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A CATEGORY OF SPECTRAL TRIPLES AND DISCRETE GROUPS WITH LENGTH FUNCTION

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Abstract

In the context of Connes' spectral triples, a suitable notion of morphism is introduced. Discrete groups with length function provide a natural example for our definitions. Connes' construction of spectral triples for group algebras is a covariant functor from the category of discrete groups with length functions to that of spectral triples. Several interesting lines for future study of the categorical properties of spectral triples and their variants are suggested.

1. Introduction

The notions of morphism, as a generalization of "coordinate transformation," and respectively of category, as a generalization of "group of transformations," are going to be central in all the attempts to reformulate the concepts of physical covariance in an algebraic context (see for instance J. Baez [1]).

In the abstract framework of A. Connes' Non-commutative Geometry [7, 11], where non-commutative manifolds are described by spectral triples, a definition of "morphism of spectral triples" is still missing in the literature. With the present short note, we intend to provide tentative definitions of "morphism" and of "category of spectral triples," and to investigate some of their properties.

Since, as typical feature of every non-commutative geometric setting, "non-commutative spaces" are described dually by the category of "spectra" (categories of representations) of their algebras of functions, defining a morphism of non-commutative spaces actually amounts to the specification of a functor between representations categories and, under this point of view, our work can also be seen as an example of "categorification" process in which sets are replaced by categories (see for example J. Baez, J. Dolan [2] or L. Ionescu [14]).

In the second part of this paper, we proceed to the construction of a natural covariant functor, from the category of discrete groups equipped with a length function, to our category of spectral triples, that shows the validity of the proposed definition of morphisms. We expect this functor to be just one particular example in a class of functors from suitable categories of "geometrical objects" to the category of spectral triples.

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Actually this work is part of a much wider research project [3] that, among several other objectives, has the purpose to study an appropriate notion of non-commutative (totally geodesic) submanifold and quotient manifold and the study of some suitable functorial relations between the categories of spectral triples and spin^c Riemannian manifolds. This program will be carried out in detail in a forthcoming paper [4]. The situations investigated here are usuful to present all the relevant structures involved without dealing with the complications arising from "spinorial calculus" on Riemannian spin^c manifolds.

Treatments of non-commutative geometry in a suitable categorical framework, mostly appealing to Morita equivalence, have already appeared in a more or less explicit form. In [8, 9, 10] A. Connes shows how to transfer a given Dirac operator using Morita equivalence bimodules and compatible connections on them, thus leading to the concept of "inner deformations" of a spectral geometry that encompasses a formula for expressing the transformed Dirac operator in the form $\widetilde{D} = D + A + JAJ^{-1}$. The categorical "ideology" becomes especially evident among the practitioners of "non-commutative algebraic geometry" (see for example M. Kontsevich and A. Rosenberg [15, 16, 24]) and morphisms between non-commutative manifolds, thought of as non-commutative spectra, have been proposed by Y. Manin [18] in terms of the notion of "Morita morphisms," i.e. functors among representations categories that are obtained by tensorization with bi-modules.

The notion of morphism of spectral triples described in the sequel is not as general as possible, and several further generalizations are undoubtedly at hand.

In a wider perspective [5] a morphism of the spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, with $j = 1, 2$, might be formalized as a "suitable" functor $\mathcal{F}\colon {}_{\mathcal{A}_2}\mathscr{M} \to {}_{\mathcal{A}_1}\mathscr{M}$, between the categories ${}_{\mathcal{A}_j}\mathscr{M}$ of $\mathcal{A}_j$-modules, having "appropriate intertwining" properties with the Dirac operators D_j.

The morphisms described in the sequel are only a very special case of that picture. However for the present purposes that level of generality would be unnecessary, and so we stick to the more restrictive definition provided by homomorphisms $\phi\colon \mathcal{A}_1 \to \mathcal{A}_2$ of algebras with an intertwining operators $\Phi\colon \mathcal{H}_1 \to \mathcal{H}_2$ between the Dirac operators.

We can thus establish our main result, stating that Connes' construction of spectral triples from group algebras is functorial in nature.

Whether these functors can be chosen to be full, if they are extendable to non-monomorphic cases and, in a broader context, which other functors into categories of spectral triples can be obtained this way seem to be interesting questions and we hope to return to these and related issues elsewhere.

2. A category of spectral triples

In this section we define a "natural" notion of morphism between spectral triples. Examples will be provided in the next section 3.

In order to facilitate the reader and to establish our notations, we start recalling the definitions and key properties related to spectral triples.

2.1. Preliminaries on spectral triples. A. Connes (see [7, 11]) has proposed a set of axioms for "non-commutative manifolds,"[1] called a (compact) spectral triple or an (unbounded) K-cycle.

- A (compact) *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is given by:
  - a unital pre-C*-algebra[2] $\mathcal{A}$;
  - a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$;
  - a (non-necessarily bounded) self-adjoint operator D on $\mathcal{H}$, called the Dirac operator, such that:
    a) the resolvent $(D - \lambda)^{-1}$ is a compact operator, $\forall \lambda \in \mathbb{C} \setminus \mathbb{R}$,[3]
    b) $[D, \pi(a)]_- \in \mathcal{B}(\mathcal{H})$, for every $a \in \mathcal{A}$, where $[x, y]_- := xy - yx$ denotes the commutator of $x, y \in \mathcal{B}(\mathcal{H})$.
- The spectral triple is called *even* if there exists a grading operator, i.e. a bounded self-adjoint operator $\Gamma \in \mathcal{B}(\mathcal{H})$ such that:

$$\Gamma^2 = \mathrm{Id}_{\mathcal{H}}; \quad [\Gamma, \pi(a)]_- = 0, \ \forall a \in \mathcal{A}; \quad [\Gamma, D]_+ = 0,$$

where $[x, y]_+ := xy + yx$ is the anticommutator of x, y.

A spectral triple that is not even is called *odd*.

- A spectral triple is *regular* if the function

$$\Xi_x : t \mapsto \exp(it|D|)x \exp(-it|D|)$$

is regular, i.e. $\Xi_x \in \mathrm{C}^\infty(\mathbb{R}, \mathcal{B}(\mathcal{H}))$,[4] for every $x \in \Omega_D(\mathcal{A})$, where[5]

$$\Omega_D(\mathcal{A}) := \mathrm{span}\{\pi(a_0)[D, \pi(a_1)]_- \cdots [D, \pi(a_n)]_- \mid n \in \mathbb{N}, \ a_0, \ldots, a_n \in \mathcal{A}\}.$$

- The spectral triple is *n-dimensional* iff there exists an integer n such that the Dixmier trace of $|D|^{-n}$ is finite nonzero.
- A spectral triple is *θ-summable* if $\exp(-tD^2)$ is a trace-class operator for every $t > 0$.
- A spectral triple is *real* if there exists an antiunitary operator $J : \mathcal{H} \to \mathcal{H}$ such that:

$$\left[\pi(a), J\pi(b^*)J^{-1}\right]_- = 0, \quad \forall a, b \in \mathcal{A};$$

[1] At least in the case of compact, finite dimensional, Riemannian, orientable, spin^c manifolds.

[2] Sometimes $\mathcal{A}$ is required to be closed under holomorphic functional calculus.

[3] As already noticed by Connes, this condition has to be weakened in the case of non-compact manifolds, cf. [13, 12, 20, 21].

[4] This condition is equivalent to $\pi(a), [D, \pi(a)]_- \in \bigcap_{m=1}^{\infty} \mathrm{Dom}\, \delta^m$, for all $a \in \mathcal{A}$, where δ is the derivation given by $\delta(x) := [|D|, x]_-$.

[5] We assume that for $n = 0 \in \mathbb{N}$ the term in the formula simply reduces to $\pi(a_0)$.

$$\big[[D, \pi(a)]_-, J\pi(b^*)J^{-1}\big]_- = 0, \quad \forall a, b \in \mathcal{A}, \quad \textit{first order condition};$$
$$J^2 = \pm \mathrm{Id}_{\mathcal{H}}; \quad [J, D]_\pm = 0;$$
$$\text{and, only in the even case,} \quad [J, \Gamma]_\pm = 0,$$

where the choice of $\pm$ in the last three formulas depends on the "dimension" n of the spectral triple modulo 8 in accordance to the following table:

n	0	1	2	3	4	5	6	7
$J^2 = \pm \mathrm{Id}_{\mathcal{H}}$	+	+	−	−	−	−	+	+
$[J, D]_\pm = 0$	−	+	−	−	−	+	−	−
$[J, \Gamma]_\pm = 0$	−		+		−		+	

- A spectral triple is called *commutative* if the algebra $\mathcal{A}$ is commutative.

