Graff, G. and Nowak-Przygodzki, P.
Osaka J. Math.
43 (2006), 485–495

PERIODICITY OF A SEQUENCE OF LOCAL FIXED POINT INDICES OF ITERATIONS

GRZEGORZ GRAFF* and PIOTR NOWAK-PRZYGODZKI*

(Received June 14, 2005)

Abstract

The classical theorem of Shub and Sullivan states that a sequence of local fixed point indices of iterations of a C^1 self-map of $\mathbb{R}^m$ is periodic. The paper generalizes this result to a wider class of maps.

1. Introduction

Let $\operatorname{ind}(f, 0)$ be a local fixed point index at 0, where f is a self-map of $\mathbb{R}^m$. If 0 is an isolated fixed point for each f^n, then $\{\operatorname{ind}(f^n, 0)\}_{n=1}^{\infty}$ is well-defined.

The sequence of indices of iterations is a very useful instrument in periodic point theory. Its applications are specially fruitful if it is known that $\{\operatorname{ind}(f^n, 0)\}_{n=1}^{\infty}$ is a periodic sequence. This knowledge can be successfully applied in detecting periodic points (cf. [8], [9]), estimating the number of orbits (cf. [1], [10]), finding orbits with the special pattern (cf. [11]) and studying dynamical properties of f in a neighborhood of a fixed point (cf. [6], [12]).

Thus, the important task is to identify classes of maps for which a sequence of local indices of iterations is periodic. Among such classes are continuous self-maps of the real line, planar homeomorphisms (cf. [5]), simplicial maps of smooth type (cf. [4], [14]). In 1974 Shub and Sullivan proved that also C^1 self-maps of $\mathbb{R}^m$ have periodic indices of iterations (cf. [13]). It is worth pointing out that in 1983 Chow, Mallet-Paret and Yorke found additional relations among the elements of $\{\operatorname{ind}(f^n, 0)\}_{n=1}^{\infty}$ in C^1-case (cf. [2]).

The core of the reasoning applied by Shub and Sullivan was to approximate $\mathrm{Id} - g^k$ by $(\mathrm{Id} + B + B^2 + \cdots + B^{k-1})(\mathrm{Id} - g)$, where B is a linear map, in such a way that $\sum_{j=0}^{k-1} B^j$ is a diffeomorphism, which implies that $\operatorname{ind}(g^k, 0) = \pm \operatorname{ind}(g, 0)$. These authors gave also an example of a continuous map f which is differentiable in all points of the unit ball except of its center 0, but $\{\operatorname{ind}(f^n, 0)\}_{n=1}^{\infty}$ is unbounded, which shows that their theorem cannot be extended on all continuous maps. In literature the theorem of Shub and Sullivan is presented as a consequence of continuous differentiability, for example in well-known monograph of dynamical systems [7] it is discussed in the

2000 Mathematics Subject Classification. Primary 37C25; Secondary 37B30.

*Research supported by KBN grant No. 1 P03A 039 29.

chapter entitled *The role of smoothness*. We show in this paper that the main idea of Shub and Sullivan, mentioned above, can be successfully adapted for a larger class of maps, which we call Shub-Sullivan class.

The paper is organized as follows. In the first section we introduce notation and formulate the main result. Next, we give the proof of it, which is based on some lemmas. Finally, we discuss how large Shub-Sullivan class is and distinguish its subclass, called *orbital class*, for which an effective estimate of the period of $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ is possible.

2. The statement of the result

For the rest of the paper we make a general assumption that we consider only such continuous maps $f: U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^m$ and U is an open neighborhood of 0, that $f(0) = 0$ and 0 is an isolated fixed point for each f^n, $n > 0$.

DEFINITION 2.1. Let $f: U \to \mathbb{R}^m$. We will say that a map f is deviated from a linear map $A: \mathbb{R}^m \to \mathbb{R}^m$ if it satisfies the following condition:

$$\left\| f(x) - f^2(x) - A(x - f(x)) \right\| = o(\|x - f(x)\|). \tag{2.1}$$

For $m \geq 1$ we define $P_m = \mathrm{Fix}(f^m) \setminus \bigcup_{0<n<m} \mathrm{Fix}(f^n)$. If $P_m(f) \neq \emptyset$, then m is called a minimal period of f. The set of all minimal periods of f is denoted by $\mathrm{Per}(f)$.

