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ON POLYNOMIAL CURVES IN THE AFFINE PLANE

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Abstract

A curve that can be parametrized by polynomials is called a polynomial curve. It is well-known that a polynomial curve has only one place at infinity. Let C be a curve with one place at infinity. Sathaye presented the following question raised by Abhyankar: Is there a polynomial curve associated with the semigroup generated by pole orders of C at infinity? In this paper, we give a negative answer to this question using Gröbner basis computation.

1. Introduction

Let C be an irreducible algebraic curve in the complex affine plane $\mathbf{C}^2$. We say that C has *one place at infinity*, if the closure of C intersects with the ∞-line in $\mathbf{P}^2$ at only one point P and C is locally irreducible at that point P.

Abhyankar–Moh [1, 4, 5] investigated properties of δ*-sequences* that are sequences of pole orders of *approximate roots* of curves with one place at infinity and obtained a criterion for a curve to have only one place at infinity. This result is called Abhyankar-Moh's semigroup theorem. Sathaye–Stenerson [14] proved that, conversely, if a sequence S of natural numbers satisfies Abhyankar–Moh's condition then there exists a curve with one place at infinity having its δ-sequence S. Suzuki [16] made clear the relationship between the δ-sequence and the dual graph of the minimal resolution of the singularity of the curve C at infinity, and gave an algebro-geometric proof of the semigroup theorem and its inverse theorem due to Sathaye–Stenerson. Fujimoto–Suzuki [6] gave an algorithm to compute the defining polynomial of the curve with one place at infinity from a given δ-sequence.

A curve that can be parametrized by polynomials is called a *polynomial curve*. It is well-known that a polynomial curve has only one place at infinity. Let C be a curve with one place at infinity, and Ω the semigroup generated by pole orders of C at infinity. Sathaye [13] presented the following question for curves with one place at infinity raised Abhyankar: Is there a polynomial curve associated with Ω? Sathaye–Stenerson [14] suggested a candidate for a negative answer to this question; however, they could not give an answer to the question since a root computation for a huge polynomial system was required.

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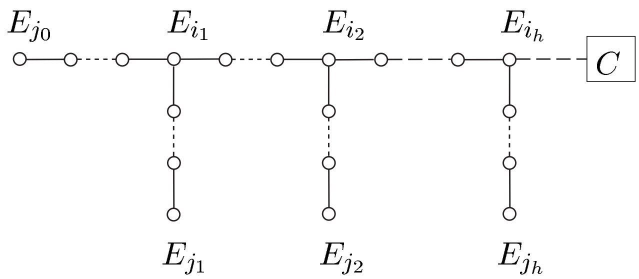
We found a negative answer to the Abhyankar's question using a computer algebra system. In this paper, we give its details.

2. Preliminaries

Through this paper, we set $\mathbf{N} = \{n \in \mathbf{Z} \mid n \geq 0\}$ and $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. Let C be a curve with one place at infinity defined by a polynomial equation $f(x, y) = 0$ in the complex affine plane $\mathbf{C}^2$. Assume that $\deg_x f = m$, $\deg_y f = n$ and $d = \gcd(m, n)$. The dual graph corresponding to the minimal resolution of the singularity of C at infinity is of the following form [16]:

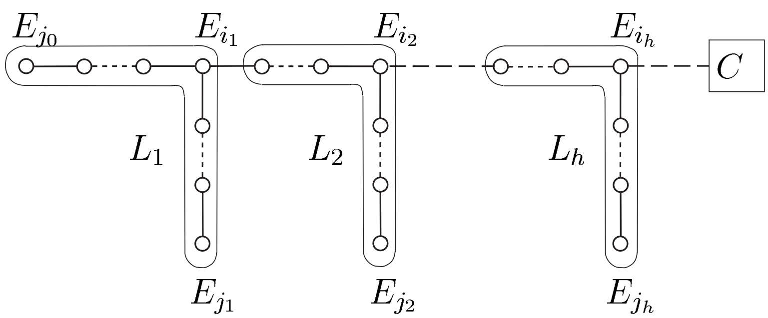


DEFINITION 1 (δ-sequence). Let f be a defining polynomial of a curve C with one place at infinity. Let δ_k $(0 \leq k \leq h)$ be the order of the pole of f on the curves corresponding to the edge nodes E_{j_k} in the above dual graph. We call the sequence $\{\delta_0, \delta_1, \dots, \delta_h\}$ the δ-sequence of C (or of f).