2.2. Morphisms of spectral triples. The objects of our category $\mathscr{S}$ will be spectral triples $(\mathcal{A}, \mathcal{H}, D)$. Given two spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, with $j = 1, 2$, a *morphism of spectral triples* is a pair

$$(\phi, \Phi) \in \mathrm{Mor}_{\mathscr{S}}[(\mathcal{A}_1, \mathcal{H}_1, D_1), (\mathcal{A}_2, \mathcal{H}_2, D_2)],$$
$$(\mathcal{A}_1, \mathcal{H}_1, D_1) \xrightarrow{(\phi, \Phi)} (\mathcal{A}_2, \mathcal{H}_2, D_2),$$

where $\phi\colon \mathcal{A}_1 \to \mathcal{A}_2$ is a $*$-morphism between the pre-C*-algebras $\mathcal{A}_1, \mathcal{A}_2$ and $\Phi\colon \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear map in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ that "intertwines" the representations $\pi_1, \pi_2 \circ \phi$ and the Dirac operators D_1, D_2:

$$\pi_2(\phi(x)) \circ \Phi = \Phi \circ \pi_1(x), \quad \forall x \in \mathcal{A}_1, \tag{2.1}$$
$$D_2 \circ \Phi = \Phi \circ D_1,$$

i.e. such that the following diagrams commute for every $x \in \mathcal{A}_1$:

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \\ {\scriptstyle D_1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle D_2} \\ \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \end{array} \qquad \begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \\ {\scriptstyle \pi_1(x)}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \pi_2\circ\phi(x)} \\ \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \end{array}$$

Of course, the intertwining relation between the Dirac operators makes sense only on the domain of D_1. In the rest of the paper, we tacitly assume that Φ carries the domain of D_1 into that of D_2.

Note also that such a definition of morphism implies quite a strong relationship between the spectra of the Dirac operators of the two spectral triples.

Loosely speaking, in the commutative case (see [4] for details), one should expect such definition to become relevant only for maps that "preserve the geodesic structures" (totally geodesic immersions and totally geodesic submersions). Clearly our definition of morphism contains as a special case the notion of (unitary) equivalence of spectral triples [11, pp.485–486].

2.3. Categories of real and even spectral triples. In the case of real spectral triples, we can define a natural notion of morphism simply by requiring that the morphisms be compatible with the real structures in the following sense: given two real spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j, J_j)$, with $j = 1, 2$, a morphism in our *category of real spectral triples* $\mathscr{S}_r$ will be a morphism of spectral triples

$$(\phi, \Phi) \in \mathrm{Mor}_{\mathscr{S}}[(\mathcal{A}_1, \mathcal{H}_1, D_1), (\mathcal{A}_2, \mathcal{H}_2, D_2)],$$
$$(\mathcal{A}_1, \mathcal{H}_1, D_1) \xrightarrow{(\phi,\Phi)} (\mathcal{A}_2, \mathcal{H}_2, D_2),$$

such that Φ also "intertwines" the real structure operators J_1, J_2:

$$J_2 \circ \Phi = \Phi \circ J_1, \tag{2.2}$$

i.e. such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \\ {\scriptstyle J_1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle J_2} \\ \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \end{array}$$

In a completely similar way, we can consider even spectral triples $(\mathcal{A}, \mathcal{H}, D, \Gamma)$ and define the *category of even spectral triples* $\mathscr{S}_e$, considering only those morphisms

$$(\mathcal{A}_1, \mathcal{H}_1, D_1) \xrightarrow{(\phi,\Phi)} (\mathcal{A}_2, \mathcal{H}_2, D_2),$$

such that Φ "intertwines" with the parity operators Γ_1, Γ_2, i.e. such that:

$$\Gamma_2 \circ \Phi = \Phi \circ \Gamma_1,$$

(2.3)
$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \\ {\scriptstyle \Gamma_1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \Gamma_2} \\ \mathcal{H}_1 & \xrightarrow{\Phi} & \mathcal{H}_2 \end{array}$$

Again, in the case of real even spectral triples $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$ we will obtain a *category of real even spectral triples* $\mathscr{S}_{re}$, choosing those morphisms that satisfy at the same time both the intertwining conditions 2.2 and 2.3 above.

Of course the category $\mathscr{S}_{re}$ of real even spectral triples is in general a non-full subcategory of both the categories $\mathscr{S}_r$ and $\mathscr{S}_e$ which are in turn non-full subcategories of $\mathscr{S}$.

REMARK 2.1. According to our definition of morphisms, an automorphism of a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$ in the categorical sense is given by a pair (ϕ, Φ) with $\phi \in \mathrm{Aut}(\mathcal{A})$ and $\Phi \in \mathcal{B}(\mathcal{H})$ implementing ϕ and commuting with D and J. If we had required from the beginning the Φ appearing in (2.1) to be isometric, we would have obtained an extension of the isometry subgroup of Aut^+, the latter being the group of diffeomorphisms preserving the K-homology class of the spectral triple introduced by A. Connes [10, Section XI].

Define $\Omega^p_D(\mathcal{A}) := \mathrm{span}\{\pi(a_0)[D, \pi(a_1)] \cdots [D, \pi(a_p)] \mid a_0, \ldots, a_p \in \mathcal{A}\}$, the space of p-forms. Every morphism $(\phi, \Phi)\colon (\mathcal{A}_1, \mathcal{H}_1, D_1) \to (\mathcal{A}_2, \mathcal{H}_2, D_2)$ of spectral triples intertwines the p-forms according to the following formula:

$$\begin{aligned}
&\Phi \circ \sum_{j=1}^{N} \pi_1\big(a_0^{(j)}\big)\big[D_1, \pi_1\big(a_1^{(j)}\big)\big] \cdots \big[D_1, \pi_1\big(a_p^{(j)}\big)\big] \\
&= \sum_{j=1}^{N} \pi_2\big(\phi\big(a_0^{(j)}\big)\big)\big[D_2, \pi_2\big(\phi\big(a_1^{(j)}\big)\big)\big] \cdots \big[D_2, \pi_2\big(\phi\big(a_p^{(j)}\big)\big)\big] \circ \Phi.
\end{aligned}$$

REMARK 2.2. Our morphisms of spectral triples are compatible with the inner deformation of the metric (see A. Connes [8, 9, 10]) in the following sense. Suppose that $(\phi, \Phi)\colon (\mathcal{A}_1, \mathcal{H}_1, D_1) \to (\mathcal{A}_2, \mathcal{H}_2, D_2)$ is a morphism of spectral triples. Let us consider the two spectral triples $(\mathcal{A}_1, \mathcal{H}_1, D_1 + A_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2 + A_2)$ obtained by Morita "self-equivalences" of $\mathcal{A}_1$ and $\mathcal{A}_2$ using the "gauge potentials" $A_1 \in \Omega^1_{D_1}(\mathcal{A}_1)$ and $A_2 \in \Omega^1_{D_2}(\mathcal{A}_2)$, respectively. We notice that (ϕ, Φ) continues to be a morphism of the "deformed" spectral triples if and only if $\Phi \circ A_1 = A_2 \circ \Phi$.

3. Discrete groups with weights

In order to prove the perfect mathematical naturality of our tentative definition of morphism of spectral triples, we provide here one interesting example of covariant functor with values in our category $\mathscr{S}$.

3.1. Preliminaries on group algebras. For the benefit of the reader, we set up the framework by recalling a few properties of group algebras (of discrete groups) and their representations.

Let G be a group equipped with the discrete topology[6].

[6]With this topology G is of course a topological group.

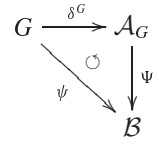
We recall that, given a group G, we can always construct its *group algebra* $\mathbb{C}[G]$, that we will denote here by $\mathcal{A}_G := \mathbb{C}[G]$. The algebra $\mathcal{A}_G$ consists of all the possible complex-valued functions on G with finite support $\mathcal{A}_G := \{f\colon G \to \mathbb{C} \mid f^{-1}\{\mathbb{C} - \{0\}\}$ is a finiteset$\}$, with sum and "scalar" multiplication by complex numbers defined pointwise: $(f + h)(x) := f(x) + g(x)$, $(\alpha f)(x) := \alpha(f(x))$, and multiplication defined by the "convolution" product: $(f * h)(z) := \sum_{\{(x,y)|xy=z\}} f(x)g(y)$. It is quickly established that $\mathcal{A}_G$, with the previously defined operations, is a complex associative unital algebra whose identity[7] is $\delta^G_e(x)$, where $\delta^G_y(x) := \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$, and that $\mathcal{A}_G$ becomes a unital associative involutive algebra with the natural involution $(f^*)(x) := \overline{f(x^{-1})}$.

Proposition 3.1. *There exists a covariant functor $\mathcal{A}$ from the category $\mathcal{G}$ of groups with homomorphisms, to the category $\mathcal{A}$ of associative complex unital involutive algebras with unital involutive algebra homomorphisms that to every group G associates the group algebra $\mathcal{A}_G$.*

Proof. We have to define the functor on morphisms i.e. given a homomorphism $\phi\colon H \to G$ between two groups H and G, we have to define a unital involutive homomorphism $\mathcal{A}_\phi\colon \mathcal{A}_G \to \mathcal{A}_H$ between the group algebras.