The main result of the paper is the generalization of Shub and Sullivan theorem to the following class of maps:

$$SS(m) = \{f: U \to \mathbb{R}^m : \exists_A \forall_{i \in \mathrm{Per}(A)} f^i \text{ is deviated from } A^i\}.$$

The period of a sequence of indices of iterations of a map in $SS(m)$ will be expressed in terms of $\mathrm{Per}(A)$. On the other hand the form of $\mathrm{Per}(A)$ is known, it can be described by a use of eigenvalues of A (cf. [1], [2], [9]), namely:

$$\mathrm{Per}(A) = \{\mathrm{LCM}(K): K \subset \sigma_1(A)\},$$

where $\sigma_1(A)$ is the set of degrees of all primitive roots of unity contained in the spectrum $\sigma(A)$, $\mathrm{LCM}(K)$ denotes the number equal to the least common multiple of all elements in K, we define $\mathrm{LCM}(\emptyset) = 1$.

Notice that the set $\mathrm{Per}(A)$ is closed for the operation of taking the least common multiple, i.e. if $s \in \mathrm{Per}(A)$ and $t \in \mathrm{Per}(A)$ then $\mathrm{LCM}(s, t) \in \mathrm{Per}(A)$. Thus $\mathrm{LCM}(\mathrm{Per}(A)) = \max\{l \in \mathrm{Per}(A)\}$.

By ν_- we will denote the number of eigenvalues of A less than -1, counting with multiplicity. Let

$$r = \begin{cases} \max\{l \in \mathrm{Per}(A)\} & \text{if} \quad 2 \mid \nu_-, \\ 2\max\{l \in \mathrm{Per}(A)\} & \text{if} \quad 2 \nmid \nu_-. \end{cases}$$

Now we may formulate our main theorem:

Theorem 2.2. *If $f \in SS(m)$, then $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ is periodic and its period is not greater than r.*

3. Proof of the main theorem

In the proof of Theorem 2.2 we will need some lemmas.

Lemma 3.1. *If for some natural i there is: $\|g(x) - g^{i+1}(x) - B(x - g^i(x))\| = o(\|x - g^i(x)\|)$, then for each natural j:*

$$\big\|g^j(x) - g^{j+i}(x) - B^j\big(x - g^i(x)\big)\big\| = o\big(\|x - g^i(x)\|\big).$$

Proof. Assume inductively that the thesis is valid for $j-1$, we prove it for j. First notice that by the assumption of the lemma for arbitrarily chosen ε there is U_ε, a neighborhood of 0, such that for every $x \in U_\varepsilon$: $\|g(x) - g^{i+1}(x)\| < \|B(x - g^i(x))\| + \varepsilon\|x - g^i(x)\| \le (\varepsilon + \|B\|)\|x - g^i(x)\|$. By this and inductive hypothesis we get for every $x \in V_\varepsilon$, where $V_\varepsilon \subset U_\varepsilon$ sufficiently small neighborhood of 0:

$$\begin{aligned}\big\|g^{j-1}(g(x)) - g^{j+i-1}(g(x)) - B^{j-1}\big(g(x) - g^{i+1}(x)\big)\big\| &< \varepsilon\big\|g(x) - g^{i+1}(x)\big\| \\ &< \varepsilon(\varepsilon + \|B\|)\big\|x - g^i(x)\big\|.\end{aligned}$$

Finally:

$$\begin{aligned}&\big\|g^j(x) - g^{j+i}(x) - B^j\big(x - g^i(x)\big) + B^{j-1}\big(g(x) - g^{i+1}(x)\big) - B^{j-1}\big(g(x) - g^{i+1}(x)\big)\big\| \\ &\le \big\|g^j(x) - g^{j+i}(x) - B^{j-1}\big(g(x) - g^{i+1}(x)\big)\big\| + \big\|B^{j-1}\big(g(x) - g^{i+1}(x)\big) - B^j\big(x - g^i(x)\big)\big\| \\ &< \varepsilon(\varepsilon + \|B\|)\big\|x - g^i(x)\big\| + \big\|B^{j-1}\big\|\varepsilon\big\|x - g^i(x)\big\|,\end{aligned}$$

where in the last inequality we used the assumption of the lemma. This ends the proof. $\square$

Taking $i = 1$ in Lemma 3.1 we obtain the following corollary:

Corollary 3.2. *If g is deviated from B, then for each natural j there is:*

$$\big\|g^j(x) - g^{j+1}(x) - B^j(x - g(x))\big\| = o(\|x - g(x)\|).$$

Lemma 3.3. *Let g be deviated from a linear map B and the map $\sum_{j=0}^{k-1} B^j$ be non-singular. Then*:

$$\operatorname{ind}(g^k, 0) = \operatorname{sgn}\det\left(\sum_{j=0}^{k-1} B^j\right) \cdot \operatorname{ind}(g, 0). \tag{3.2}$$

