THE CLOSURE OF A SURFACE BRAID REPRESENTED BY A 4-CHART WITH AT MOST ONE CROSSING IS A RIBBON SURFACE

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Abstract

We show that the closure of a surface braid represented by a 4-chart with at most one crossing is a ribbon surface.

1. Introduction

Kamada [3] gave a method to describe a surface braid by an oriented labeled planar graph, called a chart, and investigated modifications of charts which induce ambient isotopies of the closure of surface braids, represented by charts, in $\mathbb{R}^4$. These modifications are called C-moves.

A surface braid of index n is represented by an n-chart whose edges are of label i with $1 \leq i < n$. The closure of a surface braid represented by a 3-chart is a ribbon surface [3].

In this paper we shall extend the Kamada's result. Namely we shall show that the closure of a surface braid represented by a 4-chart with at most one crossing is a ribbon surface.

An *n-chart* is an oriented labeled planar graph, which may be empty or have closed edges without vertices called *hoops*, satisfying the following four conditions (see Fig. 1):

(1) Every vertex has degree $1, 4$, or 6.

(2) The labels of edges are in $\{1, 2, \ldots, n-1\}$.

(3) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled i and $i + 1$ alternately for some i, where the orientation and label of each arc are inherited from the edge containing the arc.

(4) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i - j| > 1$.

A vertex of degree $1, 4$ and 6 is called a *black vertex*, a *crossing*, and a *white vertex* respectively.
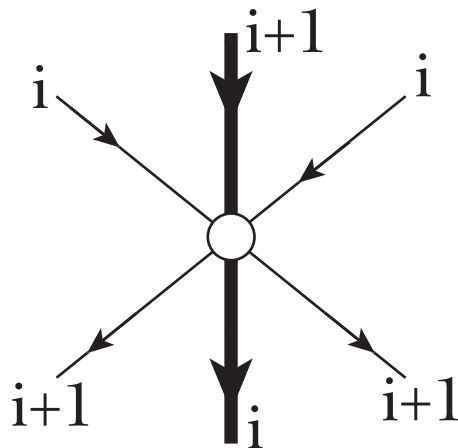
Among six short arcs in a small neighborhood of a white vertex, a middle arc of each consecutive three arcs oriented inward or outward is called a *middle arc* of the

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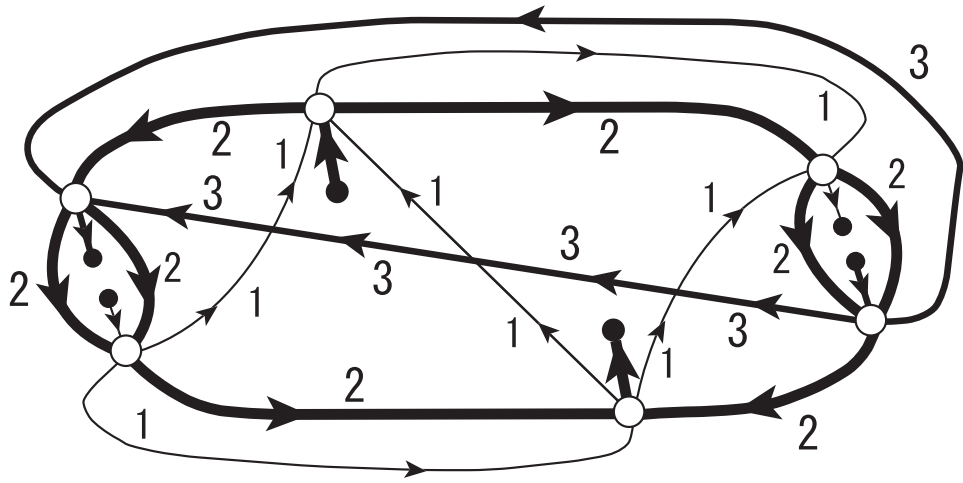
○ White vetex   ● Black vertex

The numbers in the figure are labels

Fig. 1.

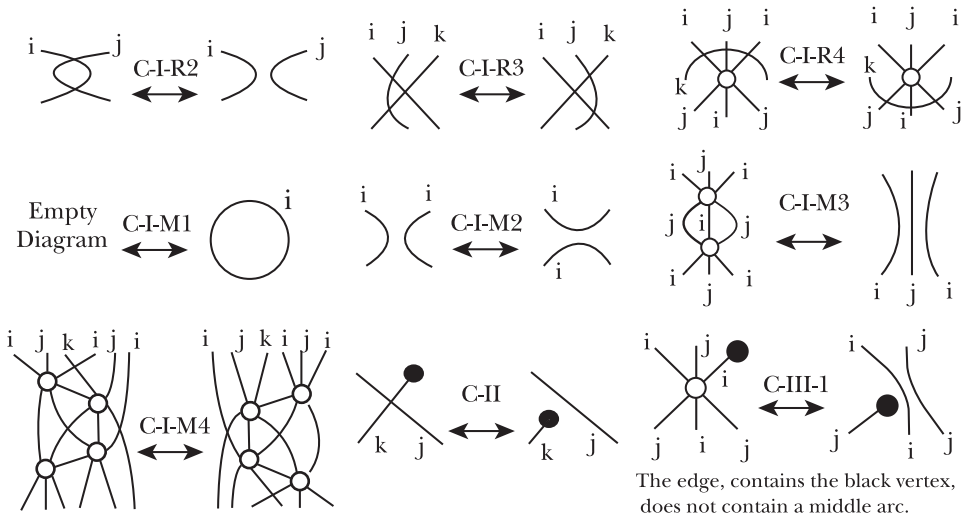The heavier arcs are middle arcs.

Fig. 2.

Fig. 3.

white vertex (see Fig. 2). There are two middle arcs in a small neighborhood of each white vertex. A middle arc of odd label is called a *distinguished arc*. Around each white vertex, there exists only one distinguished arc.

NOTE. (1) For each chart, the number of distinguished arcs is equal to the number of white vertices. *Exploiting this fact is the main idea of this paper*.

An edge is called a *free edge* if its two vertices are black vertices. An edge is called a *terminal edge* if it contains one black vertex and one white vertex. An edge is called a *loop* if it contains only one vertex.

A *C-move* is a local modification of a chart in a disk as shown in Fig. 3 (see [3], [2] for the precise definition). We often use C-I-M2 moves and C-III-1 moves. Two charts are *C-move equivalent* if there exists a finite sequence of C-moves which turns one of the two charts into the other.