First of all notice that every group G can be naturally "embedded" inside its group algebra by $\delta^G\colon G \to \mathcal{A}_G$, $z \mapsto \delta^G_z$. The map δ^G is injective, unital (i.e., $e \mapsto \delta^G_e$), multiplicative (i.e. $\delta^G_x * \delta^G_y = \delta^G_{xy}$), involutive (i.e. $(\delta^G_x)^* = \delta^G_{x^{-1}}$).

Then recall that for a given group G, $(\mathcal{A}_G, \delta^G)$ is a free object over G in the category of unital associative involutive algebras i.e. every unital multiplicative involutive function $\psi\colon G \to \mathcal{B}$ from G to a unital associative involutive algebra $\mathcal{B}$ can be "lifted" to a unital involutive algebra homomorphism Ψ that makes the following diagram commutative:



Finally take in the above diagram respectively $\mathcal{B} := \mathcal{A}_H$, $\psi\colon G \to \mathcal{A}_H$ defined by $\psi := \delta^H \circ \phi$ in order to get the desired morphism of unital involutive algebras $\mathcal{A}_\phi := \Psi$. The association $\phi \mapsto \mathcal{A}_\phi$ is "functorial" i.e. respects compositions and identity functions. □

Proposition 3.2. *On the complex vector space $\mathcal{A}_G$ there exists a natural inner*

[7]Here e denotes the identity element of G.

product given by:

$$\langle f \mid h \rangle := \sum_{x \in G} \overline{f(x)} h(x).$$

With this inner product $\mathcal{A}_G$ is a pre-Hilbert space. The completion of $\mathcal{A}_G$ with respect to the previous inner product is a Hilbert space.

The Hilbert space constructed in Proposition 3.2 is naturally identified with the Hilbert space $l^2(G) := L^2(G, \mu)$, where μ is the counting measure on the discrete group G. In the following we will always denote this Hilbert space by $\mathcal{H}_G := l^2(G)$.

Proposition 3.3. *There is a natural unital representation $\pi_G^0 \colon \mathcal{A}_G \to \mathcal{L}(\mathcal{A}_G)$ of the group algebra $\mathcal{A}_G$ over itself by left action* (*by convolution*). *The representation is faithful.*

Proof. To every element $f \in \mathcal{A}_G$ we associate the element $\pi_G^0(f) \colon \mathcal{A}_G \to \mathcal{A}_G$ given by $(\pi_G^0(f))(h) := f * h$, for every $h \in \mathcal{A}_G$.

From the definition of π_G^0 it is clear that $\pi_G^0(f) \in \mathcal{L}(\mathcal{A}_G)$ and that $f \mapsto \pi_G^0(f)$ is a linear function: $\pi_G^0 \in \mathcal{L}(\mathcal{A}_G; \mathcal{L}(\mathcal{A}_G))$.

By direct calculation, π_G^0 is multiplicative and unital hence a representation.

The injectivity of π_G^0 follows from the triviality of the kernel (as in any unital left-regular representation): if $f \neq 0$, then $\pi_G^0(f)(\delta_e^G) = f * \delta_e^G = f \neq 0$. □

Corollary 3.4. *There is a natural faithful representation $\pi_G \colon \mathcal{A}_G \to \mathcal{B}(\mathcal{H}_G)$ of the group algebra $\mathcal{A}_G$ as bounded operators on the Hilbert space $\mathcal{H}_G$.*

Proof. The operator $\pi_G^0(f) \in \mathcal{L}(\mathcal{A}_G)$ is a bounded operator on the pre-Hilbert space $\mathcal{A}_G$. To prove this note that if $f = \sum_{x \in G} f(x) \delta_x^G$, by the linearity of π_G^0, we have $\pi_G^0(f) = \sum_{x \in G} f(x) \pi_G^0(\delta_x^G)$ so that it is enough to prove the boundedness of the operators $\pi_G^0(\delta_x^G)$ for all $x \in G$. This follows immediately from the fact that $\pi_G^0(\delta_x^G)$ is an isometry of the inner product space $\mathcal{A}_G$:

$$\left\| \pi_G^0 \left(\delta_x^G \right) (h) \right\|^2 = \left\| \delta_x^G * h \right\|^2 = \sum_{z \in G} \left| h \left(x^{-1} z \right) \right|^2 = \sum_{z \in G} |h(z)|^2 = \|h\|^2.$$

By linear extension theorem, $\pi_G^0(f)$ extends to a bounded operator $\pi_G(f)$ on $\mathcal{H}_G$ with the same norm. □

The representation $\pi_G \colon \mathcal{A}_G \to \mathcal{B}(\mathcal{H}_G)$, in Corollary 3.4, is nothing but the *left-regular representation* $\lambda_G \colon \mathbb{C}[G] \to \mathcal{B}(l^2(G))$.

Proposition 3.5. *The exists a natural antilinear involution $J_G \colon \mathcal{H}_G \to \mathcal{H}_G$ on the Hilbert space $\mathcal{H}_G$.*

Proof. On the pre-Hilbert space $\mathcal{A}_G$, the algebra involution $*\colon \mathcal{A}_G \to \mathcal{A}_G$, defined by $f^*(x) := \overline{f(x^{-1})}$, is antilinear and isometric:

$$\langle f^* \mid g^* \rangle = \sum_{x\in G} \overline{\overline{f(x^{-1})}}\,\overline{g(x^{-1})} = \sum_{x\in G} \overline{g(x^{-1})} f(x^{-1}) = \sum_{x\in G} \overline{g(x)} f(x) = \langle g \mid f \rangle.$$

By linear extension theorem (for antilinear maps), there exists a unique antilinear extension $J_G\colon \mathcal{H}_G \to \mathcal{H}_G$ to the closure $\mathcal{H}_G$ of $\mathcal{A}_G$. The map J_G is antilinear, involutive, isometric. □

3.2. Preliminaries on weighted groups.

DEFINITION 3.6. By a *weight* on a group G we mean a real-valued function $\omega\colon G \to \mathbb{R}$. Given two weighted groups (G, ω_G) and (H, ω_H), we say that a function $\phi\colon G \to H$ is a *weighted homomorphism* if:

$$\phi\colon G \to H \quad \text{is a group homomorphism and} \quad \omega_G = \phi^\bullet(\omega_H) := \omega_H \circ \phi.$$

A weight is called *proper* if for every $k \in \mathbb{N}$, $\omega_G^{-1}([-k, +k])$ is a finite set in G.

Note that proper weights exist only on countable groups.

REMARK 3.7. A special case of weight on a group G, is given by the notion of a *length function* on a group[8] [6] i.e. a function $l_G\colon G \to \mathbb{R}$ such that:

$$\begin{aligned} &l_G(xy) \le l_G(x) + l_G(y), \quad \forall x, y \in G,\\ &l_G(x^{-1}) = l_G(x), \quad \forall x \in G,\\ &l_G(x) = 0 \iff x = e, \quad \text{where } e \in G \text{ is the identity element of } G. \end{aligned}$$

Of course a length function is always positive since: $0 = l_G(e) = l_G(xx^{-1}) \le l_G(x) + l_G(x^{-1}) = 2l_G(x)$ for all $x \in G$.

A weighted homomorphism of groups with length is called an *isometry*. The previous conditions actually imply that every isometry is injective:

$$\phi(x) = e_H \implies l_H(\phi(x)) = l_H(e_H) = 0 \implies l_G(x) = 0 \implies x = e_G.$$

Proposition 3.8. *The class of* (*proper*) *weighted groups with weighted homomorphisms forms a category. The class of groups equipped with a* (*proper*) *length function when the morphisms are the isometries, is a full subcategory.*

[8]Here we follow the definition used by M. Rieffel [22, Section 2].

Proof. The composition of weighted homomorphisms (respectively isometries) $\phi\colon G \to H$ and $\psi\colon H \to K$ is a weighted homomorphism (isometry):

$$(\psi \circ \phi)^\bullet(\omega_K) = \phi^\bullet(\psi^\bullet(\omega_K)) = \phi^\bullet(\omega_H) = \omega_G.$$

For every object (H, ω_H), the identity isomorphism $\iota\colon H \to H$ is a weighted homomorphism (isometry) that satisfies $\psi \circ \iota = \psi$, and $\iota \circ \phi = \phi$ for every composable weighted homomorphisms ϕ, ψ. □

Of course the category of normed spaces with linear norm-preserving maps is a (non-full) subcategory of the category of Abelian groups with length function (the length function being the norm) and isometries as defined above coincide with the well-known concept of norm-preserving maps in normed spaces.

Proposition 3.9. *There is a covariant functor $\mathcal{A}$ from the category $\mathscr{G}_i$ of groups with injective homomorphism as arrows, to the category $p\mathscr{H}_i$ of pre-Hilbert spaces with isometries.*

In the same way, we have a covariant functor $\mathcal{H}$ from the category $\mathscr{G}_i$ of groups with injective morphism to the category $\mathscr{H}_i$ of Hilbert spaces with isometries.

