example among the surfaces of degree $\leq 6$, that is linearly 2-normal and smooth in $\mathbf{P}^5(\mathbf{C})$. Then we study the monomial rational 3-folds satisfying a Laplace equation of order two. We study all the monomial examples with a computer program. We conclude with a conjecture, generalizing the above mentioned Togliatti example.

## 1. Preliminaries

We use terminology and preliminaries of [7].

By variety we mean a projective integral scheme over the field of complex numbers embedded in some complex projective space $\mathbf{P}^n$. We denote by $V_k$ a variety of dimension $k$. A linear space of dimension $s$ will be called an $s$-plane. At every smooth point $p$ of a variety $V \subseteq \mathbf{P}^n$ we define the projective tangent space to $V$ at $p$, denoted by $T(V, p)$. The projective tangent space to $V_k$ at $p$ is the $k$-plane containing all tangent lines to $V_k$ at $p$.

Let $V_{sm}$ be the quasi projective variety of smooth points of a variety $V$. The variety $\mathrm{Tan}(V) = \overline{\bigcup_{p \in V_{sm}} T(V, p)}$ is called the tangent variety to $V$.

Let $V_k \subseteq \mathbf{P}^n$ and let $p$ be a general point of $V_k$. Let

$$\underline{x} = \underline{x}(t_1, \dots, t_k) = \underline{x}(\underline{t})$$

be a local parametrization of $V_k$, centered at $p$.

DEFINITION 1.1. The $d$th-osculating space to $V_k$ at $p$ is the subspace of $\mathbf{P}^n$ spanned by $p$ and by all the derivative points of degree less than or equal to $d$ of a local parametrization of $V_k$, evaluated at $p$.

This definition does not depend on the parametrization. We denote the $d$th-osculating space to $V$ at $p$ by $T^d(V, p)$. Let $V_o \subseteq V$ be the quasi projective variety of points where $T^d(V, p)$ has maximal dimension. The variety

$$\mathrm{Tan}^d(V) = \overline{\bigcup_{p \in V_0} T^d(V, p)}$$

is called the variety of $d$th-osculating spaces to $V$.

Let $V_k$ be a $k$ dimensional variety. Let $k_d = \binom{k+d}{k} - 1$. Obviously

$$\dim \left(T^d(V_k, p)\right) \leq \min(n, k_d).$$

If, for general $p$,

$$\dim T^d(V_k, p) = k_d - \delta,$$

with $d \geq 2$, we say that $V_k$ satisfies $\delta$ Laplace equations of order $d$.

Let $I = (i_1, \ldots, i_k)$ be a multiindex, that is a $k$-tuple of non negative integers. We shall denote by $|I|$ the sum of the components of $I$, i.e. $|I| = i_1 + \cdots + i_k$. If $x_1, \ldots, x_k$ are variables, we shall denote by $\underline{x}^I$ the monomial

$$\underline{x}^I = x_1^{i_1} \cdot \cdots \cdot x_k^{i_k}.$$

If $\underline{x}(t_1, \ldots, t_k) = (x_1(t_1, \ldots, t_k), \ldots, x_n(t_1, \ldots, t_k))$ is a vector function, we shall denote by $x_I$ the partial derivative

$$\underline{x}_I = \frac{\partial^{|I|}\underline{x}(t_1, \ldots, t_k)}{\partial t_1^{i_1} \cdots \partial t_k^{i_k}}.$$

Let $V_k$ be a variety which satisfies $\delta$ Laplace equations of order $d$ and let $\underline{x}(t_1, \ldots, t_k)$ be any local parametrization of $V_k$. $V_k$ is locally given by the set of equations:

$$\sum_{0 \leq |I| \leq d} A_I^{(h)} \underline{x}_I = 0 \qquad h = 1, \ldots, \delta. \tag{1}$$

We say that $V_k$ represents the system of differential equations (1), or that $V_k$ is an integral variety for it.