For each chart Γ, let $c(\Gamma)$, $w(\Gamma)$, and $f(\Gamma)$ be the number of crossings, the number of white vertices, and the number of free edges respectively. The triad $(c(\Gamma), w(\Gamma), -f(\Gamma))$ is called the *complexity* of the chart. A chart is called a *minimal* chart, if its complexity is minimal among the charts C-move equivalent to the chart with respect to the lexical order of triads of integers.

NOTE. (2) Any terminal edge in a minimal chart contains a middle arc of its white vertex.

Our main theorem is the following.

Main Theorem.   *Any minimal* 4-*chart with at most one crossing does not have a white vertex.*

The closure of a surface braid is a *ribbon surface* if and only if it is ambient isotopic to the closure of a surface braid represented by a chart without white vertices [3]. Thus, our theorem says that the closure of a surface braid represented by a 4-chart with at most one crossing is a ribbon surface.

To make the argument simple, we assume that *the charts lie on the* 2-*sphere instead of the plane.*

For each 4-chart Γ, let Γ' be the graph obtained from the chart Γ by omitting the hoops, the free edges. A complementary domain of a connected component of the graph Γ' is called a *room*. Choose a connected component G of the graph Γ'. Let G' be the subgraph of G that consists of the set of all the edges of label 2 and all their vertices. A complementary domain of the graph G' is called a *house*. A house containing a crossing is called a *special house*. A *non-special* house does not contain any crossing. The chart in Fig. 1 has 14 rooms and 4 houses one of which is a special house with 3 crossings.

Terminology.   (1) If a vertex or an edge is contained in the closure of a room or a house, then we say that the vertex or the edge *belongs* to the room or the house, or that the room or the house *possesses* the vertex or the edge.
(2) *Vertices* or *edges* are words for charts. But *points* and *sets* are not words for charts.

Now the following is the outline of the proof of Main Theorem.

We suppose that there exists a minimal 4-chart with a white vertex and at most one crossing. If each connected component of the chart represents a ribbon surface, then the 4-chart represents a ribbon surface. Thus, to prove Main Theorem by contradiction, we take a connected 4-chart with at most one crossing and *minimal number of white vertices* among all 4-charts which do not represent ribbon surface. This means that the chart satisfies the following condition.

Connectedness Condition.   If a 4-chart is C-move equivalent to the 4-chart above, and if it has the same complexity as the 4-chart above, then it is never disconnected if we ignore hoops and free edges.

Proposition 1.   *For any minimal* 4-*chart, any non-special house possesses no terminal edges of label* 2.

Proposition 2.   *For any minimal* 4-*chart with at most one crossing*, *the special house does not possess any terminal edge of label* 2.

Proposition 3. *For any minimal 4-chart with at most one crossing, if the chart has a white vertex but no terminal edge of label 2, then there exists a non-special house with connected boundary which possesses no distinguished arc.*

Proposition 4. *For any minimal 4-chart, if a non-special house, with connected boundary, possesses no terminal edge of label 2, then the house possesses a distinguished arc.*

Proposition 3 and Proposition 4 contradict each other. Together with Proposition 1 and 2, this means that Main Theorem has been proved.

2. Reducible triplet and proof of Proposition 1

We investigate rooms in minimal charts.

Let A be a terminal edge belonging to a room R. Note that any room is an open disk. Let X_R be the closure of the room R. Let $\bar{D}$ be a disk and $\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n$ be points on the boundary of the disk, $\partial \bar{D}$, which are situated in a counterclockwise order on the boundary of the disk. The points split the boundary of the disk into n arcs $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n$ where the boundary points of the arc $\bar{A}_i$ are $\bar{P}_i$ and $\bar{P}_{i+1}$ we understand the cyclic order $\bar{P}_{n+1} = \bar{P}_1$. Let $g \colon \bar{D} \to X_R$ be a continuous map of the disk $\bar{D}$ onto the closure X_R of the room such that the following four conditions are satisfied (see Fig. 4):

(1) The map g maps the interior of the disk $\bar{D}$ onto the room R homeomorphically; hence the map g maps the boundary of the disk onto the boundary of the room R.
(2) The restriction of the map g to the interior of the disk is orientation preserving.
(3) The map g maps the interior of each arc $\bar{A}_i$ onto the interior of an edge belongs to the room homeomorphically, where the interior of an arc means the maximal open arc contained in the arc and the interior of an edge means that the set of the points in the arc different from the vertices.
(4) $g(\bar{A}_1) = g(\bar{A}_n) = A$.
Then the set

$$\left\{ g \colon \bar{D} \to X_R \,;\; \bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n \,;\; \bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n \right\}$$

is called an *associated set* for the room R with respect to the terminal edge A.

NOTES. (3) The labels of $g(\bar{A}_k)$ and $g(\bar{A}_{k+1})$ are same if and only if $g(\bar{A}_k)$ is a terminal edge (and hence $g(\bar{A}_k) = g(\bar{A}_{k+1})$).
(4) If parities of the labels of $g(\bar{A}_k)$ and $g(\bar{A}_{k+1})$ are same and if $g(\bar{A}_k) \neq g(\bar{A}_{k+1})$, then $g(\bar{A}_k)$ and $g(\bar{A}_{k+1})$ have a common crossing.
(5) If $g(\bar{A}_k)$ does not contain any crossing, then the parities of the labels of the three edges $g(\bar{A}_{k-1})$, $g(\bar{A}_k)$, $g(\bar{A}_{k+1})$ are not same.
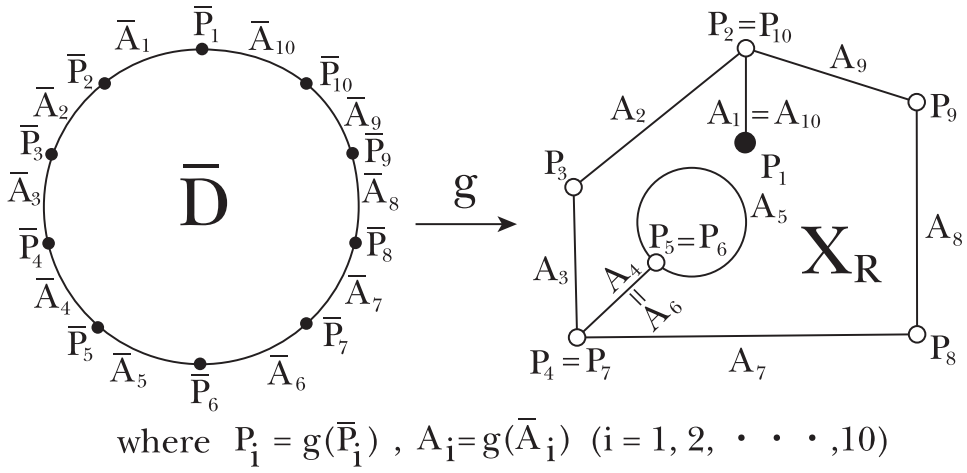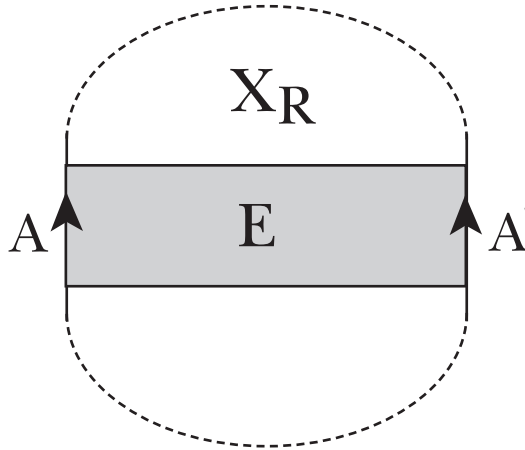
where $P_i = g(\overline{P}_i)$, $A_i = g(\overline{A}_i)$ $(i = 1, 2, \cdots, 10)$