The functors $\mathcal{A}$ and $\mathcal{H}$ are left exact.

Proof. The functor on objects is defined by $G \mapsto \mathcal{A}_G \in p\mathscr{H}_i$ and by $G \mapsto \mathcal{H}_G \in \mathscr{H}_i$ respectively.

To define the functor on morphisms, we first note that for any given group G, the set $\{\delta_x^G \mid x \in G\}$ is a (Hamel) basis for the vector space $\mathcal{A}_G$ that is orthomormal with respect to the inner product in $\mathcal{A}_G$.

If the function $\phi\colon G \to H$ is a monomorphism, the induced (linear) map $\mathcal{A}_\phi\colon \mathcal{A}_G \to \mathcal{A}_H$ becomes an isometry because it maps δ_x^G to $\delta_{\phi(x)}^H$ i.e. it sends an orthonormal basis to an orthonormal set.

Since $\mathcal{A}_\phi$ is an isometry, it is bounded as a map from $\mathcal{A}_G$ to $\mathcal{H}_H$ and it can be uniquely extended to an isometry $\mathcal{H}_\phi\colon \mathcal{H}_G \to \mathcal{H}_H$.

The associations $\phi \mapsto \mathcal{A}_\phi$ and $\phi \mapsto \mathcal{H}_\phi$ satisfy all the functorial properties. □

The following theorem is a well-known result of A. Connes [6, Lemma 5]:

Theorem 3.10. *To every pair (G, ω_G) where G is a discrete countable group and ω_G is a weight function on G, we can associate a triple $(\mathcal{A}_G, \mathcal{H}_G, D_{\omega_G})$ given as follows*:

- *$\mathcal{A}_G$ is the group algebra of G as defined above in* Subsection 3.1.
- *$\mathcal{H}_G$ is the Hilbert space of G as defined above in* Proposition 3.2.
- *The representation of the algebra $\mathcal{A}_G$ on $\mathcal{H}_G$ is the left-regular representation $\pi_G\colon \mathcal{A}_G \to \mathcal{B}(\mathcal{H}_G)$ defined above in* Corollary 3.4.

• *The Dirac operator D_{ω_G} is the pointwise multiplication operator by the weight function ω_G, i.e.*

$$(D_{\omega_G}\xi)(x) := \omega_G(x)\xi(x), \quad \forall x \in G,$$

naturally defined on the domain $\left\{\xi \in \mathcal{H}_G \;\middle|\; \sum_{x\in G}|\omega_G(x)\,\xi(x)|^2 < \infty\right\}$.

The triple $(\mathcal{A}_G, \mathcal{H}_G, D_{\omega_G})$ is a spectral triple if and only if the weight ω_G is proper and such that, for all $x \in G$, the differences[9] $[\omega_G - \tau_x(\omega_G)]\colon G \to \mathbb{R}$, *are bounded real-valued functions.*

Proof. $\mathcal{A}_G$ is a pre-C* algebra: defining $\|f\| := \|\pi_G(f)\|_{\mathcal{H}_G}$, we see that the C*-property $\|f^*f\| = \|f\|^2$ is immediate[10].

The Dirac operator D_{ω_G} is self-adjoint and has compact resolvent if and only if ω_G is proper.

Every element $f \in \mathcal{A}_G$ can be written as $f = \sum_{x\in G} f(x)\delta_x^G$.

It follows that $\pi_G(f) = \sum_{x\in G} f(x)\pi_G(\delta_x^G)$ and we have:

$$\|[D_{\omega_G}, \pi_G(f)]\| = \left\|\sum_{x\in G} f(x)\left[D_{\omega_G}, \pi_G\left(\delta_x^G\right)\right]\right\| \le \sum_{x\in G}|f(x)| \cdot \left\|\left[D_{\omega_G}, \pi_G\left(\delta_x^G\right)\right]\right\|,$$

so that, in order to show the boundedness of $[D_{\omega_G}, \pi_G(f)]$ it is enough to show the boundedness of $[D_{\omega_G}, \pi_G(\delta_x^G)]$ for all $x \in G$.

Now, from the fact that $\pi_G(\delta_x^G)$ is unitary in $\mathcal{H}_G$, we have:

$$\begin{aligned}\left\|\left[D_{\omega_G}, \pi_G\left(\delta_x^G\right)\right]\right\| &= \left\|D_{\omega_G}\pi_G\left(\delta_x^G\right) - \pi_G\left(\delta_x^G\right)D_{\omega_G}\right\| \\ &= \left\|\left(D_{\omega_G} - \pi_G\left(\delta_x^G\right)D_{\omega_G}\pi_G\left(\delta_x^G\right)^{-1}\right)\pi_G\left(\delta_x^G\right)\right\| \\ &= \left\|D_{\omega_G} - \pi_G\left(\delta_x^G\right)D_{\omega_G}\pi_G\left(\delta_x^G\right)^{-1}\right\|\end{aligned}$$

and since, by direct calculation, we get $\pi_G(\delta_x^G)D_{\omega_G}\pi_G(\delta_x^G)^{-1} = D_{\tau_x(\omega_G)}$, where $\tau_x(\omega_G)\colon y \mapsto \omega_G(x^{-1}y)$, we see that $\|[D_{\omega_G}, \pi_G(\delta_x^G)]\| = \|D_{\omega_G} - D_{\tau_x(\omega_G)}\|$. Since $\|D_{\omega_G} - D_{\tau_x(\omega_G)}\| = \|\omega_G - \tau_x(\omega_G)\|_\infty := \sup\{|\omega_G(y) - \omega_G(x^{-1}y)|\colon y \in G\}$, the assertion is proved[11]. □

[9] Where $\tau_x(\omega_G)\colon y \mapsto \omega_G(x^{-1}y)$ is the "left x-translated" of ω_G.

[10] It must be pointed out that, denoting by $C_r^*(G)$ the closure of $\mathcal{A}_G$ in the norm defined above, the correspondence $G \mapsto C_r^*(G)$ is not functorial, in general. It becomes so, for the full subcategory of amenable groups. In the case of non amenable groups we do not have finite dimensional spectral triples (see A. Connes [6, Theorem 19]).

[11] Let $u\colon G \times G \to \mathbb{T}$ be a normalized 2-cocycle on G and consider the left-regular representation of G twisted by u, defined by $(\pi_u^G(\delta_x^G))\delta_y^G := u(y^{-1}x^{-1}, x)\delta_{xy}^G$. Then up to minor modifications the same argument shows that the Dirac operator D_{ω_G} also gives a spectral triple over the "twisted group algebra" generated by the $\pi_u^G(\delta_x^G)$'s.

REMARK 3.11. In the case of length funtions on groups, the last condition of Theorem 3.10 is automatically satisfied:

$$l_G(y) - \tau_x(l_G)(y) = l_G(y) - l_G(x^{-1}y) \leq l_G(x).$$

Lemma 3.12. *$\mathcal{A}_G \subset \mathcal{H}_G$ is an invariant core for the operator D_{ω_G}.*

Proof. Suppose that[12] $(\xi, \eta) \in D_{\omega_G}$. Since $\mathcal{A}_G$ is dense in $\mathcal{H}_G$, there is a sequence $\xi_n \xrightarrow{n\to\infty} \xi$ with $\xi_n \in \mathcal{A}_G$. We show that it is possible to choose the sequence $\xi_n \in \mathcal{A}_G$ in such a way that $D_{\omega_G}(\xi_n) \xrightarrow{n\to\infty} \eta$. In fact, selecting an arbitrary well ordering $n \mapsto x_n \in G$ in the support set of ξ, we can always define $\xi_n := \sum_{j=0}^{n} \xi(x_j)\delta^G_{x_j}$ and check that $\xi_n \xrightarrow{n\to\infty} \xi$ and also $D_{\omega_G}(\xi_n) \xrightarrow{n\to\infty} \eta$ so that $(\xi, \eta) \in \overline{D_{\omega_G|\mathcal{A}_G}}$ i.e. $\mathcal{A}_G$ is a core for D_{ω_G}.