REMARK 1.2. Since, for general $p \in V_k$,

$$k_d - \delta = \dim T^d(V_k, p) \geq \dim T(V_k, p) = k$$

then

$$\delta \leq k_d - k.$$

If $n < k_d$, then $V_k \subseteq \mathbf{P}^n$ represents at least $k_d - n$ Laplace equations of order $d$. These Laplace equations are called trivial.

We shall denote by $\mathcal{L}_{k,r} := |\mathcal{O}_{\mathbf{P}^k}(r)|$ the complete linear system of hypersurfaces of degree $r$ in $\mathbf{P}^k$. Its dimension is $k_r$. By

$$v^*(k, r) \colon \mathcal{L}_{k,1} \to \mathcal{L}_{k,r}$$

we denote the dual Veronese map, defined by:

$$v^*(k, r)(H) = rH$$

The image of $v^*(k,r)$, denoted by $V^*(k,r)$, is projectively equivalent, on the complex field, to the usual Veronese variety $V(k,r)$.

A linear subspace $\mathcal{L} \subseteq \mathcal{L}_{k,r}$ represents a linear system of hypersurfaces of degree $r$. Among them we have the linear systems consisting of all reducible hypersurfaces of type $(r-d)H + D$, where $H$ is a fixed hyperplane and $D$ is a variable hypersurface of degree $d$. We denote these linear systems by $(r-d)H + \mathcal{L}_{k,d}$. The following theorem is well known:

**Theorem 1.3.**

$$T^d(V^*(k,r), rH) = (r-d)H + \mathcal{L}_{k,d}.$$

Proof. See [7]. □

**Corollary 1.4.** *For all $d \leq r$, we have*

$$\dim T^d(V^*(k,r), rH) = \dim \mathcal{L}_{k,d} = k_d.$$

Corollary 1.4 is equivalent to the fact that $V^*(k,r)$ does not satisfy any Laplace equation of order $d$, for $d \leq r$.

It is very convenient to fix a homogeneus coordinate system in $\mathbf{P}^k$, $\{x_o \ldots x_k\}$ and a dual homogeneous coordinate system $\{u_o \ldots u_k\}$. We associate to a linear system $\mathcal{L} \subseteq \mathcal{L}_{k,r}$ of hypersurfaces of degree $r$ its apolar system, denoted by $Ap(\mathcal{L})$ and defined as follows: a hypersurface of equation $\sum (r!/I!) a_I x^I = 0$ and degree $r$ belongs to $Ap(\mathcal{L})$ if and only if $\sum (r!/I!) a_I b_I = 0$ for all hypersurfaces $\sum (r!/I!) b_I x^I = 0$ belonging to $\mathcal{L}$.

Let $\rho = \langle \Gamma_0, \ldots, \Gamma_t \rangle$ be a subspace of $\mathcal{L}_{k,r}$ and let $V^*_\rho(k,r)$ be the projection of $V^*(k,r)$, from $\rho$. When $k$ and $r$ are fixed we shall denote $V^*_\rho(k,r)$ by $V_\rho$ for simplicity. Let $Ap(\rho) = \langle C_0, \ldots, C_{k_r - t - 1} \rangle$. The rational map:

$$\mathcal{L}_{k,1} \dashrightarrow V^*_\rho(k,r) \subseteq \mathbf{P}^{k_r - t - 1},$$

can be written in a suitable local coordinates system, in the following form:

$$x_0, \ldots, x_k \to f_0(x), \ldots, f_{k_r - t - 1}(x),$$

where $f_i(x)$ is a homogeneous equation for $C_i$.

DEFINITION 1.5. Let $\rho \subseteq \mathbf{P}^n$ be an $s$-plane. We denote by $V_\rho$ the projection of a variety $V \subseteq \mathbf{P}^n$ to $\mathbf{P}^{n-s-1}$ from $\rho$.