Fig. 4.

Fig. 5.

Let X_R be the closure of a room R. Let A and A' be different edges in X_R. Then the pair of edges (A, A') is said to be *admissible with respect to a disk E* in X_R provided that the following three conditions are satisfied (see Fig. 5):

(1) The disk E does not meet any edges except the two edges A and A'.

(2) The disk E meets each of the two edges by an arc on ∂E.

(3) If we orient the disk so that the orientation of the arc $A \cap \partial E$ induced from the one of the disk coincides with the orientation induced from the one of the edge A, then the orientation of the arc $A' \cap \partial E$ induced from the one of the disk does not coincide with the orientation induced from the one of the edge A'.
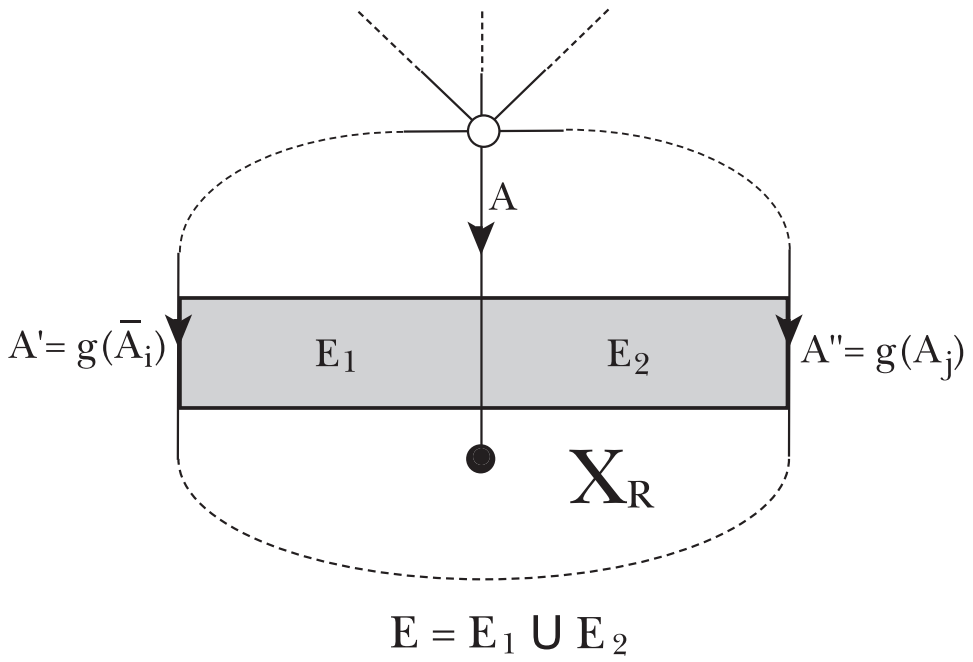
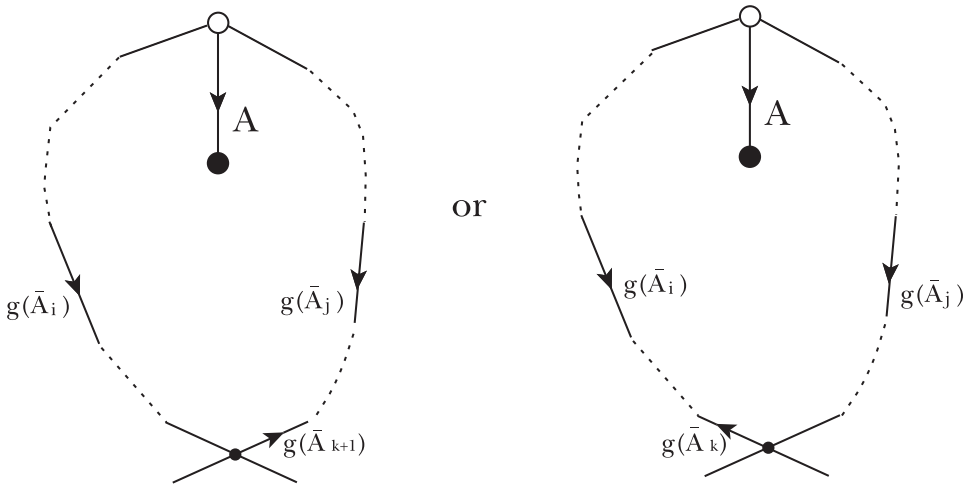$$E = E_1 \bigcup E_2$$

Fig. 6.

Fig. 7.

Let A', A, and A'' be edges belong to a room R such that A is a terminal edge of label 2, and the labels of A' and A'' are odd, where the two edges A' and A'' are possibly same. Let $\{g\colon \bar{D} \to X_R\,;\ \bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n\,;\ \bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n\}$ be an associated set for the room R with respect to the terminal edge A. Without loss of generality, we can assume that $A' = g(\bar{A}_i)$, $A'' = g(\bar{A}_j)$, and $i < j$. Then the triplet (A', A, A'') is said to be *semi-reducible with respect to a disk E* in X_R if it satisfies the following condition (1). The triplet (A', A, A'') is said to be *reducible with respect to a disk E* in X_R if it satisfies the following conditions (1) and (2).