Of course, since ξ_n has finite support, $D_{\omega_G}(\xi_n)$ also has finite support and so $D_{\omega_G}(\xi_n) \in \mathcal{A}_G$. In particular $\mathcal{A}_G$ is an invariant subspace for D_{ω_G}. □

Lemma 3.13. *Given the weight $\omega_G : G \to \mathbb{R}$ on the group G, the following conditions are equivalent*[13]:

$\forall x \in G \quad \omega_G - \tau_x(\omega_G)$ *is constant*;

$\omega_G = \alpha + \phi$, *where α is a constant and $\phi : G \to \mathbb{R}$ is a homomorphism*;

$\forall x \in G, \quad \omega_G - \tau'_x(\omega_G)$ *is constant*;

$\omega_G\left(xzy^{-1}\right) - \omega_G\left(zy^{-1}\right) = \omega_G(xz) - \omega_G(z), \quad \forall x, y, z \in G.$

Proof. By direct calculation if $\omega_G = \alpha + \phi$ then $\omega_G - \tau_x(\omega_G)$ and $\omega_G - \tau'_x(\omega_G)$ are constant. That $\omega_G - \tau_x(\omega_G)$ being constant is equivalent to

$$\omega_G(xg) = \omega_G(g) - \phi\left(x^{-1}\right), \tag{3.1}$$

for some function $\phi : G \to \mathbb{R}$. Taking $x = g^{-1}$ in the previous equation we have $\phi(g) = \omega_G(g) - \omega_G(e_G)$. Hence equation (3.1) implies

$$\phi(xg) = \phi(g) - \phi\left(x^{-1}\right) \tag{3.2}$$

and (taking $g = e_G$) $\phi(x) = -\phi(x^{-1})$. Substituting this in equation (3.2), we see that ϕ is a homomorphism so that $\omega = \alpha + \phi$ with $\alpha := \omega(e_G)$. The same proof applies to the case $\omega_G - \tau'_x(\omega_G)$ being constant. The last equation is easily reduced to equivalence to the constancy of $\omega_G - \tau'_x(\omega_G)$ by substitutions. □

[12]Operators and their graphs are denoted with the same symbol.

[13]By definition, $\tau'_x(\omega_G)$: $y \mapsto \omega_G(yx^{-1})$ is the "right" translation of ω_G by x.

In view of their relevance for the construction of spectral triples, weights satisfying the last condition in Theorem 3.10 deserve a special name.

DEFINITION 3.14. A weight ω_G on the group G is said to be a *Dirac weight* if $\omega_G - \tau_x(\omega_G)$ are bounded functions, for every $x \in G$.

The following proposition is essentially a restatement of the results already obtained by M. Rieffel [23, see the end of Section 2].

Proposition 3.15. *Given a proper Dirac-weighted countable group* (G, ω_G)*, the anti-unitary operator* J_G *defined in* Proposition 3.5 *is a real structure on the spectral triple* $(\mathcal{A}_G, \mathcal{H}_G, D_{\omega_G})$ *if and only if either* ω_G *is a constant function or* ω_G *is a homomorphism of groups.*

Proof. We have $J_G^2 = \mathrm{Id}_{\mathcal{H}_G}$.

By linear extension, the condition $(J_G D_{\omega_G})(\xi) = \pm(D_{\omega_G} J_G)(\xi)$ for $\xi \in \mathcal{H}_G$ holds if and only if $(J_G D_{\omega_G})(\delta_x^G) = \pm(D_{\omega_G} J_G)(\delta_x^G)$, which is also equivalent to:

$$\omega_G\left(x^{-1}\right) = \pm\omega_G(x) \quad \forall x \in G. \tag{3.3}$$

There is no problem at all to verify the property

$$(J_G \pi_G(g) J_G) \circ \pi_G(f)(\xi) = \pi_G(f) \circ (J_G \pi_G(g) J_G)(\xi), \quad \forall \xi \in \mathcal{H}_G. \tag{3.4}$$

In fact, for all $f, g, \xi \in \mathcal{A}_G$:

$$\begin{aligned}[\pi_G(f) \circ (J_G \pi_G(g) J_G)](\xi) &= f * J_G(g * (J_G(\xi))) = f * \left(J_G^2(\xi) * J_G(g)\right) \\ &= f * \xi * (J_G(g)) = J_G(g * (J_G(\xi)) * (J_G(f))) \\ &= (J_G \pi_G(g) J_G)(f * \xi) = [(J_G \pi_G(g) J_G) \circ \pi_G(f)](\xi)\end{aligned}$$

and by linear extension theorem (since J_G, $\pi_G(f)$, and $\pi_G(g)$ are all bounded) condition (3.4) holds for all $\xi \in \mathcal{H}_G$.

We now prove that the first order condition

$$[D_{\omega_G}, \pi_G(f)]_- \circ (J_G \pi_G(g) J_G)(\xi) = (J_G \pi_G(g) J_G) \circ [D_{\omega_G}, \pi_G(f)]_-(\xi), \tag{3.5}$$

for all $f, g \in \mathcal{A}_G$ and all $\xi \in \mathcal{H}_G$, holds if and only if $\omega_G - \tau_x(\omega_G)$ are constant functions. Since all the operators involved are bounded on the Hilbert space $\mathcal{H}_G$, by linear extension theorem, it is enough to check the first order condition only for every $\xi \in \mathcal{A}_G$.

Let $f = \sum_{x \in G} f(x)\delta_x^G$, $g = \sum_{y \in G} g(y)\delta_y^G$ and $\xi = \sum_{z \in G} \xi(z)\delta_z^G$ be three elements in $\mathcal{A}_G$. Substitution in equation (3.5) above and (anti-)linearity yield

$$\sum_{x,y,z \in G} \overline{f(x)} g(y) \xi(z) \cdot \left[D_{\omega_G}, \delta_x^G\right]_- \circ \left(J_G \pi_G\left(\delta_y^G\right) J_G\right)\left(\delta_z^G\right)$$

$$= \sum_{x,y,z\in G} \overline{f(x)} g(y) \xi(z) \cdot \left(J_G \pi_G\left(\delta_y^G\right) J_G\right) \circ \left[D_{\omega_G}, \delta_x^G\right]_- \left(\delta_z^G\right).$$

This last equation holds if and only if, for all $x, y, z \in G$:

$$\left[D_{\omega_G}, \delta_x^G\right]_- \circ \left(J_G \pi_G\left(\delta_y^G\right) J_G\right)\left(\delta_z^G\right) = \left(J_G \pi_G\left(\delta_y^G\right) J_G\right) \circ \left[D_{\omega_G}, \delta_x^G\right]_- \left(\delta_z^G\right).$$

By direct calculation we have:

$$\begin{aligned}
&\left[D_{\omega_G}, \delta_x^G\right]_- \circ \left(J_G \pi_G\left(\delta_y^G\right) J_G\right)\left(\delta_z^G\right) = \omega_G\left(xzy^{-1}\right) - \omega_G\left(zy^{-1}\right) \delta^G_{xzy^{-1}},\\
&\left(J_G \pi_G\left(\delta_y^G\right) J_G\right) \circ \left[D_{\omega_G}, \delta_x^G\right]_- \left(\delta_z^G\right) = \omega_G(xz) - \omega_G(z) \delta^G_{xzy^{-1}}.
\end{aligned}$$

Hence our result is that the first order condition (3.5) holds if and only if

$$\omega_G\left(xzy^{-1}\right) - \omega_G\left(zy^{-1}\right) = \omega_G(xz) - \omega_G(z), \quad \forall x, y, z \in G.$$

and this, by lemma 3.13, is equivalent to the fact that $\omega_G = \alpha + \phi$ where $\alpha\colon G \to \mathbb{R}$ is constant and $\phi\colon G \to \mathbb{R}$ is a homomorphism of groups.

Now, equation (3.3) above, in the plus case, is equivalent to $\phi = 0$ and so to $\omega_G = \alpha$ being a constant. In the minus case, it is equivalent to $\alpha = 0$ and so to $\omega_G = \phi$ being a homomorphism of groups. □

REMARK 3.16. The spectral triple $(\mathcal{A}_G, \mathcal{H}_G, D_{\omega_G})$ is regular (see M. Rieffel [23, end of Section 2]). For instance, $[|D_{\omega_G}|, \pi_G(f)]$ and $[|D_{\omega_G}|, [D, \pi_G(f)]]$ are bounded for all $f \in \mathcal{A}_G$ as a consequence of the following estimates which can be obtained by repeating the argument in the proof of Theorem 3.10:

$$\begin{aligned}
&\left\|\left[D_{|\omega_G|}, \pi_G(\delta_x^G)\right]\right\| \le \|\tau_x(|\omega_G|) - |\omega_G|\|_\infty \le \|\tau_x(\omega_G) - \omega_G\|_\infty;\\
&\left\|\left[D_{|\omega_G|}, \left[D_{\omega_G}, \pi_G(\delta_x^G)\right]\right]\right\| \le \|\tau_x(\omega_G) - \omega_G\|_\infty^2,
\end{aligned}$$

and more generally

$$\begin{aligned}
&\left\|\left[|D_{\omega_G}|, \ldots, \left[|D_{\omega_G}|, \pi_G\left(\delta_x^G\right)\underbrace{\left]\cdots\right]}_{n}\right\| \le \|\tau_x(\omega_G) - \omega_G\|_\infty^n.\right.\\
&\left\|\left[|D_{\omega_G}|, \ldots, \left[|D_{\omega_G}|, \left[D_{\omega_G}, \pi_G\left(\delta_x^G\right)\right]\underbrace{\left]\cdots\right]}_{n}\right\| \le \|\tau_x(\omega_G) - \omega_G\|_\infty^{n+1}.\right.
\end{aligned}$$