**Lemma 1.6.** *Let $\rho$ be an $s$-plane of $\mathbf{P}^n$, let $\pi : \mathbf{P}^n \dashrightarrow \mathbf{P}^{n-s-1}$ be the projection from $\rho$ and let $P$ be a general point of $V$. Then $T^d(V_\rho, \pi(P))$ is the projection of $T^d(V,P)$, from $\rho$.*

Proof. See [7]. □

**Corollary 1.7.** *Let $V_k \subseteq \mathbf{P}^n$ be a variety satisfying $t$ Laplace equations of order $d$ and let $\rho \subseteq \mathbf{P}^n$ be an $s$-plane intersecting the general $d$-osculating space to $V$ in a space of dimension $\delta$. Then $V_\rho$ satisfies $t+\delta+1$ Laplace equations of order $d$.*

For $V = V^*(k,r)$, linear spaces $\rho \subseteq \mathcal{L}_{k,r}$ can be identified with linear systems of hypersurfaces of degree $r$. Then:

**Proposition 1.8.** *$V^*_\rho(k,r)$ satisfies $1+\epsilon$ Laplace equations of order $d$, if and only if the linear system $\rho$ satisfies the following condition $(\epsilon, d)$:*

*For the general hyperplane $H \subseteq \mathbf{P}^k$, $\rho$ contains $(1+\epsilon)$ linearly independent hypersurfaces of the linear system*

$$(r-d)H + \mathcal{L}_{k,d}.$$

We define $\rho$ a Togliatti system.

REMARK 1.9. Let $V_k \subseteq \mathbf{P}^n$ be a rational variety of dimension $k$. Then $V_k$ is obtained by projecting a suitable Veronese variety $V^*(k,r)$ from a linear subspace $\rho \subseteq \mathcal{L}_{k,r}$. By Theorem 1.3, the condition $(\epsilon, d)$ is equivalent to the fact that $\rho$ intersects each osculating space to $V^*(k,r)$ in $(1+\epsilon)$ independent points. Since every rational variety is obtained by projecting a suitable Veronese variety, then the problem to classify rational varieties which satisfy Laplace equations is equivalent to the problem of classifying linear spaces in special position with respect to $\mathrm{Tan}^d(V^*(k,r))$.

Since we are interested in varieties satisfying $1+\epsilon$ non trivial Laplace equations of order $d$, we can always assume that:

$$\dim\rho \leq k_r - k_d - 1 + \epsilon. \tag{2}$$

**Theorem 1.10.** *Let $\rho \subseteq \mathcal{L}_{k,r}$ be a $s$-plane satisfying the condition $(\epsilon, d)$. Then $\dim\rho \geq k+\epsilon$.*

Proof. See [7]. □

In the following section we shall be mainly concerned with the cases where $d=2$ and $k=2$.

There are two natural problems to consider.

The first one is to find sharp bounds for the dimension of a Togliatti system for given values of $k, r, d, \epsilon$. For example if $r = 3, d = 2, \epsilon = 0$, we have:

$$(*) \qquad k \leq \dim \rho \leq k_3 - k_2 - 1.$$

The second problem is to classify all possible Togliatti systems of given dimension, at least when the dimension is close to the bounds. We proved [7] that:

i) for $r = 3, d = 2, \epsilon = 0$, if $\dim\rho = k$, then $\rho = Q + \mathcal{L}_{k,1}$;

ii) for $r = 3, d = 2, \epsilon = 0, k > 2$, if $\dim \rho = k + 1$, then $\rho = \langle Q + \mathcal{L}_{k,1}, C\rangle$, where $C$ is a cubic;

iii) if $r \geq 4, d = 2, \epsilon = 0$, then $\dim \rho \geq k + 2$.

We remark that all the ruled surfaces in $\mathbf{P}^r$ (also not developable) satisfy a Laplace equation. In fact the ruled surfaces have parametric representation $x_i = a_i(v) + ub_i(v)$ $(i = 0, \dots, r)$.

REMARK 1.11. We remember that:

1) $V^*_\rho(2, r)$ satisfies one Laplace equation of order two if and only if for the general line $L \subseteq \mathbf{P}^2$, $\rho$ contains one and only one hypersurface of the linear system $(r - 2)L + \mathcal{L}_{2,2}$.