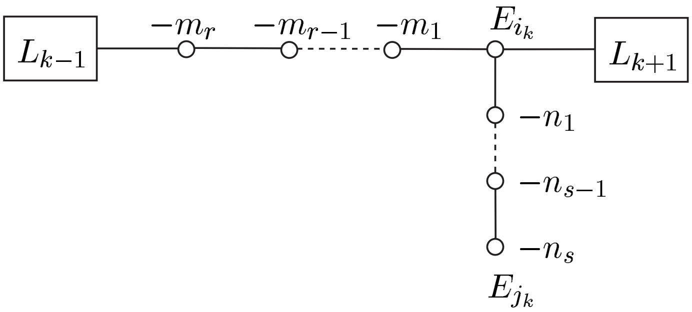
We have the following fact, since $\deg_x f = m$ and $\deg_y f = n$.

FACT 1. $\delta_0 = n$, $\delta_1 = m$.

We set L_k for each k $(1 \leq k \leq h)$, the linear branches as shown in the following figure:



DEFINITION 2 ((p, q)-sequence). Now, we assume that the weights of L_k are of the following form:



We define the natural numbers p_k, q_k, a_k, b_k satisfying

$$(p_k, a_k) = 1, \quad (q_k, b_k) = 1, \quad 0 < a_k < p_k, \quad 0 < b_k < q_k,$$

$$\frac{p_k}{a_k} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{m_3 - \ddots - \cfrac{1}{m_r}}} \quad \text{and} \quad \frac{q_k}{b_k} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \ddots - \cfrac{1}{n_s}}}.$$

We call the sequence $\{(p_1, q_1), (p_2, q_2), \ldots, (p_h, q_h)\}$ the (p, q)-sequence of C (or of f).

The following Abhyankar-Moh's semigroup theorem and its converse theorem by Sathaye–Stenerson are results for δ-sequence.

Theorem 1 (Abhyankar–Moh [1, 4, 5]). *Let C be an affine plane curve with one place at infinity. Let $\{\delta_0, \delta_1, \ldots, \delta_h\}$ be the δ-sequence of C and $\{(p_1, q_1), \ldots, (p_h, q_h)\}$ the (p, q)-sequence of C. We set $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$ $(1 \le k \le h+1)$. We have then,*

(i) $q_k = d_k/d_{k+1},\ d_{h+1} = 1\ (1 \le k \le h)$,

(ii) $d_{k+1} p_k = \begin{cases} \delta_1 & (k = 1) \\ q_{k-1}\delta_{k-1} - \delta_k & (2 \le k \le h) \end{cases}$,

(iii) $q_k \delta_k \in \mathbf{N}\delta_0 + \mathbf{N}\delta_1 + \cdots + \mathbf{N}\delta_{k-1}\ (1 \le k \le h)$.

Theorem 2 (Sathaye–Stenerson [14]). *Let $\{\delta_0, \delta_1, \ldots, \delta_h\}$ $(h \ge 1)$ be a sequence of $h+1$ natural numbers. We set $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$ $(1 \le k \le h+1)$ and $q_k = d_k/d_{k+1}$ $(1 \le k \le h)$. Furthermore, suppose that the following conditions are satisfied:*

(1) $\delta_0 < \delta_1$,
(2) $q_k \geq 2$ $(1 \leq k \leq h)$,
(3) $d_{h+1} = 1$,
(4) $\delta_k < q_{k-1}\delta_{k-1}$ $(2 \leq k \leq h)$,
(5) $q_k\delta_k \in \mathbf{N}\delta_0 + \mathbf{N}\delta_1 + \cdots + \mathbf{N}\delta_{k-1}$ $(1 \leq k \leq h)$.
Then, there exists a curve with one place at infinity having the δ*-sequence* $\{\delta_0, \delta_1, \dots, \delta_h\}$.