REMARK 3.17. On the real spectral triple $(\mathcal{A}_G, \mathcal{H}_G, D_{\omega_G}, J_G)$, it is impossible to introduce a grading operator Γ_G, (unless ω_G is the zero function[14]). This is because if ω_G is a non-zero constant the equation $D_{\omega_G}\Gamma = -\Gamma D_{\omega_G}$ cannot be satisfied. On the other hand, if ω_G is a homomorphism, then we are in the case $J_G D_{\omega_G} = -D_{\omega_G} J_G$

[14]In the case ω_G equal to zero, a convenient grading is given by $\Gamma(\delta_x^G) := \delta^G_{x^{-1}}$.

which, from the table at the end of Section 2.1, is incompatible with the existence of a grading. Of course, "doubling" in an appropriate way the Hilbert space $\mathcal{H}_G$, we can easily get another graded real spectral triple:

- the pre-C*-algebra is the same group algebra $\mathcal{A}_G$;
- the Hilbert space is given by the direct sum $\mathcal{H}_G \oplus \mathcal{H}_G$;
- the representation of $\mathcal{A}_G$ in $\mathcal{H}_G \oplus \mathcal{H}_G$ is the direct sum representation $\pi_G \oplus \pi_G$ i.e. for all $f \in \mathcal{A}_G$ and $\xi, \eta \in \mathcal{H}_G$:

$$[\pi_G \oplus \pi_G(f)](\xi \oplus \eta) := \begin{bmatrix} \pi_G(f) & 0 \\ 0 & \pi_G(f) \end{bmatrix} \cdot \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} [\pi_G(f)](\xi) \\ [\pi_G(f)](\eta) \end{bmatrix};$$

- the Dirac operator is given by:

$$D_{\omega_G} \oplus (-D_{\omega_G}) = \begin{bmatrix} D_{\omega_G} & 0 \\ 0 & -D_{\omega_G} \end{bmatrix};$$

- the grading operator is given by:

$$\Gamma_G := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

- the real structure is given by:

$$J_G \oplus J_G = \begin{bmatrix} J_G & 0 \\ 0 & J_G \end{bmatrix}.$$

3.3. The functor: monomorphism case.

Theorem 3.18. *There exists a covariant functor $\mathfrak{F}$, from the category $\mathcal{G}_i^{\omega}$ of proper Dirac-weighted countable groups with weighted monomorphisms to the category of spectral triples, that to every (G, ω_G) associates $(\mathcal{A}_G, \mathcal{H}_G, D_{\omega_G})$.*

Proof. We only need to prove existence of a functor on monomorphisms $\phi\colon G \to H$. It is our purpose to show that the pair $(\mathcal{A}_\phi, \mathcal{H}_\phi)$ defined in Proposition 3.9 is a morphism

$$(A_G, \mathcal{H}_G, D_{\omega_G}) \xrightarrow{(\mathcal{A}_\phi, \mathcal{H}_\phi)} (A_H, \mathcal{H}_H, D_{\omega_H})$$

of spectral triples.

This amounts to showing that for every $f \in \mathcal{A}_G$ and for every $\xi \in \mathcal{H}_G$:

$$\mathcal{H}_\phi \circ \pi_G(f)(\xi) = \pi_H(\mathcal{A}_\phi(f)) \circ \mathcal{H}_\phi(\xi);$$

and that:

$$\mathcal{H}_\phi \circ D_{\omega_G}(\xi) = D_{\omega_H} \circ \mathcal{H}_\phi(\xi). \tag{3.6}$$

The first property follows from the fact that, for every $f \in \mathcal{A}_G$ and for every $\xi \in \mathcal{A}_G \subset \mathcal{H}_G$ we have:

$$(\mathcal{H}_\phi \circ \pi_G(f))(\xi) = \mathcal{H}_\phi(f * \xi) = \mathcal{A}_\phi(f * \xi) = \mathcal{A}_\phi(f) * \mathcal{A}_\phi(\xi) = \pi_H(\mathcal{A}_\phi(f)) \circ \mathcal{H}_\phi(\xi).$$

Since, for every $f \in \mathcal{A}_G$, the bounded operators $\mathcal{H}_\phi \circ \pi_G(f)$ and $\pi_H(\mathcal{A}_\phi(f)) \circ \mathcal{H}_\phi$ coincide on the dense subspace $\mathcal{A}_G$ of $\mathcal{H}_G$, the identity follows.

The second property is obtained from the fact that, for every $\xi \in \mathcal{A}_G \subset \mathcal{H}_G$:

$$\begin{aligned}
\mathcal{H}_\phi \circ D_{\omega_G}(\xi) &= \mathcal{H}_\phi \left(\sum_{z \in G} \omega_G(z)\xi(z)\delta_z^G \right) = \sum_{z \in G} \omega_G(z)\xi(z)\mathcal{H}_\phi \left(\delta_z^G\right) \\
&= \sum_{z \in G} \omega_G(z)\xi(z)\delta_{\phi(z)}^H = \sum_{z \in G} \omega_H(\phi(z))\xi(z)\delta_{\phi(z)}^H \\
&= D_{\omega_H} \left(\sum_{z \in G} \xi(z)\delta_{\phi(z)}^H \right) = D_{\omega_H} \left(\sum_{z \in G} \xi(z)\mathcal{H}_\phi \left(\delta_z^G\right) \right) \\
&= D_{\omega_H} \circ \mathcal{H}_\phi \left(\sum_{z \in G} \xi(z)\delta_z^G \right) = D_{\omega_H} \circ \mathcal{H}_\phi(\xi),
\end{aligned}$$

so that the two operators $\mathcal{H}_\phi \circ D_{\omega_G}$ and $D_{\omega_H} \circ \mathcal{H}_\phi$ coincide on the dense subspace $\mathcal{A}_G$ of $\mathcal{H}_G$. From the fact that $\mathcal{A}_G$ is an invariant subspace for D_{ω_G} and $\mathcal{H}_\phi$, and from Lemma 3.12 above, we see that $\mathcal{A}_G$ is a core for both operators and the equality (3.6) follows. □

Proposition 3.19. *Under the same assumptions as in* Theorem 3.18, *if the weight ω_G is a group homomorphism or a constant, then $(\mathcal{A}_\phi, \mathcal{H}_\phi)$ is a morphism of real spectral triples i.e.*:

$$\mathcal{H}_\phi \circ J_G = J_H \circ \mathcal{H}_\phi.$$

Proof. For every element $\xi = \sum_{x \in G} \xi(x)\delta_x^G \in \mathcal{A}_G \subset \mathcal{H}_G$ we have:

$$\begin{aligned}
\mathcal{H}_\phi \circ J_G \left(\sum_{x \in G} \xi(x)\delta_x^G \right) &= \sum_{x \in G} \mathcal{H}_\phi \circ J_G \left(\xi(x)\delta_x^G\right) = \sum_{x \in G} \overline{\xi(x)}\mathcal{H}_\phi \left(J_G \left(\delta_x^G\right)\right) \\
&= \sum_{x \in G} \overline{\xi(x)}\mathcal{H}_\phi \left(\delta_{x^{-1}}^G\right) = \sum_{x \in G} \overline{\xi(x)}\delta_{\phi(x)^{-1}}^H \\
&= \sum_{x \in G} J_H \left(\xi(x)\delta_{\phi(x)}^H\right) = \sum_{x \in G} J_H \circ \mathcal{H}_\phi \left(\xi(x)\delta_x^G\right)
\end{aligned}$$

$$= J_H \circ \mathcal{H}_\phi \left(\sum_{x \in G} \xi(x) \delta_x^G \right).$$

This means that both the operators $\mathcal{H}_\phi \circ J_G$ and $J_H \circ \mathcal{H}_\phi$ coincide on the dense subspace $\mathcal{A}_G$ of the Hilbert space $\mathcal{H}_G$ and, since they are bounded, it follows by linear extension theorem, that they are equal on all of $\mathcal{H}_G$. □

REMARK 3.20. With the same notations developed in Remark 3.17, it is easily established that $(\mathcal{A}_\phi, \mathcal{H}_\phi \oplus \mathcal{H}_\phi)$ is a morphism of graded spectral triples (with real structure, when available). The association $G \mapsto \mathcal{A}_G$, $\phi \mapsto (\mathcal{A}_\phi, \mathcal{H}_\phi \oplus \mathcal{H}_\phi)$ is functorial from the category $\mathcal{G}_i^\omega$ to the category $\mathcal{S}$ of spectral triples.