(1) The edge A splits the disk E into two disks, say E_1 and E_2, so that the pair (A', A) is admissible with respect to one of the split disks E_1 and E_2, and the pair (A, A'') is also admissible with respect to the other split disk (see Fig. 6).

(2) If the intersection $g(\bar{A}_k) \cap g(\bar{A}_{k+1})$ is a crossing for some k with $i \leq k < j$, then the triplet $(g(\bar{A}_k), A, g(\bar{A}_{k+1}))$ is not semi-reducible (see Fig. 7).

Lemma 1. *For any minimal 4-chart*, *there is no reducible triplet*.

Proof. We prove the lemma by contradiction. Suppose that there exists a reducible triplet (A', A, A''). Let R be the room possessing the reducible triplet and let $\{g\colon \bar{D} \to X_R\,;\ \bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n\,;\ \bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n\}$ be an associated set for the room R with respect to the terminal edge A. We may assume that $g(\bar{A}_i) = A'$, and $g(\bar{A}_j) = A''$ with $i < j$. We need the following claim to prove Lemma 1.

Claim 1. *Let $(g(\bar{A}_s), A, g(\bar{A}_t))$ be a reducible triplet with $i \leq s < t \leq j$ and $t - s$ minimal. Then only one of the following three cases occurs*:

CASE 1. $g(\bar{A}_s) = g(\bar{A}_t)$, $t = s + 1$.

CASE 2. $g(\bar{A}_s) \neq g(\bar{A}_{s+1}) \neq g(\bar{A}_{s+2})$, $t = s + 2$, *and the label of $g(\bar{A}_{s+1})$ is* 2.

CASE 3. $g(\bar{A}_s) \neq g(\bar{A}_{s+1}) = g(\bar{A}_{s+2})$, $t = s + 3$, *and $g(\bar{A}_{s+1})$ is a terminal edge of label* 2.

Proof of Claim. Since $(g(\bar{A}_s), A, g(\bar{A}_t))$ is reducible, the label of A is 2 and the labels of $g(\bar{A}_s)$ and $g(\bar{A}_t)$ are odd.

First of all, the label of $g(\bar{A}_r)$ $(s < r < t)$ is 2. For, if the label of $g(\bar{A}_r)$ is odd, then, by considering the orientation of the edge $g(\bar{A}_r)$, we find that the triplet $(g(\bar{A}_s), A, g(\bar{A}_r))$ or $(g(\bar{A}_r), A, g(\bar{A}_t))$ is reducible (see Fig. 8). This contradicts the condition that $t - s$ is minimum.

Suppose $g(\bar{A}_s) = g(\bar{A}_{s+1})$. Then $g(\bar{A}_s)$ is a terminal edge by Note (3). Hence the triplet $(g(\bar{A}_s), A, g(\bar{A}_{s+1}))$ is reducible. Since $(s + 1) - s = 1 \leq t - s$, we must have $t = s + 1$. This means Case 1 occurs.

Suppose $g(\bar{A}_s) \neq g(\bar{A}_{s+1})$. If the label of $g(\bar{A}_{s+1})$ is odd, then $t = s + 1$ by the fact mentioned at the first of the proof. Since parity of $g(\bar{A}_s)$ and $g(\bar{A}_{s+1})$ is same, $g(\bar{A}_s)$ and $g(\bar{A}_{s+1})$ must have a common crossing by Note (4). This contradicts the second condition for reducible triplet. Thus the label of $g(\bar{A}_{s+1})$ must be 2. Further suppose
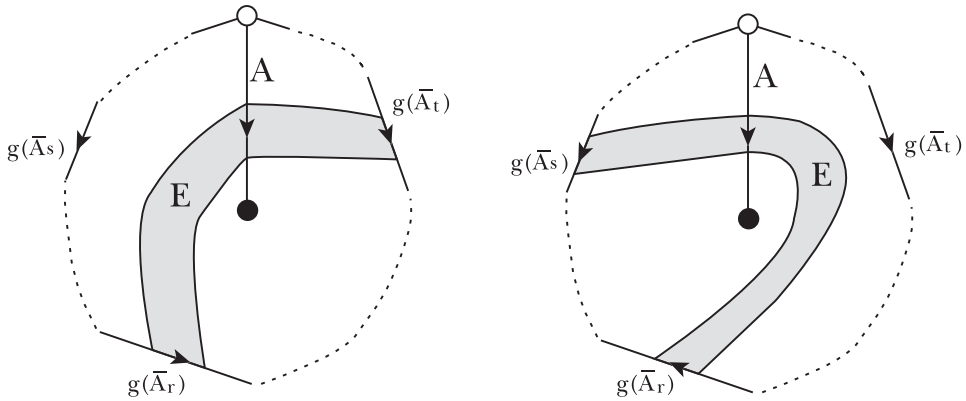
Fig. 8.

that $g(\bar{A}_{s+1}) \neq g(\bar{A}_{s+2})$. Then the label of $g(\bar{A}_{s+2})$ is odd. Therefore $t = s + 2$ by the fact mentioned at the first of the proof. This means Case 2 occurs.

Suppose $g(\bar{A}_s) \neq g(\bar{A}_{s+1}) = g(\bar{A}_{s+2})$. Then the label of $g(\bar{A}_{s+1})$ is 2. The edge $g(\bar{A}_{s+1}) = g(\bar{A}_{s+2})$ is a terminal edge by Note (3). Hence the label of $g(\bar{A}_{s+3})$ is odd. Thus $t = s + 3$ by the fact mentioned at the first of the proof. This means Case 3 occurs. Therefore Claim 1 has been proved.   □

Now continue the proof of Lemma 1. By Claim 1 it is enough to show that the three cases never occur.

Case 1. Suppose that $g(\bar{A}_s) = g(\bar{A}_{s+1})$, and that the triplet $(g(\bar{A}_s), A, g(\bar{A}_{s+1}))$ is reducible. Then the edge $g(\bar{A}_s)$ is a terminal edge by Note (3). Let v be the white vertex of the terminal edge. Then $g(\bar{A}_s) = g(\bar{A}_{s+1})$ must contain a middle arc of the white vertex v by Note (2). Hence the edge $g(\bar{A}_{s+2})$ does not contain a middle arc of v. Since the label of the edge $g(\bar{A}_{s+1})$ is odd, the label of $g(\bar{A}_{s+2})$ is 2. Thus by a C-I-M2 move between A and $g(\bar{A}_{s+2})$, we have a new terminal edge of label 2 which contains the white vertex v but does not contain a middle arc of v (see Fig. 9). By a C-III-1 move around the white vertex v, we can decrease the number of white vertices. This contradicts the minimal complexity of the chart.