Suzuki [16] gave an algebro-geometric proof of the above two theorems by a consideration of the resolution graph at infinity. Further, Suzuki gave an algorithm for mutual conversion of a dual graph and a δ-sequence.

3. Construction of defining polynomials of curves

We shall assume that $f(x, y)$ is monic in y. We define approximate roots by Abhyankar's definition.

DEFINITION 3 (approximate roots). Let $f(x, y)$ be a defining polynomial, monic in y, of a curve with one place at infinity. Let $\{\delta_0, \delta_1, \dots, \delta_h\}$ be the δ-sequence of f. We set $n = \deg_y f$, $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$ and $n_k = n/d_k$ $(1 \leq k \leq h+1)$. Then, for each k $(1 \leq k \leq h+1)$, a pair of polynomials $(g_k(x, y), \psi_k(x, y))$ satisfying the following conditions is uniquely determined:
(i) g_k is monic in y and $\deg_y g_k = n_k$,
(ii) $\deg_y \psi_k < n - n_k$,
(iii) $f = g_k^{d_k} + \psi_k$.
We call this g_k the k-th approximate root of f.

We can easily get the following fact from the definition of approximate roots.

FACT 2. We have

$$g_1 = y + \sum_{j=0}^{\lfloor p/q \rfloor} c_k x^k, \quad g_{h+1} = f$$

where $c_k \in \mathbf{C}$, $p = \deg_x f/d$, $q = \deg_y f/d$, $d = \gcd\{\deg_x f, \deg_y f\}$ and $\lfloor p/q \rfloor$ is the maximal integer l such that $l \leq p/q$.

DEFINITION 4 (Abhyankar-Moh's condition). We call the conditions (1)–(5) concerning $\{\delta_0, \delta_1, \dots, \delta_h\}$ in Theorem 2 Abhyankar-Moh's condition.

In [6], we presented the following theorem to give normal forms of defining polynomials of curves with one place at infinity, and detailed a method of construction of their defining polynomials by computer.

Theorem 3 ([6]). *Let* $\{\delta_0, \delta_1, \ldots, \delta_h\}$ $(h \geq 1)$ *be a sequence of natural numbers satisfying Abhyankar-Moh's condition* (*see* Definition 4). *Set* $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$ $(1 \leq k \leq h+1)$ *and* $q_k = d_k/d_{k+1}$ $(1 \leq k \leq h)$.
(1) *We define* g_k $(0 \leq k \leq h+1)$ *as follows*:

$$\begin{cases} g_0 = x, \\ g_1 = y + \displaystyle\sum_{j=0}^{\lfloor p/q \rfloor} c_j x^j, \quad c_j \in \mathbf{C}, \ p = \frac{\delta_1}{d_2}, \ q = \frac{\delta_0}{d_2}, \\ g_{i+1} = g_i^{q_i} + a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{i-1}} g_0^{\bar{\alpha}_0} g_1^{\bar{\alpha}_1} \cdots g_{i-1}^{\bar{\alpha}_{i-1}} \\ \qquad + \displaystyle\sum_{(\alpha_0,\alpha_1,\ldots,\alpha_i)\in\Lambda_i} c_{\alpha_0\alpha_1\cdots\alpha_i} g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_i^{\alpha_i}, \\ \qquad a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{i-1}} \in \mathbf{C}^*, \ c_{\alpha_0\alpha_1\cdots\alpha_i} \in \mathbf{C} \quad (1 \leq i \leq h), \end{cases}$$

where $(\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{i-1})$ *is the sequence of* i *non-negative integers satisfying*

$$\sum_{j=0}^{i-1} \bar{\alpha}_j \delta_j = q_i \delta_i, \quad \bar{\alpha}_j < q_j \quad (0 < j < i)$$

and

$$\Lambda_i = \left\{ (\alpha_0, \alpha_1, \ldots, \alpha_i) \in \mathbf{N}^{i+1} \;\middle|\; \alpha_j < q_j \ (0 < j < i), \quad \alpha_i < q_i - 1, \ \sum_{j=0}^{i} \alpha_j \delta_j < q_i \delta_i \right\}.$$

Then, $g_0, g_1, \ldots, g_h$ *are approximate roots of* f $(= g_{h+1})$, *and* f *is the defining polynomial, monic in* y, *of a curve with one place at infinity having the* δ*-sequence* $\{\delta_0, \delta_1, \ldots, \delta_h\}$.
(2) *The defining polynomial* f, *monic in* y, *of a curve with one place at infinity having the* δ*-sequence* $\{\delta_0, \delta_1, \ldots, \delta_h\}$ *is obtained by the procedure of* (1), *and the values of parameters* $\{a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{i-1}}\}_{1\leq i\leq h}$ *and* $\{c_{\alpha_0\alpha_1\cdots\alpha_i}\}_{0\leq i\leq h}$ *are uniquely determined for* f.