Proof. We use the following well-known fact: let $s, h\colon U \to \mathbb{R}^n$ be continuous maps, $s^{-1}(0) = \{0\}$ and $\|s(x) - h(x)\| < \|s(x)\|$ for $x \in U \setminus \{0\}$, then $h^{-1}(0) = \{0\}$ and $\deg_0(s) = \deg_0(h)$, where $\deg_0(s)$ denotes the topological degree of s at 0. This statement is a consequence of the observation that the linear homotopy between s and h has no zeros in $U \setminus \{0\}$.

Now we take $s(x) = \sum_{j=0}^{k-1} B^j(x - g(x))$, $h(x) = x - g^k(x)$.

We define the map w_j by the following equation:

$$g^j(x) - g^j(g(x)) = B^j(x - g(x)) + w_j(x.)$$

We have:

$$x - g^k(x) = \sum_{j=0}^{k-1}\left(g^j(x) - g^{j+1}(x)\right) = \left[\sum_{j=0}^{k-1} B^j\right](x - g(x)) + \sum_{j=0}^{k-1} w_j(x). \tag{3.3}$$

By Corollary 3.2 for every ε and $j < k$ there is a neighborhood $U_{\varepsilon,j}$ of 0 such that for each $x \in U_{\varepsilon,j}$ there is:

$$\|w_j(x)\| < \varepsilon\|x - g(x)\|. \tag{3.4}$$

Thus, for $x \in U_\varepsilon$-sufficiently small neighborhood of 0 there is: $\left\|\sum_{j=0}^{k-1} w_j(x)\right\| < k\varepsilon\|x - g(x)\|$. As the linear map $\sum_{j=0}^{k-1} B^j$ is non-singular, we get for ε which is small enough:

$$\left\|\sum_{j=0}^{k-1} w_j(x)\right\| < \left\|\left(\sum_{j=0}^{k-1} B^j\right)(x - g(x))\right\|,$$

which is equivalent to $\|s(x) - h(x)\| < \|s(x)\|$ for $x \in U_\varepsilon \setminus \{0\}$.

Finally by the multiplicativity of topological degree we get:

$$\begin{aligned}\operatorname{ind}(g^k, 0) &= \deg_0(\mathrm{Id} - g^k) = \deg_0\left[\left(\sum_{j=0}^{k-1} B^j\right)(x - g(x))\right]\\ &= \operatorname{sgn}\det\left(\sum_{j=0}^{k-1} B^j\right)\deg_0(\mathrm{Id} - g) = \operatorname{sgn}\det\left(\sum_{j=0}^{k-1} B^j\right)\operatorname{ind}(g, 0).\end{aligned}$$ □

Now, for a given linear map A we define the function $q\colon \mathbb{N} \to \mathbb{N}$ by: $q(n) = \max\{l \in \mathrm{Per}(A)\colon l \mid n\}$.

The proof of the following lemma may be found in [9] (cf. also Proposition 3.2.30 in [6]).

Lemma 3.4. *Let $b = n/(q(n))$, then for each natural n: $\sum_{j=0}^{b-1} A^{q(n)j}$ is non-singular and*

$$\operatorname{sgn}\det\left(\sum_{j=0}^{b-1} A^{q(n)j}\right) = \begin{cases} -1 & \textit{if} \quad 2 \mid n,\ 2 \nmid q(n),\ 2 \nmid \nu_-, \\ 1 & \textit{in the opposite case.} \end{cases}$$

Proof of Theorem 2.2. By the assumption of the theorem for every $i \in \mathrm{Per}(A)$ f^i is deviated from A^i. On the other hand for every natural n, $q(n) \in \mathrm{Per}(A)$, thus the thesis of Lemma 3.3 is valid for $k = b$, the map $g = f^{q(n)}$ and $B = A^{q(n)}$. Applying also Lemma 3.4 we get:

$$\operatorname{ind}(f^n, 0) = \operatorname{ind}\big(g^{n/q(n)}, 0\big) = \begin{cases} -\operatorname{ind}\big(f^{q(n)}, 0\big) & \text{if} \quad 2 \mid n,\ 2 \nmid q(n),\ 2 \nmid \nu_-, \\ \operatorname{ind}\big(f^{q(n)}, 0\big) & \text{in the opposite case.} \end{cases}$$