Case 2. Suppose that $g(\bar{A}_s) \neq g(\bar{A}_{s+1}) \neq g(\bar{A}_{s+2})$, $t = s + 2$, and that the label of $g(\bar{A}_{s+1})$ is 2. Further suppose that the triplet $(g(\bar{A}_s), A, g(\bar{A}_{s+2}))$ is reducible. Now $(g(\bar{A}_s), g(\bar{A}_{s+1}))$ or $(g(\bar{A}_{s+1}), g(\bar{A}_{s+2}))$ is not admissible. Suppose $(g(\bar{A}_s), g(\bar{A}_{s+1}))$ is not admissible. Let $v = g(\bar{P}_{s+1})$. Since $(g(\bar{A}_s), g(\bar{A}_{s+1}))$ is not admissible, $g(\bar{A}_{s+1})$ does not contain a middle arc of v. By a C-I-M2 move between A and $g(\bar{A}_{s+1})$, we have a new terminal edge of label 2 which contains the white vertex v but does not contain a middle arc of v (see Fig. 10). We get the same contradiction as above by applying a C-III-1 move. Similar for the other case.
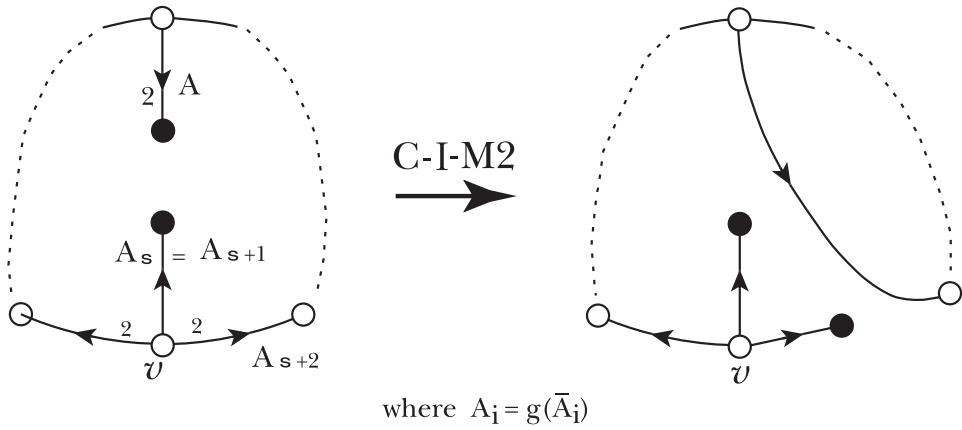
where $A_i = g(\bar{A}_i)$

Fig. 9.

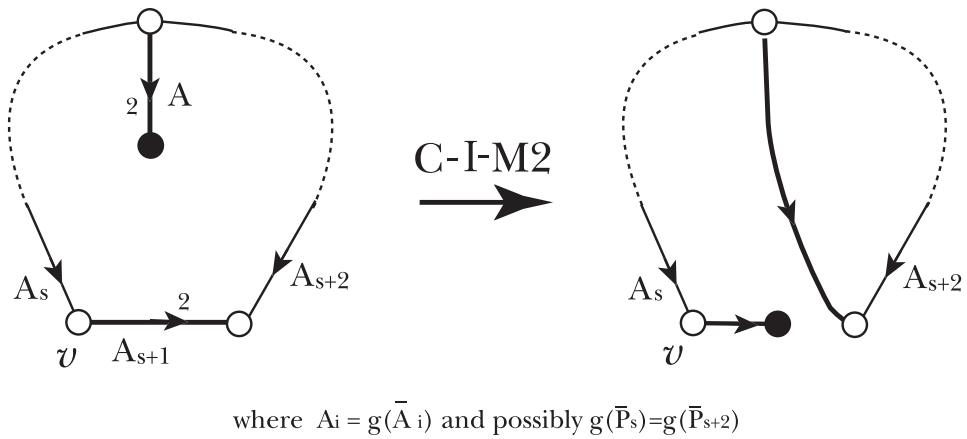where $A_i = g(\bar{A}_i)$ and possibly $g(\bar{P}_s)=g(\bar{P}_{s+2})$

Fig. 10.

Case 3. Suppose that $g(\bar{A}_s) \neq g(\bar{A}_{s+1}) = g(\bar{A}_{s+2})$, $t = s+3$. Further suppose that the edge $g(\bar{A}_{s+1})$ is a terminal edge of label 2, and that the triplet $(g(\bar{A}_s), A, g(\bar{A}_{s+3}))$ is reducible. Since the edge $g(\bar{A}_{s+1})$ is a terminal edge, it must contain the middle arc of its white vertex. Hence the triplet $(g(\bar{A}_s), g(\bar{A}_{s+1}), g(\bar{A}_{s+3}))$ is semi-reducible. Thus by a C-I-M2 move between A and $g(\bar{A}_{s+1})$, we have a new free edge without increasing the number of white vertices and crossings (see Fig. 11). This contradicts the minimal complexity of the chart.

We get a contradiction for every case. Therefore Lemma 1 has been proved. □
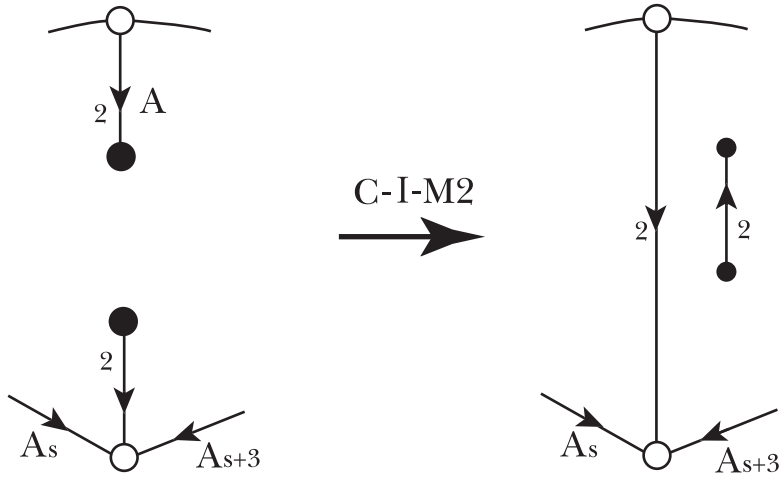
where $A_i = g(\overline{A}_i)$

Fig. 11.