Let we consider a rational surface $V_2$, not ruled, representing a Laplace equation, belonging to $\mathbf{P}^n$ with $n \geq 5$. It is a projection of $V(2, r)$, projection by a space of dimension $\rho = N - n - 1$ on $\mathbf{P}^n$, with $N = r(r + 3)/2$. In the general point $A$ of $V_2$ the second osculating space has dimension four.

If we consider the linear system of hyperplanes of $\mathbf{P}^n$ containing $T^2(V_2, A)$, every hyperplane intersects $V_2$ in a curve having in $A$ a triple point. If we consider the linear system of plane curves $C^r$ that represents $V_2$ we have, in the point $A' \in \mathbf{P}^2$ corresponding to $A$, a curve having in $A'$ a triple point.

2) For the general point $A' \in \mathbf{P}^2$ there exists a curve of degree $r$ having in $A'$ a triple point.

The two points of view are equivalent (see [10] for the proof).

## 2. Rational surfaces linearly 2-normal in $\mathbf{P}^n$

DEFINITION 2.1. A variety $V_k \subseteq \mathbf{P}^n$ , is *linearly* 2-*normal* if and only if:

1) $V_k$ verifies a not trivial Laplace equation of order two.

2) $V_k$ is not projection of a variety that satisfies a Laplace equation of order two.

An interesting problem is to classify linearly 2-normal rational surfaces. An example is Togliatti triangle. In [9] Togliatti proves that this is the unique example, among the linearly 2-normal surfaces of degree $\leq 6$, to be smooth in $\mathbf{P}^5$.

Let $S$ be a surface, let $|D|$ be a complete linear system of curves on $S$, and let $P_1, \dots, P_r$ be points of $S$. Then we will consider the sublinear system $\delta$ consisting of divisors $D \in |D|$ which pass through the points $P_1, \dots, P_r$ and we denote it by $|D - P_1 - \cdots - P_r|$. We say that $P_1, \dots, P_r$ are the assigned base points of $\delta$.

We consider now $\pi : S' \to S$ the morphism obtained by blowing up $P_1, \dots, P_r$, and let $E_1, \dots, E_r$ be the exceptional curves; then there is a natural one-to-one correspondence between the elements of $\delta$ on $S$ and the elements of the complete linear system $\delta' = |\pi^* D - E_1 - \cdots - E_r|$ on $S'$ given by $D \to \pi^* D - E_1 - \cdots - E_r$, because the latter divisor is effective on $S'$ if and only if $D$ passes through $P_1, \dots, P_r$.

The new linear system $\delta'$ on $S'$ may or may not have base points. We call any base point of $\delta'$, considered as an infinitely near point of $S$, an unassigned base point of $\delta$.

These definitions also make sense if some of the $P_i$ themselves are infinitely near points of $S$, or if they are given with multiplicities greater than 1. If, for example, $P_2$ is infinitely near $P_1$ (we say also that $P_1$ and $P_2$ coincide), then for $D \in \delta$ we require that $D$ contains $P_1$, and that $\pi^* D - E_1$ contain $P_2$, where $\pi_1$ is the blowing-up of $P_1$. On the other hand, if $P_1$ is given with multiplicity $r \geq 1$, then we require that $D$ have at least an $r$-fold point at $P_1$, and in the definition of $\delta'$, we take $\pi^* D - rE_1$, (for more details see [6]).

Let $|\mathcal{L}| = \mathcal{L}_{2,r} := |\mathcal{O}^2_{\mathbf{P}}(r)|$ be the system of all the curves of degree $r$ in $\mathbf{P}^2$.

Let $\mathcal{L}_{2,3}(-P_1 - \cdots - P_r)$ be the linear system of plane cubic curves with assigned (ordinary) base points $P_1, \dots, P_r$, and assume that no three of the $P_i$ are collinear, and no six of them lie on a conic (i.e. the points are in generic position). Let $\pi : X \to \mathbf{P}^2$ be the morphism obtained by blowing up $P_1, \dots, P_r$. For each $r = 0, 1, \dots, 6$, we obtain an embedding of $X$ in $\mathbf{P}^{9-r}$ as a surface of degree $9 - r$, hose canonical sheaf $\omega_X$ is isomorphic to $\mathcal{O}_X(-1)$.