Let us consider now the case when $2 \mid n$, $2 \nmid q(n)$ and $2 \nmid \nu_-$. Remind that for $2 \nmid \nu_-$ we defined $r = 2\max\{l \in \mathrm{Per}(A)\}$. It is obvious that $q(n) = q(n + r)$, thus $2 \nmid q(n)$ iff $2 \nmid q(n + r)$ and, as $2 \mid r$, $2 \mid n$ iff $2 \mid (n + r)$. Finally, again by Lemma 3.3 and Lemma 3.4:

$$\operatorname{ind}\big(f^{n+r}, 0\big) = -\operatorname{ind}\big(f^{q(n+r)}, 0\big) = -\operatorname{ind}\big(f^{q(n)}, 0\big) = \operatorname{ind}\big(f^n, 0\big).$$

Analogously we prove the periodicity in the remaining case. □

4. On the Shub-Sullivan class

Let $U \subset \mathbb{R}^m$ be a neighborhood of 0. We define the following classes of maps.

By a standard way, we denote by $C^1(m)$ smooth maps on U.

By $\widetilde{C}(m)$ we denote maps $f\colon U \to \mathbb{R}^m$ such that f is differentiable in U and the derivative Df is continuous at 0. In the forthcoming definitions A is a linear self-map of $\mathbb{R}^m$.

$$Z(m) = \{f : \exists_A \| f(x) - f(y) - A(x - y)\| = o(\|x - y\|)\}.$$

$$OC(m) = \left\{ f : \exists_A \forall_{i \in \mathrm{Per}(A)} \big\| f(x) - f^{i+1}(x) - A\big(x - f^i(x)\big)\big\| = o\big(\big\|x - f^i(x)\big\|\big) \right\},$$

which we call *orbital class*. Remind that by $SS(m)$ we denote:

$$SS(m) = \left\{ f : \exists_A \forall_{i \in \mathrm{Per}(A)} \big\| f^i(x) - f^{2i}(x) - A^i\big(x - f^i(x)\big)\big\| = o\big(\big\|x - f^i(x)\big\|\big) \right\}.$$

DEFINITION 4.1. Let f belong to one of the classes given above, then we will say that a map A, which appears in the definition of the class, corresponds to f.

Theorem 4.2. *The following inclusions hold and are proper*:

$$C^1(m) \subset \widetilde{C}(m) \subset Z(m) \subset OC(m) \subset SS(m).$$

Proof. First we show that all inclusions hold.

1. $C^1(m) \subset \widetilde{C}(m)$ is obvious.

2. $\widetilde{C}(m) \subset Z(m)$.

Let $f \in \widetilde{C}(m)$, $f = Df(0) + s$. Then, by the assumption, $\|Ds(x)\| = \|Df(x) - Df(0)\| \to 0$, when x converges to 0. By Mean Value Theorem for x, y in a small enough and closed ball with the center at 0 and ξ in an open line segment joining x and y there is: $\|s(x) - s(y)\| \leq \|Ds(\xi)\| \|x - y\|$, thus $\|s(x) - s(y)\| = o(\|x - y\|)$. Taking $A = Df(0)$ we finally obtain: $\|f(x) - f(y) - A(x - y)\| = o(\|x - y\|)$.

REMARK 4.3. Notice that if $f \in Z(m)$ and A corresponds to f, then $f(x) = A(x) + s(x)$, where $s \in Z(m)$ and $Ds(0) = 0$. As a consequence f must be differentiable at 0.

3. $Z(m) \subset OC(m)$.

If $f \in Z(m)$, then there is a linear map A such that $\|f(x) - f(y) - A(x - y)\| = o(\|x - y\|)$. Let $i \in \mathrm{Per}(A)$ and let us take $y = f^i(x)$. We get $\|f(x) - f^{i+1}(x) - A(x - f^i(x))\| = o(\|x - f^i(x)\|)$, thus $f \in OC(m)$.

4. $OC(m) \subset SS(m)$.

If $f \in OC(m)$, then there is a linear map A such that for each $i \in \mathrm{Per}(A)$ there is: $\|f(x) - f^{i+1}(x) - A(x - f^i(x))\| = o(\|x - f^i(x)\|)$, so f satisfies the assumption of Lemma 3.1. Now, in the thesis of this lemma we put $g = f$, $B = A$ and $j = i$ and state that $f \in SS(m)$.

This ends the proof of first part of Theorem 4.2. Now we show that all inclusions are proper.

1′. $\widetilde{C}(m) \setminus C^1(m) \neq \emptyset$ is obvious.