An automorphism α of G induces by functoriality an automorphism $\mathcal{A}_\alpha$ of the group algebra $\mathcal{A}_G$ implemented by the unitary $\mathcal{H}_\alpha$ on the Hilbert space $\mathcal{H}_G$ and, if α is also weighted, $(\mathcal{A}_\alpha, \mathcal{H}_\alpha)$ is an automorphism of the spectral triple $(\mathcal{A}_G, \mathcal{H}_G, D_{\omega_G})$. In particular, if $\alpha := \mathrm{ad}_g$, $g \in G$, is inner, $\mathcal{H}_\alpha = \pi_G(g) J \pi_G(g) J$ is the image of g through the inner regular representation of G.[15]

Equivalence classes of monomorphisms categorically correspond to subobjects, in our case, subgroups. Every subgroup H of the weighted group (G, ω_G) comes naturally equipped with a weight function $\omega_H := \omega_G|_H$ obtained by restriction of the original weight function on G and the inclusion map $\iota: H \to G$ is a morphism in $\mathcal{G}_i^\omega$. By Proposition 3.9 and Theorem 3.18, $(\mathcal{A}_\iota, \mathcal{H}_\iota)$ is a monomorphism from the spectral triple $(A_H, \mathcal{H}_H, D_{\omega_H})$ to the spectral triple $(A_G, \mathcal{H}_G, D_{\omega_G})$. Similarly, one has the following:

Corollary 3.21. *The functor* $\mathfrak{F}: \mathcal{G}_i^\omega \to \mathcal{S}$ *is left exact*: *every monomorphism of groups gives rise to a monomorphism of spectral triples.*

Note however that the functor $\mathfrak{F}$ is not full: there are morphisms (even monomorphisms) of spectral triples over group algebras that are not obtained from monomorphisms of groups. This fact might call for suitable modifications of our setting that could entail better functorial correspondences.

3.4. Preliminaries on charged groups and co-isometries. Before proceeding further, we need to collect a few more facts about weights and lengths on groups.

DEFINITION 3.22. A *charged group* is a weighted group (G, ω_G) such that the function $|\omega_G|: x \mapsto |\omega_G(x)|$ is a length function on G.

A homomorphism $\phi: G \to H$ between charged groups is called *isometric* if $\phi^\bullet(|\omega_H|) = |\omega_G|$.

[15]Note that $(\mathcal{A}_G, \mathcal{H}_G, D_{\omega_G})$, with $\mathcal{A}_G$ acting on $\mathcal{H}_G$ by the (linearization of the) inner regular representation of G, is a spectral triple too.

REMARK 3.23. Every group with length function (G, l_G) is a charged group.

An isometric homomorphism between charged groups is continuous with respect to the metric topologies induced by the length functions.

Every weighted homomorphism $\phi\colon G \to H$ between charged groups (G, ω_G) and (H, ω_H) is isometric[16].

The category of charged groups with isometric weighted morphisms is a full subcategory of the category of weighted groups with weighted monomorphisms.

DEFINITION 3.24. Let (G, ω_G) and (H, ω_H) be two weighted groups. A homomorphism of groups $\phi\colon G \to H$ is called a *co-weighted homomorphism* if there exists a weighted homomorphism $\rho\colon H \to G$ such that $\phi \circ \rho = \iota_H$. A co-weighted homomorphism between two charged groups is said to be *co-isometric* if $|\omega_H(\phi(g))| \leq |\omega_G(g)|$ for all $g \in G$.

Lemma 3.25. *Let (G, l_G) be a group with length function and let H be a normal subgroup of G. The function $l_{G/H}\colon G/H \to \mathbb{R}$ defined by*

$$l_{G/H}(xH) := \inf\{l_G(xh) \mid h \in H\}$$

is a length on G/H called the quotient length.

Proof. Using the normality of H in G we see that $\{xyhk \mid h, k \in H\} = \{xhyk \mid h, k \in H\}$. Hence this calculation follows:

$$\begin{aligned}
&l_{G/H}(xyH) \leq l_G(xhyk) \leq l_G(xh) + l_G(yk) \quad \forall h, k \in H\\
&\Longrightarrow l_{G/H}(xyH) - l_G(xh) \leq l_G(yk), \quad \forall h, k \in H\\
&\Longrightarrow l_{G/H}(xyH) - l_G(xh) \leq l_{G/H}(yH), \quad \forall h \in H\\
&\Longrightarrow l_{G/H}(xyH) - l_{G/H}(yH) \leq l_G(xh), \quad \forall h \in H\\
&\Longrightarrow l_{G/H}(xyH) - l_{G/H}(yH) \leq l_{G/H}(xH).
\end{aligned}$$

Since H is normal in G, we have $\{x^{-1}h \mid h \in H\} = \{(xh)^{-1} \mid h \in H\}$ and so:

$$\begin{aligned}
l_{G/H}\left(x^{-1}H\right) &= \inf\left\{l_G\left(x^{-1}h\right) \;\middle|\; h \in H\right\} = \inf\left\{l_G\left((xh)^{-1}\right) \;\middle|\; h \in H\right\}\\
&= \inf\{l_G(xh) \mid h \in H\} = l_{G/H}(xH).
\end{aligned}$$

Finally, we have $0 \leq l_{G/H}(H) = \inf\{l_G(h) \mid h \in H\} \leq l_G(e_G) = 0$. □

Lemma 3.26. *Let $\phi\colon G \to H$ be a homomorphism between two groups and l_G a length function on G. We can define the push-forward $\phi_\bullet(l_G)\colon \phi(G) \to \mathbb{R}$ as follows*:

$$(\phi_\bullet(l_G))(h) := \inf\{l_G(g) \mid g \in G,\ \phi(g) = h\}, \quad \forall h \in \phi(G).$$

[16]Of course, ϕ is continuous.

The push-forward is a length function on $\phi(G)$.

Proof. Under the natural isomorphism $\phi(G) \simeq G/\operatorname{Ker}\phi$, the function $\phi_\bullet(l_G)$ coincides with the function $l_{G/\operatorname{Ker}\phi}$ defined above. □

REMARK 3.27. A co-weighted homomorphism $\phi\colon G \to H$ between two charged groups is a co-isometry if and only if $|\omega_H| = \phi_\bullet(|\omega_G|)$.

Lemma 3.28. *There is a category $\mathscr{G}_c$ whose objects are groups and whose morphisms are epimorphisms.*

There is a category whose objects are charged groups and whose morphisms are co-isometric homomorphisms.

Proof. The composition of epimorphisms is another epimorphism and the composition of co-isometric homomorphisms is a co-isometric homomorphism. The identity map of every group is a co-isometric homomorphism (hence epimorphism) that plays the role of the identity in the category. □

Corollary 3.29. *There is a category $\mathscr{H}_c$ whose objects are Hilbert spaces and whose morphisms are co-isometries.*

DEFINITION 3.30. A *covariant relator* from the category $\mathscr{A}$ to the category $\mathscr{B}$ is a pair $(\mathcal{R}_{\mathrm{Ob}}, \mathcal{R}_{\mathrm{Mor}})$ of relations, $\mathcal{R}_{\mathrm{Ob}} \subset \mathrm{Ob}_{\mathscr{A}} \times \mathrm{Ob}_{\mathscr{B}}$ between objects and $\mathcal{R}_{\mathrm{Mor}} \subset \mathrm{Mor}_{\mathscr{A}} \times \mathrm{Mor}_{\mathscr{B}}$ between morphisms, such that:

$$(A, B) \in \mathcal{R}_{\mathrm{Ob}} \Longrightarrow (\iota_A, \iota_B) \in \mathcal{R}_{\mathrm{Mor}}$$

and, whenever α_1, α_2 are composable morphisms in $\mathscr{A}$ and whenever β_1, β_2 are composable morphisms in $\mathscr{B}$:

$$(\alpha_2, \beta_2), (\alpha_1, \beta_1) \in \mathcal{R}_{\mathrm{Mor}} \Longrightarrow (\alpha_2 \circ \alpha_1, \beta_2 \circ \beta_1) \in \mathcal{R}_{\mathrm{Mor}}.$$

REMARK 3.31. A covariant functor is a covariant relator such that both $\mathcal{R}_{\mathrm{Ob}}$ and $\mathcal{R}_{\mathrm{Mor}}$ are functions. Contravariant relators are defined in a similar way interchanging the order of compositions.

Proposition 3.32. *There is a contravariant relator $\mathcal{H}$ from the category $\mathscr{G}_c$ of groups with epimorphisms to the category $\mathscr{H}_c$ of Hilbert spaces with isometries.*

Proof. $\mathcal{H}_{\mathrm{Ob}}$ is the function that to every object G in $\mathscr{G}_c$ associates the Hilbert space $\mathcal{H}_G \in \mathscr{H}_c$.

We now define the relation $\mathcal{H}_{\mathrm{Mor}}$. As we already know from Proposition 3.1, every homomorphism $\phi\colon G \to H$ is associated to a linear map $\mathcal{A}_\phi\colon \mathcal{A}_G \to \mathcal{A}_H$ of pre-Hilbert spaces.