4. Abhyankar's question and Sathaye-Stenerson's conjecture

DEFINITION 5 (planar semigroup). Let $\{\delta_0, \delta_1, \ldots, \delta_h\}$ $(h \geq 1)$ be a sequence of natural numbers satisfying Abhyankar-Moh's condition. A semigroup generated by $\{\delta_0, \delta_1, \ldots, \delta_h\}$ is said to be a planar semigroup.

DEFINITION 6 (polynomial curve). Let C be an algebraic curve defined by $f(x, y) = 0$, where $f(x, y)$ is an irreducible polynomial in $\mathbf{C}[x, y]$. We call C a polynomial curve, if C has a parametrization $x = x(t)$, $y = y(t)$, where $x(t)$ and $y(t)$ are polynomials in $\mathbf{C}[t]$.

The following question was introduced by Sathaye [13].

Abhyankar's Question. Let Ω be a planar semigroup. Is there a polynomial curve with a δ-sequence generating Ω?

Moh [11] showed that there is no polynomial curve with the δ-sequence $\{6, 8, 3\}$. But this is not a negative answer to the Abhyankar's question since there is a polynomial curve $(x, y) = (t^3, t^8)$ with the δ-sequence $\{3, 8\}$ that generates the same semigroup as above. Sathaye–Stenerson [14] proved that the semigroup generated by $\{6, 22, 17\}$ has no other δ-sequence generating the same semigroup, and proposed the following conjecture on this question.

Sathaye-Stenerson's Conjecture. There is no polynomial curve having the δ-sequence $\{6, 22, 17\}$.

By Theorem 3, the defining polynomial of the curve with one place at infinity having the δ-sequence $\{6, 22, 17\}$ is as follows:

$$\begin{aligned} f = &\left(g_2^2 + a_{2,1}x^2 g_1\right) + c_{5,0,0}x^5 + c_{4,0,0}x^4 + c_{3,0,0}x^3 + c_{2,0,0}x^2 \\ &+ c_{1,1,0}xg_1 + c_{1,0,0}x + c_{0,1,0}g_1 + c_{0,0,0} \end{aligned}$$

where

$$\begin{aligned} g_1 &= y + c_3x^3 + c_2x^2 + c_1x + c_0, \\ g_2 &= \left(g_1^3 + a_{11}x^{11}\right) + c_{10,0}x^{10} + c_{9,0}x^9 + c_{8,0}x^8 + (c_{7,1}g_1 + c_{7,0})x^7 \\ &\quad + (c_{6,1}g_1 + c_{6,0})x^6 + (c_{5,1}g_1 + c_{5,0})x^5 + (c_{4,1}g_1 + c_{4,0})x^4 \\ &\quad + (c_{3,1}g_1 + c_{3,0})x^3 + (c_{2,1}g_1 + c_{2,0})x^2 + (c_{1,1}g_1 + c_{1,0})x + c_{0,1}g_1 + c_{0,0}. \end{aligned}$$

Since a curve has one place at infinity and genus zero if and only if it has polynomial parametrization (see [2] or [3]), $\{6, 22, 17\}$ is a negative answer to the Abhyankar's question if it can be shown that the above type curve does not include a polynomial curve.

We summarize elementary facts about polynomial parametrizations (see [8], [9]).

DEFINITION 7 (proper polynomial parametrization). A polynomial parametrization $(x, y) = (u(t), v(t))$, where $u, v \in \mathbf{C}[t]$, is called proper if and only if t may be expressed as a rational function in x, y.

FACT 3. Any polynomial curve has a proper polynomial parametrization.