2′. $Z(m) \setminus \widetilde{C}(m) \neq \emptyset$. We put $s(1/n) = 1/n^2$ and extend s to a piecewise linear map $s\colon [0, 1] \to \mathbb{R}$. The function s is not differentiable at any neighborhood of 0, but satisfies $|s(x) - s(y)| = o(|x - y|)$. Now define $s_m(x)\colon B(0, 1) \to \mathbb{R}^m$ by the formula: $s_m(x) = s(\|x\|)x$, where $B(0, 1)$ is the unit ball in $\mathbb{R}^m$. There is: $\|s_m(x) - s_m(y)\| = \|s(\|x\|)(x - y) + [s(\|x\|) - s(\|y\|)]y\| \leq \|s(\|x\|)(x - y)\| + o(\|x\| - \|y\|)\|y\| = o(\|x - y\|)$. For any linear map A, we put $f(x) = A(x) + s_m(x)$. Then $f \in Z(m)$, but $f \notin \widetilde{C}(m)$.

3′. $OC(m) \setminus Z(m) \neq \emptyset$.

Let $f : U \to \mathbb{R}^m$ be given by the formula:

$$f(x, y) = \big(x + x^2 + \|y\|, y\big),$$

where $(x, y) \in \mathbb{R} \times \mathbb{R}^{m-1}$. Let us take as A identity map, $\mathrm{Per}(A) = \{1\}$.

We have: $f(x, y) - f^2(x, y) = -((x + x^2 + \|y\|)^2 + \|y\|, 0)$, $A((x, y) - f(x, y)) = -(x^2 + \|y\|, 0)$. Thus:

$$\begin{aligned}&\left\| f(x, y) - f^2(x, y) - A((x, y) - f(x, y))\right\| \\ &= \left|2x^3 + 2x\|y\| + 2x^2\|y\| + x^4 + \|y\|^2\right| = o\big(x^2 + \|y\|\big) = o(\|(x, y) - f(x, y)\|).\end{aligned}$$

We see that $f \in OC(m)$ but, as f is not differentiable at 0, by Remark 4.3, $f \notin Z(m)$. It is not difficult to observe that $g\colon U \to \mathbb{R}$, given by the formula $g(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $g(0) = 0$ satisfies: $g \in OC(1) \setminus Z(1)$.

4′. $SS(m) \setminus OC(m) \neq \emptyset$. Let $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} -\dfrac{1}{2}x - x^2 & \text{if} \quad x \geq 0, \\ -2x + x^2 & \text{if} \quad x < 0. \end{cases}$$

For $x \geq 0$ there is: $x - f(x) = (3/2)x + o(x)$, $f(x) - f^2(x) = -(3/2)x + o(x)$, $f^2(x) - f^4(x) = -(9/4)x^2 + o(x^2)$, $x - f^2(x) = -(9/4)x^2 + o(x^2)$.

For $x < 0$ there is: $x - f(x) = 3x + o(x)$, $f(x) - f^2(x) = -3x + o(x)$, $f^2(x) - f^4(x) = (9/2)x^2 + o(x^2)$, $x - f^2(x) = (9/2)x^2 + o(x^2)$.

The assumption that f is in Shub-Sullivan class implies that f must be deviated from some linear map A, the only possibility here is $A(x) = -x$. We have then: $\mathrm{Per}(A) = \{1, 2\}$. Then two conditions equivalent to $f \in SS(1)$ are satisfied: $|f(x) - f^2(x) - (-1)(x - f(x))| = o(|x - f(x)|)$ and $|f^2(x) - f^4(x) - (-1)^2(x - f^2(x))| = o(|x - f^2(x)|)$.

On the other hand $f \notin OC(1)$ as we conclude by the following calculation.

For $x \geq 0$ there is: $f(x) - f^3(x) = (9/8)x^2 + o(x^2)$, for $x < 0$ there is: $f(x) - f^3(x) = -9x^2 + o(x^2)$. In both cases we take $i = 2$ and see that the condition $|f(x) - f^3(x) - A(x - f^2(x))| = o(|x - f^2(x)|)$ is satisfied only for A such that $A(x) = -(1/2)x$ if $x \geq 0$ and $A(x) = -2x$ if $x < 0$.