Proposition 1. *For any minimal 4-chart, any non-special house possesses no terminal edges of label* 2.

Proof. Let H be a non-special house in a minimal 4-chart. Suppose that a terminal edge A of label 2 belongs to the house H. Then the terminal edge belongs to a room R in the house. Let v be the white vertex on the terminal edge. Since the chart is minimal, the terminal edge contains a middle arc of the white vertex v. Let A' and A'' be the edges of odd label belonging to the room R such that both of the two edges contain the white vertex v. Then the triplet (A', A, A'') is semi-reducible. But the house contains no crossing. Thus the triplet is reducible. This contradicts Lemma 1. Thus the non-special house does not possess any terminal edge of label 2. Therefore Proposition 1 has been proved.   □

3. Special rooms and proof of Proposition 2

A *special pair* is an admissible pair with a common crossing. A semi-reducible triplet is called a *special triplet* if it contains a special pair (see Fig. 12). A *special room* is a room possessing a special pair. A *non-special room* does not contain any special pair.

Lemma 2. *In a minimal chart, only a special room is able to possess a terminal edge of label* 2.
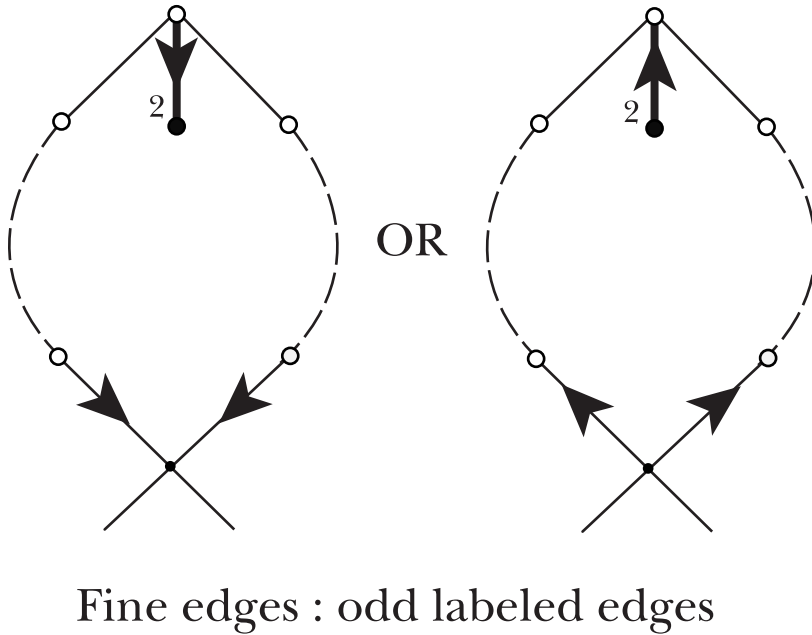
Fine edges : odd labeled edges

Fig. 12.

Proof. We prove the lemma by contradiction. Suppose that a non-special room R possesses a terminal edge A of label 2. Let $\{g \colon \bar{D} \to X_R; \bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n; \bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n\}$ be an associated set for the room with respect to the terminal edge A. Suppose that the room R possesses a crossing v. Let $v = g(\bar{A}_k) \cap g(\bar{A}_{k+1})$. Since the room is not special, the pair $(g(\bar{A}_k), g(\bar{A}_{k+1}))$ is not special. Hence the triplet $(g(\bar{A}_k), A, g(\bar{A}_{k+1}))$ is not semi-reducible. Therefore the triplet $(g(\bar{A}_2), A, g(\bar{A}_{n-1}))$ is reducible. This contradicts Lemma 1.   □

Lemma 3. *Let A be a terminal edge of label 2 belonging to a room R, and $\{g \colon \bar{D} \to X_R; \bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n; \bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n\}$ an associated set for the room with respect to the terminal edge A. Then in a minimal chart, the followings are satisfied.* (1) *If $g(\bar{A}_2)$ does not contain a crossing, then the edge $g(\bar{A}_2)$ contains a distinguished arc of the vertex $g(\bar{P}_3)$ and the edge $g(\bar{A}_3)$ contains a middle arc of the vertex $g(\bar{P}_4)$.* (2) *If $g(\bar{A}_{n-1})$ does not contain a crossing, then the edge $g(\bar{A}_{n-1})$ contains a distinguished arc of the vertex $g(\bar{P}_{n-1})$ and the edge $g(\bar{A}_{n-2})$ contains a middle arc of the vertex $g(\bar{P}_{n-2})$.*

Proof. Suppose that $g(\bar{A}_2)$ does not contain a crossing. Then the edge $g(\bar{A}_2)$ is not a terminal edge. Hence the edge $g(\bar{A}_3)$ is of label 2.

We show that $(g(\bar{A}_2), g(\bar{A}_3))$ is admissible. If not, $g(\bar{A}_3)$ does not contain a middle arc of $g(\bar{P}_3)$. Apply a C-I-M2 move between A and $g(\bar{A}_3)$ to get a new terminal edge without a middle arc of the vertex $g(\bar{P}_3)$. We can apply a C-III-1 move to dismiss the vertex $g(\bar{P}_3)$. This contradicts the minimal complexity of the chart. Therefore $(g(\bar{A}_2), g(\bar{A}_3))$ is admissible.

Since $(g(\bar{A}_2), g(\bar{A}_3))$ is admissible, the point $g(\bar{P}_4)$ must be a white vertex. For, if not, apply a C-I-M2 move between A and $g(\bar{A}_3)$. If $g(\bar{P}_4)$ is a black vertex, then we get a new free edge. This means the complexity decreases. Since $g(\bar{A}_3)$ is of label 2, $g(\bar{P}_4)$ is not a crossing. Hence $g(\bar{P}_4)$ is a white vertex.

Since $(g(\bar{A}_2), g(\bar{A}_3))$ is admissible and the point $g(\bar{P}_4)$ is a white vertex, the edge $g(\bar{A}_3)$ must contain a middle arc of the white vertex $g(\bar{P}_4)$. For, if not, apply a C-I-M2 move between A and $g(\bar{A}_3)$. Then we have a new terminal edge which does not contain a middle arc of vertex $g(\bar{P}_4)$ without changing complexity of the chart. Now we can apply a C-III-1 move to dismiss the vertex $g(\bar{P}_4)$. This contradicts the minimal complexity of the chart.