If ϕ is an epimorphism, the linear map $\mathcal{A}_\phi$ is continuous if and only if $\operatorname{Ker}\phi$ is a finite subgroup of G:

$$
\begin{aligned}
\|\mathcal{A}_\phi(f)\|_H^2 &= \left\| \mathcal{A}_\phi \left(\sum_{x\in G} f(x)\delta_x^G \right) \right\|_H^2 = \left\| \sum_{x\in G} f(x)\delta_{\phi(x)}^H \right\|_H^2 \\
&= \left\| \sum_{y\in H} \left(\sum_{x\in\phi^{-1}(y)} f(x) \right) \delta_y^H \right\|_H^2 = \sum_{y\in H} \left| \sum_{x\in\phi^{-1}(y)} f(x) \right|^2 \\
&\le \sum_{y\in H} \left(\sum_{x\in\phi^{-1}(y)} |f(x)| \right)^2 \le \sum_{y\in H} \sum_{x\in\phi^{-1}(y)} \operatorname{card}(\operatorname{Ker}\phi)|f(x)|^2 \\
&= \sum_{x\in G} \operatorname{card}(\operatorname{Ker}\phi)|f(x)|^2 = \operatorname{card}(\operatorname{Ker}\phi)\|f\|_G^2.
\end{aligned}
\tag{3.7}
$$

It follows that in general the operator $\mathcal{A}_\phi$ is an unbounded operator from the Hilbert space $\mathcal{H}_G$ to the Hilbert space $\mathcal{H}_H$.

The operator $\mathcal{A}_\phi$ is densely defined because its domain contains the dense subspace $\mathcal{A}_G \subset \mathcal{H}_G$. Hence there exists an adjoint operator $\mathcal{A}_\phi^*$ that, in the case of finite $\operatorname{Ker}\phi$, coincides with the "pull-back" operator $f \mapsto f\circ\phi$, for $f \in \mathcal{A}_H$. Unfortunately, when $\operatorname{Ker}\phi$ is not finite, $\mathcal{A}_\phi$ is not a closable operator.

Let us now denote by $\mathcal{P}$ the set of linear isometric operators $\mathcal{K} \subset \mathcal{H}_G \times \mathcal{H}_H$ such that $\mathcal{K} \subset \mathcal{A}_\phi$. The family $\mathcal{P}$ is an inductive partially ordered set and as such, by Zorn's lemma, it admits a maximal element. Every maximal operator $\mathcal{K}$ in the family $\mathcal{P}$ is necessarily surjective and so its adjoint $\mathcal{K}^*\colon \mathcal{H}_H \to \mathcal{H}_G$ is an isometry and $\mathcal{K}^{**} = \overline{\mathcal{K}}$ is a partial isometry with range $\mathcal{H}_H$.

Every maximal partial isometry $\mathcal{K} \in \mathcal{P}$ has a closure that is the adjoint operator $\mathcal{H}_\psi^*$ of an isometric operator $\mathcal{H}_\psi$, where $\psi\colon H \to G$ is a monomorphism that is right inverse to the epimorphism $\phi\colon G \to H$.

We now define a contravariant relator $\mathcal{H}_{\mathrm{Mor}}$ on morphisms by saying that $(\phi, \mathcal{K}^*) \in \mathcal{H}_{\mathrm{Mor}}$ if and only if $\mathcal{K}$ is a maximal isometry in $\mathcal{A}_\phi$.

$\mathcal{H}$ will be a contravariant functor if and only if the set of maximal partial isometries in $\mathcal{A}_\phi$ has cardinality one, which is equivalent to the fact that there exists only one "splitting homomorphism" for ϕ i.e. there exists a unique $\psi\colon H \to G$ such that $\phi\circ\psi = \iota_H$.

The same considerations can be applied to the full subcategory of (proper) weighted groups. □

REMARK 3.33. The relation between the kernel of ϕ and the kernel of $\mathcal{A}_\phi$ is given by:

$$
\operatorname{Ker}(\phi) = G \cap \left[\delta_e^G + \operatorname{Ker}(\mathcal{A}_\phi)\right], \quad \operatorname{Ker}(\mathcal{A}_\phi) = \delta_e^G + \operatorname{span}(\operatorname{Ker}(\phi)).
$$

3.5. The functor: coisometric case.

Theorem 3.34. *There exists a contravariant relator $\mathfrak{R}$ from the category $\mathcal{G}_c^\omega$ of proper Dirac-weighted groups with co-weighted homomorphisms to the category of spectral triples $\mathcal{S}$, that to every (G, ω_G) associates $(\mathcal{A}_G, \mathcal{H}_G, D_{\omega_G})$.*

Proof. The relator on objects $\mathfrak{R}_{\mathrm{Ob}}$ coincides with the functor defined in Theorem 3.18.

We want to see that, on morphisms, the relator $\mathfrak{R}_{\mathrm{Mor}}$ is defined in the same way as in Proposition 3.32 i.e. $\mathfrak{R}_{\mathrm{Mor}}$ associates to every splitting weighted epimorphism $\phi\colon G \to H$ the family of pairs $(\mathcal{A}_\psi, \mathcal{H}_\psi)$, where ψ is any weighted (mono)morphism $\psi\colon H \to G$ such that $\phi \circ \psi = \iota_H$.

Let $\phi\colon G \to H$ be a co-weighted epimorphism of proper weighted groups. For sure (see Proposition 3.1) we have that $\mathcal{A}_\psi : \mathcal{A}_G \to \mathcal{A}_H$ is an involutive unital homomorphism of the group algebras.

If the homomorphism $\phi\colon G \to H$ admits "right inverses" i.e. if there exist weighted morphisms $\psi\colon H \to G$ such that $\phi \circ \psi = \iota_H$, from Proposition 3.32 we know that, for any such "right inverse" ψ, the function $\mathcal{H}_\psi : \mathcal{H}_G \to \mathcal{H}_H$ is an isometry of Hilbert spaces. From the same Proposition 3.32 we also know that the pair $(\mathfrak{R}_{\mathrm{Ob}}, \mathfrak{R}_{\mathrm{Mor}})$ where the second relation is given by $\mathfrak{R}_{\mathrm{Mor}} := \{(\mathcal{A}_\psi, \mathcal{H}_\psi) \mid \phi \circ \psi = \iota_H,\ \phi \in \mathrm{Mor}_{\mathcal{G}_c^\omega}\}$ is a contravariant relator.

From theorem 3.18, $(\mathcal{A}_\psi, \mathcal{H}_\psi)$ is a morphism in the category of spectral triples $\mathcal{S}$, i.e. for all $f \in \mathcal{A}_H$ and for all $\xi \in \mathcal{A}_H$:

$$\mathcal{H}_\psi \circ \pi_H(f)(\xi) = \pi_G(\mathcal{A}_\psi(f)) \circ \mathcal{H}_\psi(\xi)$$
$$D_{\omega_G} \circ \mathcal{H}_\psi(\xi) = \mathcal{H}_\psi \circ D_{\omega_H}(\xi).$$

□

REMARK 3.35. The relator becomes a functor in case that it is possible to select canonically a splitting of the co-isometry (for example in the case of Hilbert spaces).

4. Conclusion and further remarks

In this work we have proposed a definition of morphism for spectral triples (and their real and even variants).

Some remarks on further generalizations are in order. We have presented here the most elementary instructive example of functorial relations between our proposed category $\mathcal{S}$ of spectral triples and other categories: in this specific case the categories $\mathcal{G}_i^\omega$ of proper Dirac-weighted groups with monomorphisms and $\mathcal{G}_c^\omega$ of proper Dirac-weighted groups with co-weighted homomorphisms. Other examples involving categories of Riemannian manifolds equipped with a spin^c structure will be dealt with in [4].

Other alternative variants of our definition of morphism of spectral triples are worth investigating. For example we might substitute the "strong" requirement of commutation of the Dirac operators with the Hilbert space maps with some milder property like[17]: $\mathcal{H}_\phi \circ [D_G, \pi_G(f)] = [D_H, \pi_H(\mathcal{A}_\phi(f))] \circ \mathcal{H}_\phi$.

As regards the specific examples of functorial relations described here, several immediate generalizations and comments come to mind. Among them, we mention:

Most of the facts presented here for the category of weighted groups can be rephrased for the category of "weighted" small categories considering the "convolution algebra" of a small category in place of the group algebra.

The notions of weight and charge on a group can be further generalized by considering functions $\omega\colon G \times G \to \mathbb{C}$ having properties formally similar to those of Hermitian forms and inner products. The Dirac operators D_ω associated to these functions ω include, in the case of finite groups, all available Dirac operators according to the classification of finite spectral triples (see, for example, T. Krajewski [17]).

The only possible choice of Dirac operator D_{ω_G} on a weighted group (G, ω_G) that is fully compatible with the requirements of a real zero dimensional spectral triple, where the real structure J is the one obtained by Tomita–Takesaki modular theory (from the cyclic separating vector $\delta^G_{e_G} \in \mathcal{H}_G$), is $D_{\omega_G} = 0$. This fact asks for some investigation on the mutual relationship between modular theory and noncommutative geometry. We hope to discuss this point elsewhere [3].

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