In m-dimensional case we take a continuous map $g\colon S^{m-1} \to \mathbb{R}$ such that $g(x)g(-x) = 1$ with $g(x) \neq 1$ at some x, and for $x \neq 0$ define $f : B(0, 1) \to \mathbb{R}^m$, where $B(0, 1)$ is the unit ball in $\mathbb{R}^m$, by:

$$f(x) = -\left(g\left(\frac{x}{\|x\|}\right) + \|x\|\right) x.$$

It is easy to see that f maps each diameter onto itself, acting in the same way as in the one-dimensional case. As a consequence: $A = -\mathrm{Id}$ and $f \in SS(m)$. On the other hand for some directions (such that $g(x) \neq 1$) f is not in the orbital class in this direction, thus $f \notin OC(m)$. $\square$

The examples which are given above illustrate that it is relatively easy to find elements of the each class that do not belong to the previous one. However, sequences

$\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ given in the examples have very simple form: a map f defined in **4**$'$ is a source type and so generates the sequence of period 1 or 2, depending on the dimension. On the other hand, it is known that in general case $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ cannot take arbitrary integer values, but must satisfy some congruences established by A. Dold (cf. [3]). In [2] there are further restrictions for possible sequences in C^1-case. Therefore a natural question is:

Problem 1. *Which integer sequences can be realized as indices of iterations for a map f in Shub-Sullivan class.*

5. Estimation of a period

Let f be a fixed map which belongs to one of the classes under consideration and let A corresponds to f (cf. Definition 4.1). Then A is unique for maps in $Z(m)$ and is not uniquely determined for maps in $OC(m)$ and $SS(m)$. This interesting phenomenon is clearly visible for f given in the example **3**$'$ of the previous section ($m \geq 2$). Not only $A = \mathrm{Id}$, but every map $A'\colon \mathbb{R} \times \mathbb{R}^{m-1} \to \mathbb{R} \times \mathbb{R}^{m-1}$ of the form: $A' = \left[\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right]$ corresponds to $f \in OC(m)$. On the other hand, the greater is the number of eigenvalues which are primitive roots of unity contained in $\sigma(A')$ the worse is the estimate for the period of $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ given in Theorem 2.2. The motivation to distinguish orbital class is the fact that for $f \in OC(m)$ we may determine "minimal" subspace $V \subset \mathbb{R}^m$ and a linear map $G\colon V \to V$ without unessential eigenvalues such that each A that corresponds to f is an extension of G (Theorem 5.3).

Let i be a natural number, we define the set:

$$\widetilde{V}_i = \left\{ x \in \mathbb{R}^m \colon x = \lim_{n\to\infty} \frac{x_n - f^i(x_n)}{\|x_n - f^i(x_n)\|}, \quad \text{where} \quad x_n \to 0 \right\}.$$

Next we define a map $\widetilde{G}\colon \bigcup_{i\in\mathbb{N}} \widetilde{V}_i \to \mathbb{R}^m$ in the following way: if $v \in \widetilde{V}_i$ i.e. there is a sequence x_n which converges to zero such that $v = \lim_{n\to\infty}(x_n - f^i(x_n))/\|x_n - f^i(x_n)\|$, then we put $\widetilde{G}(v) = \lim_{n\to\infty}(f(x_n) - f^{i+1}(x_n))/\|x_n - f^i(x_n)\|$. We say that $\widetilde{G}(v)$ is well-defined if this limit exists and is independent both of the choice of x_n and i.

Let now $V = \mathrm{span} \bigcup_{i\in\mathbb{N}} \widetilde{V}_i$. If $\widetilde{G}$ is well-defined on $\bigcup_i \widetilde{V}_i$ and extends to a linear map on V, then we denote this extension by G and say that G is well-defined.

Remind that $A\colon \mathbb{R}^m \to \mathbb{R}^m$ corresponds to $f \in OC(m)$ if

$$\forall_{i\in\mathrm{Per}(A)} \left\| f(x) - f^{i+1}(x) - A(x - f^i(x)) \right\| = o\left(\left\| x - f^i(x) \right\|\right).$$

Define $V_i = \mathrm{span}\, \widetilde{V}_i$.

Lemma 5.1. *If A corresponds to $f \in OC(m)$, then for each $i \in \mathrm{Per}(A)$ there is*: $A(V_i) \subset V_i$.