Now we show that $g(\bar{A}_2)$ contains a distinguished arc of the vertex $g(\bar{P}_3)$ by contradiction. Suppose that $g(\bar{A}_2)$ does not contain a middle arc of the vertex $g(\bar{P}_3)$. Then $g(\bar{A}_2)$ is not a loop. Hence $g(\bar{P}_2) \neq g(\bar{P}_3)$. Since $g(\bar{A}_2)$ does not contain a distinguished arc of the vertex $g(\bar{P}_3)$ and the pair $(g(\bar{A}_2), g(\bar{A}_3))$ is admissible, the edge $g(\bar{A}_3)$ contains a middle arc of $g(\bar{P}_3)$. Let A' be the odd labeled edge with the vertex $g(\bar{P}_3)$ different from the edge $g(\bar{A}_2)$ such that the pair $(A', g(\bar{A}_3))$ is admissible. Now apply a C-I-M2 move between A and $g(\bar{A}_3)$. And then operate a C-I-M2 move between A' and $g(\bar{A}_{n-1})$ (possibly $A' = g(\bar{A}_{n-1})$). Then we can use a C-I-M3 move to dismiss the two white vertices $g(\bar{P}_2)$ and $g(\bar{P}_3)$ without increasing the number of crossings. This contradicts the minimal complexity of the chart. Therefore the edge $g(\bar{A}_2)$ contains a distinguished arc of the vertex $g(\bar{P}_3)$. This proves (1). The proof of (2) is similar.   □

Proposition 2. *For any minimal* 4-*chart with at most one crossing, the special house does not possess any terminal edge of label* 2.

Proof.   We prove the proposition by contradiction. Suppose that there exists a terminal edge A of label 2 in the special house H of a minimal 4-chart with at most one crossing. By Lemma 2, the terminal edge A must belong to a special room R.

Since the number of crossings is at most one, situation (1) or situation (2) in Lemma 3 occurs. Now we use the notation in Lemma 3.

Suppose that situation (1) occurs. There are two cases.

CASE 1.   *The edge* $g(\bar{A}_3)$ *belongs to only one room*.

This means that if we take out the edge $g(\bar{A}_3)$ from the chart, then the chart is disconnected. So apply a C-I-M2 move between A and $g(\bar{A}_3)$ to get a new disconnected chart without increasing the number of white vertices such that each connected component has a white vertex. This contradicts Connectedness Condition.
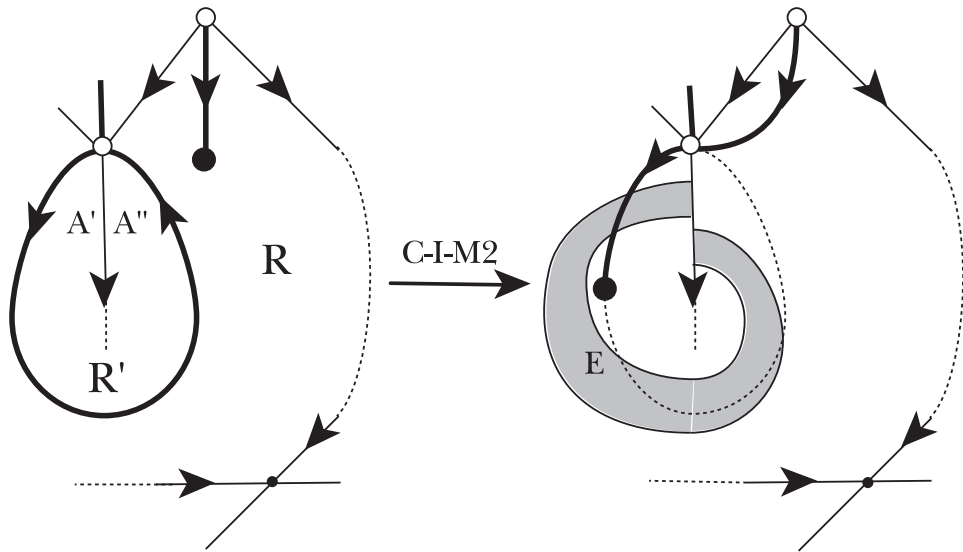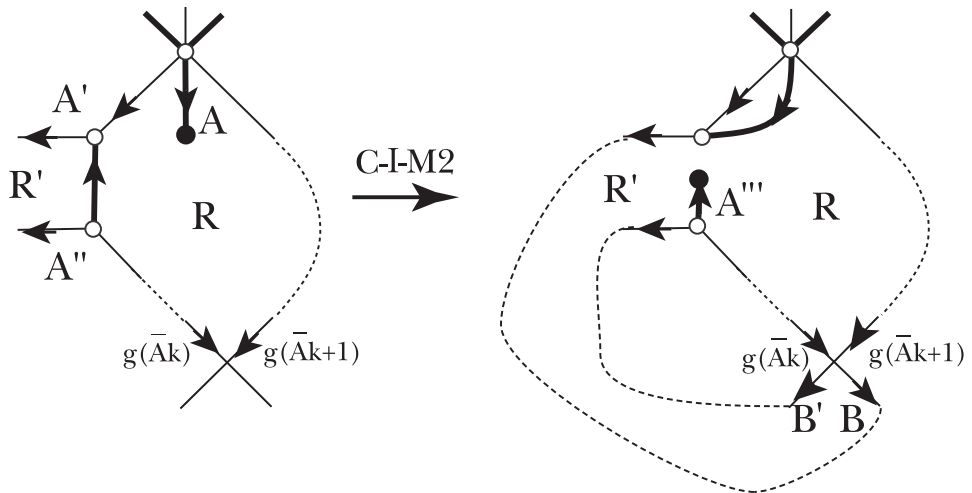
Fig. 13.

Fig. 14.

CASE 2. *The edge $g(\bar{A}_3)$ belongs to two rooms.*
Let R' be the other room different from the room R. We need the following claim.