FACT 4. Let C be a polynomial curve defined by an irreducible polynomial equation $f(x, y) = 0$ in the complex affine plane $\mathbf{C}^2$. Let $(x, y) = (u(t), v(t))$ be a proper polynomial parametrization of C. Then $\deg_t u = \deg_y f$ and $\deg_t v = \deg_x f$.

Now we assume that there exists a polynomial curve having the δ-sequence $\{6, 22, 17\}$. Thus, the defining polynomial f of C has the above form using the approximate roots g_1 and g_2. By Fact 1 and Fact 4, this curve has the following polynomial parametrization:

$$\begin{cases} x = t^6 + a_1 t^5 + a_2 t^4 + a_3 t^3 + a_4 t^2 + a_5 t + a_6 \\ y = t^{22} + b_1 t^{21} + b_2 t^{20} + b_3 t^{19} + \cdots + b_{20} t^2 + b_{21} t + b_{22} \end{cases}$$

The following lemma presented in [14] plays a vital role to generate polynomial systems corresponding to δ-sequences.

Lemma 1. *Let C be a polynomial curve defined by $f(x, y) = 0$ having the proper polynomial parametrization $(u(t), v(t))$ and the δ-sequence $\{\delta_0, \delta_1, \delta_2\}$. Let g_2 be the second approximate root of f. Then $\deg_t g_2(u(t), v(t)) = \delta_2$.*

Proof. This follows immediately from the form of f $(= g_3)$ obtained by Theorem 3. $\square$

By this lemma, all formal terms with t-degree more than 17 in $g_2(x(t), y(t))$ must be eliminated. We get the polynomial system I from the coefficients of these terms. Furthermore, we can successively eliminate some variables by using polynomials with the form: $cz - h$ $(w_1, w_2, \dots, w_s)$ in I, where $c \in \mathbf{C}^*$, $z, w_1, w_2, \dots, w_s$ are variables and $h \in \mathbf{C}[w_1, w_2, \dots, w_s]$. As a result, we obtain the polynomial system with 11 variables and 17 polynomials.

$\{6, 22, 17\}$ is a negative answer to the Abhyankar's question if the polynomial system I does not have a root. For such a huge polynomial system it is suitable to compute the Gröbner basis of the ideal. However, it has been impossible to compute the Gröbner basis of I under *well-known* term orderings, even using a computer with 8 GB of memory.

5. A negative answer to Abhyankar's question

We find a *lighter* candidate for a negative answer to the Abhyankar's question. Let C be a curve with one place at infinity defined by a polynomial equation $f(x, y) = 0$ in the complex affine plane $\mathbf{C}^2$. Let M be the surface obtained by the minimal resolution of the singularity of C at infinity, and E the exceptional curve on M. We assume that $E_0, E_1, \dots, E_{i_h}$ are irreducible components of E, where the numbering of indices is by the ordering generated in the process to get M. The holomorphic 2-form $\omega = dx \wedge dy$ in $\mathbf{C}^2$ extends to a meromorphic 2-form on M. The canonical divisor $K = (\omega)$ has the support on E. We get $K = \sum_{l=0}^{i_h} k_l E_l$, where k_l is the zero order of ω on E_l. We call the zero order k_{i_h} of ω on E_{i_h} *k-number*. We obtain the following fact, since the proper transform of C intersects only E_{i_h} on M.

FACT 5. $K \cdot C = k_{i_h}$.

The k-number corresponding to the δ-sequence $\{6, 22, 17\}$ is 20. We classified δ-sequences with genus ≤ 50 into groups that generate the same semigroups. Furthermore, we listed δ-sequences with the following three properties: (i) There is no other δ-sequence that generates the same semigroup. (ii) The number of generators is 3. (iii) k-number ≥ -1. Then, we obtained $\{6, 15, 4\}, \{4, 14, 9\}, \{6, 15, 7\}, \{6, 21, 4\}, \{6, 10, 11\}, \{4, 18, 13\}, \ldots$. We got $\{6, 21, 4\}$ as a negative answer to the Abhyankar's question using Gröbner basis computations for polynomial systems corresponding to these δ-sequences. We show its details below.