A Del Pezzo surface is defined to be a surface $X$ of degree $d$ in $\mathbf{P}^d$ such that $\omega_X \cong \mathcal{O}_X(-1)$. So the previous construction gives Del Pezzo surfaces of degrees $d = 3, 4, \dots, 9$.

A classical result states that every Del Pezzo surface is either one given by the previous construction for a suitable choice of points $P_i \in \mathbf{P}^2$, or the 2-uple embedding of a quadric surface in $\mathbf{P}^3$, which is a Del Pezzo surface of degree eight in $\mathbf{P}^8$.

We remark:

The not ruled irreducible surfaces $V^n$, of degree $n$, with elliptic sections are rational and are projections of the *Del Pezzo surfaces*.

We study this class of surfaces. First of all we examine the not ruled surfaces of degree seven in $\mathbf{P}^5$ whose hyperplane sections have genus one in $\mathbf{P}^5$ and which satisfy one Laplace equation.

**Theorem 2.2.** *Let $V^7$ be the image of $\mathbf{P}^2$ in $\mathbf{P}^5$, under the embedding defined by the linear system $\mathcal{M} \subset \mathcal{L}_{2,3}(-P - Q)$. If $P$ and $Q$ do not coincide the surface is not ruled. It doesn't verify a Laplace equation.*

Proof. Let $V^7 \subseteq \mathbf{P}^5$ be a surface of degree seven in $\mathbf{P}^5$ whose hyperplane sections have genus one in $\mathbf{P}^5$ and which satisfy one Laplace equation. $V^7$ is the image of $\mathbf{P}^2$ in $\mathbf{P}^5$ embedded by a subsystem of dimension five of $\mathcal{L}_{2,3}(-P-Q)$. Let $\mathcal{M}$ be this system.

$P$ is a base point of $\mathcal{M}$; hence the lines for $P$ are images of a pencil of conics, let $S$ be, of $V^7$. Let $A$ be a general point of $V^7$. We consider $T^2(V^7, A)$, it is a linear subspace of dimension four and cuts $V^7$ in a curve $L$ having in $A$ a triple point. The image on $\mathbf{P}^2$ is a cubic with a triple point in the image $A'$ of $A$. We have that $C_{A'}$ is made by three lines containing $A'$. Hence the cubic contains as a part the line $PA'$, we have $C_{A'} = \langle P, A' \rangle + R_{A'}$, where $R_{A'}$ is a conic through $A'$. Hence the curve $L$ contains the conic of the pencil $S$, $\Gamma_A$, that passes through $A$. We consider the linear system residual to the pencil $S$ with respect to the system of hyperplane sections. Let $\mathcal{F}$ be. Let $\mathcal{F}'$ be the corresponding net of conics. For each $A' \in \mathbf{P}^2$, $R_{A'} \in \mathcal{F}'$. But, $R_{A'}$ is singular at $A'$, hence $\mathcal{F}'$ is a net whose jacobian is not determined. Therefore $\mathcal{F}'$ is composed with a pencil, with base point $Q$. Hence through the general point of the surface, we find a hyperplane section of degree six, made by three conics. Contradiction.

If $P$ and $Q$ coincide, the surface is ruled, because the image of the line $\langle P, A' \rangle$ in the surface is a line. $\square$

We consider now the not ruled surfaces of degree 8 in $\mathbf{P}^5$ whose hyperplane sections have genus one in $\mathbf{P}^5$ and which satisfy one Laplace equation.

We have to consider two cases:

1) $\mathcal{M} \subset \mathcal{L}_{2,3}(-P)$.
2) $\mathcal{M} \subset \mathcal{L}_{2,4}(-2P-2Q)$.

Now we examine the first one.