Proof. Let $v \in \widetilde{V}_i$, then $v = \lim_{n\to\infty}(x_n - f^i(x_n))/\|x_n - f^i(x_n)\|$. From the fact that A corresponds to f we get:

$$\lim_{n\to\infty} \frac{f(x_n) - f^{i+1}(x_n)}{\|x_n - f^i(x_n)\|} = \lim_{n\to\infty} A\left(\frac{x_n - f^i(x_n)}{\|x_n - f^i(x_n)\|}\right) = A(v). \tag{5.5}$$

On the other hand:

$$\begin{aligned}&\lim_{n\to\infty} \frac{f(x_n) - f^{i+1}(x_n)}{\|x_n - f^i(x_n)\|}\\ &= \lim_{n\to\infty} \frac{f(x_n) - f^{i+1}(x_n)}{\|f(x_n) - f^{i+1}(x_n)\|}\,\frac{\|f(x_n) - f^{i+1}(x_n)\|}{\|x_n - f^i(x_n)\|} = w\alpha,\end{aligned}$$

where $w \in \widetilde{V}_i$, $\alpha \in \mathbb{R}_+$, which is a consequence of the obvious fact that for a given sequence $\{a_n\} \subset \mathbb{R}^m$ and $a_0 \neq 0$, there is: $a_n \to a_0$ iff $a_n/\|a_n\| \to a_0/\|a_0\|$ and $\|a_n\| \to \|a_0\|$. Finally, we get that $A(v) \in \operatorname{span} \widetilde{V}_i$. $\square$

Lemma 5.2. *If A corresponds to $f \in OC(m)$, then*:

$$V = \operatorname{span} \bigcup_{i\in \mathrm{Per}(A)} \widetilde{V}_i = \sum_{i\in \mathrm{Per}(A)} V_i.$$

Proof. Remind that $q(i) = \max\{l \in \mathrm{Per}(A) : l \mid i\}$. We define $b = i/(q(i))$, because $q(i) \mid i$, so b is a natural number. We show that $\widetilde{V}_i \subset V_{q(i)}$.

Let $v \in \widetilde{V}_i$, $v = \lim_{n\to\infty}(x_n - f^i(x_n))/\|x_n - f^i(x_n)\|$.

By the formulas 3.3 and 3.4 in the proof of Lemma 3.3 we obtain for $k = b$, $g = f^{q(i)}$ and $B = A^{q(i)}$ and each natural i:

$$\left\| \frac{x_n - f^i(x_n)}{\|x_n - f^{q(i)}(x_n)\|}\,\frac{\|x_n - f^i(x_n)\|}{\|x_n - f^i(x_n)\|} - \left[\sum_{j=0}^{b-1} A^{jq(i)}\right]\left(\frac{x_n - f^{q(i)}(x_n)}{\|x_n - f^{q(i)}(x_n)\|}\right)\right\| < \varepsilon. \tag{5.6}$$

Because the sequence $z_n = (x_n - f^{q(i)}(x_n))/\|x_n - f^{q(i)}(x_n)\|$ is contained in a sphere, we may choose its subsequence which is convergent. Without loss of generality assume that z_n is convergent to $w \in \widetilde{V}_{q(i)}$. Then, by Lemma 3.4, $\sum_{j=0}^{b-1} A^{jq(i)}(w) \neq 0$. Repeating the argument from the proof of Lemma 5.1, we get that $(x_n - f^i(x_n))/\|x_n - f^i(x_n)\| \to v \in \widetilde{V}_i$ and $(\|x_n - f^i(x_n)\|)/\|x_n - f^{q(i)}(x_n)\| \to \alpha \in \mathbb{R}_+$. By the inequality (5.6) $v\alpha = \sum_{j=0}^{b-1} A^{jq(i)}(w)$ and by Lemma 5.1 A preserves $V_{q(i)}$, thus $v \in \operatorname{span} \widetilde{V}_{q(i)} = V_{q(i)}$. $\square$

Theorem 5.3. *Let $f \in OC(m)$. A linear map $A\colon \mathbb{R}^m \to \mathbb{R}^m$ corresponds to f if and only if G is well-defined and $A_{|V} = G$.*

Proof. Assume that A corresponds to f.

By Lemma 5.2 $V = \operatorname{span} \bigcup_{i \in \operatorname{Per}(A)} \widetilde{V}_i$. Let $v \in \widetilde{V}_i$ for some natural $i \in \operatorname{Per}(A)$, $v = \lim_{n\to\infty}(x_n - f^i(x_n))/\|x_n - f^i(x_n)\|$. Then, by the formula (5.5), $\widetilde{G}(v)$ is well-defined and $\widetilde{G}(v) = A(v)$, which shows that $\widetilde{G}$ agrees with A on each $\widetilde{V}_i$. Thus, as A is a linear map, we obtain the thesis.