First, we need to prove the uniqueness of $\{6, 21, 4\}$ since the above-mentioned classification is for genus ≤ 50. Let $\{\delta_0, \delta_1, \ldots, \delta_h\}$ be a sequence of natural numbers satisfying Abhyankar-Moh's condition, where $h \geq 1$. Set $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$ $(1 \leq k \leq h + 1)$ and $q_k = d_k/d_{k+1}$ $(1 \leq k \leq h)$.

Lemma 2. *For any k $(1 \leq k \leq h)$, $d_{k+1} \neq \delta_k$.*

Proof. Assume that there exists a natural number k $(1 \leq k \leq h)$ such that $d_{k+1} = \delta_k$. We get $q_k\delta_k = (d_k/d_{k+1})\delta_k = d_k$. From this and $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$, $q_k|\delta_i$ for each i $(0 \leq i \leq k - 1)$. By Abhyankar-Moh's condition (5), it follows that there exists an integer k_0 $(0 \leq k_0 \leq k - 1)$ such that $q_k\delta_k = \delta_{k_0}$. However, it must be $k_0 = k - 1$ from $q_k\delta_k = d_k$ and Abhyankar-Moh's condition (2). Thus, we obtain $d_k = \delta_{k-1}$ and $\delta_{k-1} > \delta_k$. We get $\delta_0 > \delta_1 > \cdots > \delta_{k-1} > \delta_k$, using the above result inductively, which is contradictory to Abhyankar-Moh's condition (1). □

DEFINITION 8 (primitive). An element of a semigroup is called primitive if it is not a sum of two nonzero elements of the semigroup.

Lemma 3 ([14]). *Let Ω be a semigroup and $\{\delta_0, \delta_1, \ldots, \delta_h\}$ a generators of Ω. If x is a primitive element of Ω, there exists a integer k $(0 \leq k \leq h)$ such that $x = \delta_k$.*

Proof. By the definition of primitive elements, this assertion is clear. □

Proposition 1. *The planar semigroup generated by $\{6, 21, 4\}$ has no other sequence satisfying Abhyankar-Moh's condition.*

Proof. Let Ω be the planar semigroup generated by $\{6, 21, 4\}$. $6, 21$ and 4 are primitive elements of Ω. Thus, by Lemma 3, $6, 21$ and 4 belong to any generating set of Ω. There are six possible cases for the order of $6, 21$ and 4.

(i) $\{\ldots, 6, \ldots, 21, \ldots, 4, \ldots\}$: By $\gcd\{6, 21, 4\} = 1$ and Abhyankar-Moh's condition (2), 4 is the last element of the sequence. By $\gcd\{6, 21\} = 3$, $\gcd\{6, 21, 4\} = 1$ and Abhyankar-Moh's condition (2), there is no element between of 6 and 21, and also between of 21 and 4. Furthermore, by Lemma 2, 6 is the first element of the sequence. Thus, we get $\{6, 21, 4\}$.

(ii) $\{\dots, 21, \dots, 6, \dots, 4, \dots\}$: We get $\{21, 6, 4\}$ in the same way as (i). But this is contradictory to Abhyankar-Moh's condition (1).

(iii) $\{\dots, 4, \dots, 21, \dots, 6, \dots\}$: By $\gcd\{4, 21\} = 1$, this case is impossible.

(iv) $\{\dots, 21, \dots, 4, \dots, 6, \dots\}$: By $\gcd\{21, 4\} = 1$, this case is impossible.

(v) $\{\dots, 6, \dots, 4, \dots, 21, \dots\}$: We get $\{6, 4, 21\}$ in the same way as (i). But this is contradictory to Abhyankar-Moh's condition (1).

(vi) $\{\dots, 4, \dots, 6, \dots, 21, \dots\}$: We get $\{4, 6, 21\}$ in the same way as (i). From $d_1 = 4$, $d_2 = \gcd\{4, 6\} = 2$, $q_1 = d_1/d_2 = 2$. Thus, $q_1\delta_1 = 12 < \delta_2$. But this is contradictory to Abhyankar-Moh's condition (4).