**Theorem 2.3.** *Let $V^8$ be the image of $\mathbf{P}^2$ in $\mathbf{P}^5$, under the embedding defined by the linear system $\mathcal{M} \subset \mathcal{L}_{2,3}(-P)$. It satisfies a Laplace equation.*

Proof. For the general $A' \in \mathbf{P}^2$, $C_{A'}$ is made by the line through $P$ and $A'$ and a residual singular conic $R_{A'}$. $R_{A'} \in \mathcal{F}'$ (by the previous theorem). We prove that $\mathcal{F}'$ is composed with a pencil. There are two possibilities, either $\mathcal{F}'$ consists of reducible conics, because the system $|R_{A'}|$ has dimension two, or the system $|R_{A'}|$ has dimension one. Also in the second case $\mathcal{F}'$ consists of reducible conics, because each pencil in $\mathcal{F}'$ contains two double lines. Hence $\mathcal{F}'$ is composed with a pencil.

We have two possibilities: the base point of the pencil is $P$, the base point of the pencil is $Q$, different from $P$. If the base point of the pencil is $P$, as before, for the general point of the surface, we find a hyperplane section of degree 6, contradiction. If the base point of the pencil is $Q$, different from $P$, we can find the suitable linear system $\mathcal{M}$. Assuming $P = (1, 0, 0)$, $Q = (0, 1, 0)$, $\mathcal{M}$ has the following generators

$$x_3^3, x_1^2x_3, x_1x_3^2, x_2^3, f(x_1, x_2, x_3), g(x_1, x_2, x_3), \tag{3}$$

where $f$ or $g$ is smooth and $f$ and $g$ passes through $P$ and for other 8 points not on the line $\langle P, Q\rangle$. The $V^8$ corresponding to this linear system satisfies our conditions, i.e. satisfies one Laplace equation and is not ruled because the lowest degree curves it contains are conics. □

Now let us consider the second case:

**Theorem 2.4.** *Let $V^8$ be the image of $\mathbf{P}^2$ in $\mathbf{P}^5$, under the embedding defined by the linear system $\mathcal{M} \subset \mathcal{L}_{2,4}(-2P - 2Q)$. It doesn't verify a Laplace equation.*

Proof. Suppose that $P$ and $Q$ do not coincide. $P$ is a double base point of $\mathcal{M}$; hence the lines for $P$ are images of a pencil of conics, let $S$ be. Let $A$ be a general point of $V^8$. We consider $T^2(V^8, A)$, it is a $\mathbf{P}^4$ and cuts $V^8$ in a curve $L$ having in $A$ a triple point. The image on $\mathbf{P}^2$ is $C_{A'} = \langle P, A'\rangle + R_{A'}$. Hence $L$ contains the conic of the pencil $S$, $\Gamma_A$, that passes through $A$. We consider the linear system linked to the pencil $S$ with respect to the system of hyperplane sections. Let $\mathcal{F}$ be. Let $\mathcal{F}'$ be the corresponding net of cubics. $\mathcal{F}' \subset \mathcal{L}_{2,3}(-P - 2Q)$. For each $A' \in \mathbf{P}^2$, $R_{A'} \in \mathcal{F}'$. But, $R_{A'}$ is singular at $A'$, hence $\mathcal{F}'$ is a net of cubics whose jacobian is not determined. Therefore, removed a possible fixed line, it is composed with a pencil, and the pencil is a pencil of lines. The line $\langle P, Q\rangle$ is a fixed component of $\mathcal{F}'$, and the pencil is the pencil of lines through $Q$. Through the general point of the surface, we find a hyperplane section of degree six. Contradiction.