Claim 2. *There must be a special pair $(g(\bar{A}_k), g(\bar{A}_{k+1}))$ for some $4 \leq k < n-1$ with $(g(\bar{A}_k), A, g(\bar{A}_{k+1}))$ semi-reducible.*

Proof of Claim 2. The triplet $(g(\bar{A}_2), A, g(\bar{A}_{n-1}))$ is not reducible by Lemma 1. Hence there exists a special pair $(g(\bar{A}_k), g(\bar{A}_{k+1}))$ for some $2 \leq k < n-1$ with $(g(\bar{A}_k), A, g(\bar{A}_{k+1}))$ semi-reducible. Since the edge $g(\bar{A}_2)$ does not contain a crossing and the edge $g(\bar{A}_3)$ is of label 2, we have that $4 \leq k$. □

Now continue Case 2. Let A' be the odd labeled edge belonging to the room R' such that it has the common vertex $g(\bar{P}_3)$ with the edge $g(\bar{A}_3)$ and that it is situated next to the edge $g(\bar{A}_3)$. Let A'' be the odd labeled edge belonging to the room R' such that it has the common vertex $g(\bar{P}_4)$ with the edge $g(\bar{A}_3)$ and that it is situated next to the edge $g(\bar{A}_3)$. Here, if the edge $g(\bar{A}_3)$ is a loop, then we take $A' = A''$ (see Fig. 13). Apply a C-I-M2 move between A and $g(\bar{A}_3)$ to get a new terminal edge A''' with the white vertex $g(\bar{P}_4)$.

The edge $g(\bar{A}_3)$ is not a loop. For, if $g(\bar{A}_3)$ is a loop, then the room R' does not possess a special pair. Hence the triplet (A', A''', A'') is reducible (see Fig. 13). This contradicts Lemma 1. Hence $g(\bar{A}_3)$ is not a loop.

The room R' does not possess the special pair $(g(\bar{A}_k), g(\bar{A}_{k+1}))$ indicated in Claim 2. The room R' possibly contain the other special pair, say (B, B'). But the triplet (B, A''', B') is not semi-reducible. This means that the triplet (A', A''', A'') is reducible (see Fig. 14). This contradicts Lemma 1.

The proof for situation (2) is similar to the one for situation (1). Therefore Proposition 2 has been proved. □

4. Distinguished arcs and proofs of Proposition 3 and 4

We investigate properties of distinguished arcs. By Proposition 1 and Proposition 2, we can assume that our minimal chart is connected and has no terminal edges of label 2.

Lemma 4. *Every house with no terminal edge of label 2 possesses an even number of distinguished arcs.*

Proof. Let X_H be the closure of a house H possessing no terminal edge of label 2. Let Y be the disk with holes obtained from X_H by cutting X_H along all edges of label 2 which do not lie on the boundary of X_H. On the boundary of Y there are two copies of each edge which do not lie on the boundary of X_H. Each copy inherits its orientation from the original edge. Now walk along the boundary component

of Y. Orientation of edges changes when we pass the white vertex which possesses a distinguished arc in the house H. Therefore the house possesses even number of distinguished arcs. $\qquad\square$

Lemma 5. *For a connected chart with no terminal edge of label* 2, *let V be the number of white vertices. For each non-negative integer i, let n_i be the number of houses which have i boundary components. Then we have*

$$V = 2(n_1 - 2) - 2\sum_{i \geq 3} n_i(i - 2).$$

Proof. Let E and F be the number of edges of label 2 and the number of houses, respectively. Note that the graph, consists of the edges of label 2 and their vertices, is a three regular graph. Thus we have

$$3V = 2E \quad \text{and} \quad F = \sum_{i \geq 1} n_i.$$

For each house with i boundary components, add $(i - 1)$ edges to make the boundaries of the house connected without adding extra vertices. Apply Euler's theorem on the 2-sphere. Then we have

$$V - \left(E + \sum_{i \geq 2} n_i(i - 1)\right) + F = 2.$$

Put the previous two equations into the last equation. Then the result follows by eliminating E and F. $\qquad\square$

Proposition 3. *For any minimal 4-chart with at most one crossing, if the chart has a white vertex but no terminal edge of label* 2, *then there exists a non-special house, with a connected boundary, which possesses no distinguished arc.*

Proof. For each integer i, let n_i be the number of houses with i connected boundaries and k_i be the number of houses with i connected boundaries but no distinguished arc. Let V be the number of white vertices. Then by Lemma 4, we have

$$V \geq 2\sum_{i \geq 1}(n_i - k_i).$$

Combining the equation in Lemma 5, we have

$$k_1 \geq 2 + \sum_{i \geq 2}(n_i - k_i) + \sum_{i \geq 3} n_i(i - 2).$$

Thus there exist at least two houses with connected boundary but no distinguished arc. The special house may be one of the two houses. Therefore one of the two houses is a desired house. □

Lemma 6. *In a minimal chart, there exists no room whose boundary consists of exactly two edges such that parities of the two edges are different and that the odd labeled edge does not contain a distinguished arc.*

Proof. Suppose that there exists such a room R with one odd labeled edge A_1 and one even labeled edge A_2. Since the odd labeled edge does not contain a distinguished arc, the edge is not a loop. Let v_1 and v_2 be the vertices of the edge A_1. Let R' be the next room which possesses the edge A_1. For $i = 1, 2$, let A'_i be the even labeled edge belonging to the room R' and containing the vertex v_i.

The pair (A'_1, A'_2) is admissible. For, if not, first apply a C-I-M2 move between A'_1 and A'_2, and then apply a C-I-M3 move to dismiss the two vertices v_1 and v_2. This contradicts the minimal complexity of the chart.

The pair (A_1, A_2) is admissible. For, if not, the pair (A'_1, A'_2) is not admissible by condition (3) for charts.

Since the pair (A'_1, A'_2) is admissible, (A_1, A'_1) or (A_1, A'_2) is admissible. If (A_1, A'_1) is admissible then the edge A_1 contains a middle arc of v_1. If (A_1, A'_2) is admissible then the edge A_1 contains a middle arc of v_2. This contradicts the condition for the odd labeled edge A_1. □

Proposition 4. *For any minimal 4-chart, if a non-special house, with connected boundary, possesses no terminal edge of label 2, then the house possesses a distinguished arcs.*

Proof. We prove the proposition by contradiction. Suppose that there exists a non-special house with a connected boundary but without distinguished arcs nor terminal edges of label 2. Among the rooms in the house, choose a room possessing the minimal number of edges. Since neither terminal edges nor crossings belong to this house, the room must possess exactly two edges. Since the house does not possess distinguished arcs, any odd labeled edge in this house does not contain a middle arc. This contradicts Lemma 6. □

Observing rooms more closely, we can show that any minimal 4-chart with at most two crossings does not have any white vertex provided that the chart represents a 2-sphere [1], [4].

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