As a consequence, the generating sequence of Ω satisfying Abhyankar-Moh's condition is only $\{6, 21, 4\}$. □

We assume that there exists a polynomial curve having the δ-sequence $\{6, 21, 4\}$. The defining polynomial of this curve is as follows:

$$f = g_2^3 + a_{2,0}x^2 + c_{1,0,1}xg_2 + c_{1,0,0}x + c_{0,0,1}g_2 + c_{0,0,0}$$

where

$$\begin{aligned} g_2 &= g_1^2 + a_7x^7 + c_{6,0}x^6 + c_{5,0}x^5 + c_{4,0}x^4 + c_{3,0}x^3 \\ &\quad + c_{2,0}x^2 + c_{1,0}x + c_{0,0}, \\ g_1 &= y + c_3x^3 + c_2x^2 + c_1x + c_0. \end{aligned}$$

By the substitution of g_1 for g_2 and changing parameters, we get

$$\begin{aligned} g_2 &= y^2 + a_7x^7 + y\bigl(c_{3,1}x^3 + c_{2,1}x^2 + c_{1,1}x + c_{0,1}\bigr) \\ &\quad + c_{6,0}x^6 + c_{5,0}x^5 + c_{4,0}x^4 + c_{3,0}x^3 + c_{2,0}x^2 + c_{1,0}x + c_{0,0}. \end{aligned}$$

We can set $a_7 = -1$ by the automorphism of $\mathbf{C}[x, y]$, $x \mapsto -a^{-1/7}x$, $y \mapsto y$. By $x \mapsto x + c_{6,0}/7$, we can remove the term $c_{6,0}x^6$. Further, by $y \mapsto y - (c_{3,1}x^3 + c_{2,1}x^2 + c_{1,1}x + c_{0,1})/2$, we can remove the terms $y(c_{3,1}x^3 + c_{2,1}x^2 + c_{1,1}x + c_{0,1})$. The proper polynomial parametrization of this curve is of the following form:

$$\begin{cases} x = t^6 + a_1t^5 + a_2t^4 + a_3t^3 + a_4t^2 + a_5t + a_6 \\ y = t^{21} + b_1t^{20} + b_2t^{19} + b_3t^{18} + \cdots + b_{19}t^2 + b_{20}t + b_{21} \end{cases}$$

By the automorphism of $\mathbf{C}[t]$, $t \mapsto t - a_1/6$, we may remove the term a_1t^5 in $x(t)$. By Lemma 1, we get $\deg_t g_2(x(t), y(t)) = 4$. All formal terms with t-degree more than 4 in $g_2(x(t), y(t))$ must be eliminated. We obtain the polynomial system J from the coefficients of these terms. Furthermore, we can successively eliminate the variables

$b_1, c_{5,0}, c_{4,0}, c_{3,0}, c_{2,0}, c_{1,0}, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{13}, b_{14}, b_{15}, b_{16}, b_{17}, b_{19}, b_{20}$ and b_{21} in this order by using polynomials with the form: $cz - h\ (w_1, w_2, \ldots, w_s)$ in J, where $c \in \mathbf{C}^*$, $z, w_1, w_2, \ldots, w_s$ are variables and $h \in \mathbf{C}[w_1, w_2, \ldots, w_s]$. As a result, we can get the polynomial system with 7 variables $\{a_2, a_3, a_4, a_5, a_6, b_{12}, b_{18}\}$ and 13 polynomials. We denote the obtained polynomial system by the same character J.

We used total degree reverse lexicographic ordering (DRL) with $a_2 \succ a_3 \succ a_4 \succ a_5 \succ a_6 \succ b_{12} \succ b_{18}$ to the Gröbner basis computation. The CPU time for the computation was 3 hours 40 minutes and the required memory 850 MB. The computation was conducted on a dual AMD AthlonMP 2200+ (1.8 GHz) machine with 4 GB memory running FreeBSD 4.7. The computer algebra system used was Risa/Asir [12].

The obtained Gröbner basis $G_{\{6,21,4\}}$ of the ideal $Id(J)$ was not $\{1\}$. However, the normal form of the coefficient p of the term with t-degree = 4 in $g_2(x(t), y(t))$ with respect to $G_{\{6,21,4\}}$ is 0. By the property of Gröbner bases for ideal membership, this shows that $p \in Id(J)$. Thus, we get $\deg_t g_2(x(t), y(t)) < 4$. Since this is contradictory to $\deg_t g_2(x(t), y(t)) = 4$, there is no polynomial curve having the δ-sequence $\{6, 21, 4\}$. Consequently, $\{6, 21, 4\}$ is a negative answer to the Abhyankar's question.