If $P$ and $Q$ coincide we have a contradiction, because for the general point $A' \in \mathbf{P}^2$ doesn't exist a curve having in $A'$ a triple point (i.e. cubic becomes curve). Let us consider the blow-up $\pi : \tilde{\mathbf{P}}^2 \to \mathbf{P}^2$ of the plane $\mathbf{P}^2$ at $P$. Let $E$ be the exceptional divisor corresponding to the blown-up point $P$. Consider a quartic $C \in \mathcal{M} \subset \mathcal{L}_{2,4}(-2P - 2Q)$. We consider the blow-up of the plane $\mathbf{P}^2$ at $P$. The strict transform $\tilde{C}$ has a double point $Q \in E$. We can consider the line $r$ passing through $P$, with the tangent direction given by $Q$. By Bezout theorem, $C_{A'}$ is made by the line $\langle P, A'\rangle$ and a cubic $R_{A'}$. $P$ must be a double point for $R_{A'}$ and, by Bezout theorem again, $R_{A'}$ contains the line through $P$ and $A'$. Hence we have $C_{A'} = 2\langle P, A'\rangle + 2r$, contradiction. □

We consider now the not ruled surfaces of degree 9 in $\mathbf{P}^5$ whose hyperplane sections have genus one in $\mathbf{P}^5$ and which satisfy one Laplace equation.

**Theorem 2.5.** *Let $V^9$ be the general projection of the general Del Pezzo surface of degree nine in $\mathbf{P}^9$, corresponding to the choice of a general subsystem $\mathcal{M}$ of $\mathcal{L}_{2,3}$*

*of dimension five. Then $V^9$ is linearly 2-normal.*

Proof. We can construct a general system without base point, i.e. you can add general. A linear system $\mathcal{M}$, without base points, is generated by the curves:

$$x_1^3, x_2^3, x_1x_2^2, x_1^2x_2, f(x_1, x_2, x_3), g(x_1, x_2, x_3), \tag{4}$$

where $f$ is any smooth cubic not containing the point $(0, 0, 1)$. Hence $\mathcal{M}$ is a system without base points and the general member of the system $\mathcal{M}$ is smooth. In this case the surface cannot be ruled, in fact the curves of minimal degree are curves of degree three. The surface verifies the Remark 1.11, hence satisfies a Laplace equation of order two. □

These examples belong to the class of examples given in ([7], example 2, p.133). For the discussion in [7], Section 4, this example is linearly 2-normal and there are no other examples of this type in this class.

## 3. The case $k = 3$, $r = 3$, $d = 2$, $\epsilon = 0$, the monomial examples

The first step in the project of classifying the linearly 2-normal rational 3-folds, for $r = 3, d = 2, \epsilon = 0$, is to consider the three-folds given by monomial equations.

Among Togliatti systems of cubics, one can consider those systems obtained by adding some cubics to a Togliatti system of lower dimension. I shall call these systems enlarged. The variety $V_\rho$ obtained by projecting the Veronese variety from an enlarged Togliatti system $\rho = \langle \rho', C_1, \dots, C_s \rangle$ can be obtained by projection of the variety $V_{\rho'}$ which already satisfies the same number of Laplace equations. A variety $V_\rho$ is called linearly 2-normal if and only if $\rho$ is not enlarged (and does not intersect any tangent space to $V$).

DEFINITION 3.1. A Togliatti system is called *monomial* if it is spanned by hypersurfaces whose equation are monomial. The apolar system of a monomial system gives a monomial map.

DEFINITION 3.2. A map:

$$\phi \colon \mathbf{C}^n \to \mathbf{P}^t$$

is called *monomial* if there exists a subset $V = \{\mathbf{m}_0, \dots, \mathbf{m}_t\} \subseteq \mathbf{Z}_{\geq 0} \times \cdots \times \mathbf{Z}_{\geq 0}$ ($n$ times) s.t. $\phi(\mathbf{x}) = (\mathbf{x}^{\mathbf{m}_0}, \dots, \mathbf{x}^{\mathbf{m}_t})$ (where $\mathbf{x}^{\mathbf{m}_i} = \left(x_1^{m_i^1} \cdots x_n^{m_i^n}\right)$, $\forall i = 0, \dots, t$).

It is easy to prove (see [8]) the following theorem:

**Theorem 3.3.** *Let $\phi\colon \mathbf{C}^n \to \mathbf{P}^t$ be a monomial map and let $V$ be the subset associated to $\phi$ as in* (3.2).