Assume that G is well-defined and $A_{|V} = G$. Suppose, contrary to our claim, that A does not correspond to f. Then there is $i \in \operatorname{Per}(A)$ and $\varepsilon > 0$, such that for every neighborhood U of 0, there is $x_n \in U$ such that:

$$\| f(x_n) - f^{i+1}(x_n) - A(x_n - f^i(x_n))\| \geq \varepsilon \|x_n - f^i(x_n)\|.$$

This is equivalent to:

$$\left\| \frac{f(x_n) - f^{i+1}(x_n)}{\|x_n - f^i(x_n)\|} - A\left(\frac{x_n - f^i(x_n)}{\|x_n - f^i(x_n)\|}\right)\right\| \geq \varepsilon. \tag{5.7}$$

We choose a subsequence of $(x_n - f^i(x_n))/\|x_n - f^i(x_n)\|$ convergent to $v \in \widetilde{V}_i$. As G is well-defined, the respective subsequence of $(f(x_n) - f^{i+1}(x_n))/\|x_n - f^i(x_n)\|$ converges to $G(v)$. Then, by (5.5) and the fact that $A_{|V} = G$, the expression in the left-hand side of the inequality (5.7) converges to $\|G(v) - A(v)\| = 0$, which leads to contradiction. □

REMARK 5.4. By the above procedure one can find the "minimal" (in the sense of Theorem 5.3) linear map G whose eigenvalues affect the period of indices of iterations for $f \in OC(m)$. Notice that, by Lemma 5.1 and Theorem 5.3, G is a self-map of V, this implies in particular that the number of essential eigenvalues is not greater than $\dim V$ (or more precisely $(\dim V)/2 + 1$).

Problem 2. *Let f be in Shub-Sullivan class. An open question remains whether a similar procedure of finding a "minimal" linear map for a given f can be realized.*

References

[1] I.K. Babenko and S.A. Bogatyi: *The behavior of the index of periodic points under iterations of a mapping*, Math. USSR Izv. **38** (1992), 1–26.

[2] S.N. Chow, J. Mallet-Paret and J.A. Yorke: *A periodic orbit index which is a bifurcation invariant*; in Geometric Dynamics (Rio de Janeiro, 1981), Springer Lect. Notes in Math. **1007**, Berlin, 1983, 109–131.

[3] A. Dold: *Fixed point indices of iterated maps*, Invent. Math. **74** (1983), 419–435.

[4] G. Graff: *Indices of iterations and periodic points of simplicial maps of smooth type*, Topology Appl. **117** (2002), 77–87.

[5] G. Graff and P. Nowak-Przygodzki: *Fixed point indices of iterations of planar homeomorphisms*, Topol. Methods Nonlinear Anal. **22** (2003), 159–166.
[6] J. Jezierski and W. Marzantowicz: Homotopy Methods in Topological Fixed and Periodic Points Theory, Kluwer Academic Pub., Dordrecht, 2005.
[7] A. Katok and A. Hasselblatt: Introduction to the Modern Theory of Dynamical Systems, Encyclopedia of Mathematics and its Applications **54**, Cambridge University Press, Cambridge, 1995.
[8] W. Marzantowicz: *Finding periodic points of a smooth mapping using Lefschetz numbers of its iterations*, Russ. Math. **41** (1997), 80–89.
[9] W. Marzantowicz and P. Przygodzki: *Finding periodic points of a map by use of a k-adic expansion*, Discrete Contin. Dynam. Systems **5** (1999), 495–514.
[10] T. Matsuoka: *The number of periodic points of smooth maps*, Ergod. Th. Dynam. Sys. **9** (1989), 153–163.
[11] T. Matsuoka and H. Shiraki: *Smooth maps with finitely many periodic points*, Mem. Fac. Sci., Kochi Univ. (Math) **11** (1990), 1–6.
[12] F.R. Ruiz del Portal and J.M. Salazar: *Fixed point index of iterations of local homeomorphisms of the plane*: *a Conley index approach*, Topology **41** (2002), 1199–1212.
[13] M. Shub and P. Sullivan: *A remark on the Lefschetz fixed point formula for differentiable maps*, Topology **13** (1974), 189–191.
[14] H-W. Siegberg: *A remark on the theorem of Shub and Sullivan*, Topology **23** (1984), 157–160.

Grzegorz Graff
Department of Algebra
Faculty of Applied Physics and Mathematics
Gdansk University of Technology
ul G. Narutowicza 11/12, 80-952, Gdansk
Poland
e-mail: graff@mifgate.pg.gda.pl

Piotr Nowak-Przygodzki
e-mail: piotrnp@wp.pl