REMARK. We computed the Gröbner bases corresponding to the δ-sequences $\{6, 15, 4\}, \{4, 14, 9\}$ and $\{6, 15, 7\}$, and obtained the normal forms of the coefficients of terms with t-degree δ_2 in $g_2(x(t), y(t))$ with respect to them. However, they were not 0 unlike the case of $\{6, 21, 4\}$.

6. Gröbner basis computation using weighted ordering

It is well-known that Gröbner basis computation is accelerated by setting weights if the input polynomial system is quasi homogeneous (see [10]). The polynomial system J corresponding to the δ-sequence $\{6, 21, 4\}$ is quasi homogeneous by the constructing method, and J become homogeneous by setting the indices of each variable as weights. We get the following weighted ordering: $b_{18} \succ b_{12} \succ a_6 \succ a_5 \succ a_4 \succ a_3 \succ a_2$ with weights $\{18, 12, 6, 5, 4, 3, 2\}$.

After various trials and errors, we obtained the Gröbner basis of the ideal $Id(J)$ by lexicographic ordering (LEX) with the above setting in a very short time and only 11 MB of memory. For verification of the results obtained by Asir and a comparison of computation time, we used another computer algebra system Singular 2.0.4 [7]. The results obtained by Singular coincided with Asir. The computation times are as follows:

δ-seq.	System	DRL	Sawada	Sawada weight DRL	Sawada weight LEX	Weight DRL	Weight LEX
$\{6, 21, 4\}$	Asir	5884	2.17	0.28	0.26	0.24	0.17
	Singular	53 h	—	0.35	0.34	0.31	0.17

'h' means hour. The time unit of values without 'h' are seconds. The line '—' means out of memory. 'Sawada' is an automatic block ordering by Dr. Sawada in AIST (see [15]). Sawada ordering is obtained by a heuristic algorithm.

We tried to compute the Gröbner basis of $\{6, 22, 17\}$-type by using weighted ordering. Let I be the polynomial system corresponding to the δ-sequence $\{6, 22, 17\}$ (see Section 4). I has 11 variables $\{a_2, a_3, a_4, a_5, a_6, b_2, b_8, b_{12}, b_{14}, b_{18}, b_{20}\}$ and 17 polynomials. Further, I is also quasi homogeneous, and becomes homogeneous by setting the indices of each variable as weights. As the above, we get the following weighted ordering: $b_{20} \succ b_{18} \succ b_{14} \succ b_{12} \succ b_8 \succ a_6 \succ a_5 \succ a_4 \succ a_3 \succ b_2 \succ a_2$ with weights $\{20, 18, 14, 12, 8, 6, 5, 4, 3, 2, 2\}$. We obtained the Gröbner basis of the ideal $Id(I)$ by LEX with the above setting. The memory used was 116 MB. The computation times were as follows:

δ-seq.	System	DRL	Sawada	Sawada weight DRL	Sawada weight LEX	Weight DRL	Weight LEX
$\{6, 22, 17\}$	Asir	—	—	303.8	382.8	2368	285.7
	Singular	—	—	92 h	92 h	326 h	78 h

Let $G_{\{6,22,17\}}$ be the obtained Gröbner basis of $Id(I)$. Let q be the coefficient of the term with t-degree $= 17$ in $g_2(x(t), y(t))$. Further, let $\bar{q}$ be the normal form of q with respect to $G_{\{6,22,17\}}$. We got that the normal form of $\bar{q}^3$ with respect to $G_{\{6,22,17\}}$ is 0 by Asir and Singular. This shows that $q \in \sqrt{Id(I)}$. This is contradictory to $\deg_t g_2(x(t), y(t)) = 17$. Consequently, the Sathaye-Stenerson's conjecture is also true.

The data files for polynomial systems that appeared in this paper are available from `http://www.fukuoka-edu.ac.jp/~fujimoto/abh2/`.

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