*Then $\phi(\mathbf{C}^n)$ satisfies $k$ linearly independent Laplace equations of order $d$ if and only if there exist $k$ linearly independent hypersurfaces of degree $d$ in $\mathbf{C}^n$ containing $V$.*

REMARK 3.4. By 3.2 a monomial map satisfies a Laplace equation of order 2 if and only if the exponents of the monomial map lie on a quadric.

REMARK 3.5. Let $\phi\colon \mathbf{C}^n \to \mathbf{P}^t$ be a monomial map. Let $\diamond$ be the maximal bounded convex polyhedron generated by $\{\mathbf{m}_0, \ldots, \mathbf{m_t}\}$ and let $X_\diamond$ be the closure of the image of $\phi(\mathbf{C}^n)$. $X_\diamond$ is not singular if and only if $\diamond$ is simple, (i.e. each vertex of $\diamond$ is incident to exactly $n$ edges) and at each vertex $\mathbf{m}$ of $\diamond$, $\left\{\mathbf{m}^{(1)} - \mathbf{m}, \ldots, \mathbf{m}^{(n)} - \mathbf{m}\right\}$ form a basis for the lattice $\mathbf{Z}^n$, where $\left\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(n)}\right\}$ are the lattice points on the $n$ edges incident to $\mathbf{m}$ and closest to $\mathbf{m}$ itself.

For more details see [8].

We remember that we are considering the case $k = 3$, $r = 3$, $d = 2$, $\epsilon = 0$. In this case the results are summarized in the following table, where we recall the results according to the dimension of $\rho$ (recall that, p.8, $3 \le \dim \rho \le 9$).

i) If $\dim \rho = 3$, then $\rho$ must be equal to $Q + \Delta_1$.

ii) If $\dim \rho = 4$, then $\rho$ is enlarged.

We have found all the monomial Togliatti systems with a computer program. This gives only five projectively distinct, not enlarged, monomial examples, for $\dim(\rho) = 5$, and fifteen for $\dim(\rho) = 6$. If $\dim \rho \le 7$ there is only one example of linearly 2-normal smooth 3-fold. With the same notations of the first paragraph, the Togliatti system is the following:

$$\rho = \left\langle x_1^3, x_2^3, x_3^3, x_4^3, x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4 \right\rangle.$$

The Laplace equation is:

$$\begin{aligned}&\xi_1^2/2\theta_{\xi_1\xi_1} + \xi_2^2/2\theta_{\xi_2\xi_2} + \xi_3^2/2\theta_{\xi_3\xi_3} + \xi_1\xi_2/2\theta_{\xi_1\xi_2} + \xi_1\xi_3/2\theta_{\xi_1\xi_3} + \xi_2\xi_3/2\theta_{\xi_2\xi_3}\\ &- \xi_1\theta_{\xi_1} - \xi_2\theta_{\xi_2} - \xi_3\theta_{\xi_3} + \theta = 0,\end{aligned}$$

where $\xi_i = x_i/x_4$, $\forall i = 1, 2, 3$.

This example is smooth. The polyhedron, associated to our example is the truncated tetrahedron, discussed in [8, p.15 (3)]. In [8] there are classified all the polyhedra corresponding to monomial maps from $\mathbf{C}^3$ whose image is smooth and satisfies one Laplace equation of order two.

It is easy to check that the truncated tetrahedron is the only polyhedron which can be isometrically embedded in the simplex with vertexes $(0, 0, 0)$, $(3, 0, 0)$, $(0, 3, 0)$,

$(0, 0, 3)$ and therefore it is the only smooth monomial example which can be obtained from $V^*(3, 3)$ for projection.

**Conjecture.** *This example can be generalized for all* $k$*, for* $r = 3, d = 2, \epsilon = 0$*, and we can say that if* $\dim\rho \leq \binom{k+2}{2} - 3$ *there is only an example of smooth projection in* $\mathbf{P}^{k(k+1)-1}$ *of* $V^*(k, 3)$*. It is projectively equivalent to the monomial map of the truncated* $(k + 1)$*-hedron.*

This conjecture is interesting because it is the natural generalization of the Togliatti example, known as Togliatti triangle ([4], [7]).

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