Hieda, Y. and Tsushima, Y.
Osaka J. Math.
43 (2006), 475–483

YOUNG DIAGRAMS AND SIMPLE CONSTITUENTS OF THE SPECHT MODULES

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(Received October 26, 2004, revised June 9, 2005)

Abstract

We discuss the simple constituents of Specht module S^λ for the symmetric group S_n defined over the field of p elements. We firstly give an easier proof to the result in [6] which asserts that there exists a simple constituent of S^λ with the shape of "a branch" of λ (Theorem 3.3), and secondly give a sufficient condition for λ to have a particular type branch as a constituent (Proposition 3.4).

1. Introduction

Let n be a natural number and p a prime. Let S_n be the symmetric group on n letters and L a field of characteristic p. Given a partition λ of n, we have an LS_n-module S^λ called the Specht module corresponding to λ, which is not simple in general. However if the partition λ is p-regular, the head of S^λ, denoted by D^λ, is simple and they cover all the non-isomorphic simple modules as λ runs through the p-regular partitions of n.

One of the main concerns about the Specht modules is to have informations about the simple constituents of them. Especially, using information only on λ, we would like to describe a p-regular partition μ for which D^μ appears as a constituent of S^λ. For this purpose, it is useful to consider the operations on the patitions λ introduced by James and Murphy [5], each of which is roughly interpreted as a rim hook removal followed by addition on the Young diagram corresponding to λ. We shall call each of the resulting partitions a branch of λ. The Jantzen-Schaper theorem tells that if D^μ is a constituent of S^λ, it follows that $\lambda = \mu$ or μ is obtained by making branches successively beginning with λ (cf. [6, Corollary 1]). One of the authors showed that if λ is p-regular, there is a p-regular branch μ of λ such that D^μ is a constituent of S^λ (cf. [6, Theorem 2]). And he gave some applications of the result in [7]. However the proof of the result cited above is rather long and complicated. In this paper we shall show a short proof to it and a result on simple constituents of the Specht modules as a byproduct of the proof.

2000 Mathematics Subject Classification. 20C30.

2. Preliminary results

A *partition* of the integer n is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of non-negative integers whose sum is n. The *Young diagram* $[\lambda]$ associated with λ is the set of the ordered pairs (i, j) of integers, called the *nodes* of $[\lambda]$, with $1 \le i \le h$ and $1 \le j \le \lambda_i$, where h denotes the largest number such that $\lambda_h \neq 0$. They are illustrated as arrays of squares. We denote by λ' the partition conjugate of λ, so $[\lambda']$ is the transposed diagram of $[\lambda]$.

Let c be a column number of $[\lambda]$ and r a positive integer. Then λ is said to be *r-singular on column c* if there is an integer $i \ge 0$ such that $\lambda_{i+1} = \lambda_{i+2} = \cdots = \lambda_{i+r} = c$, and is *$r$-regular on column c* if otherwise. We also say that λ is *r-singular* if it is r-singular on some column, and is r-regular if otherwise. For the convenience of later arguments, we understand that every partition is r-regular on column 0. We denote by $P(n)$ and $P(n)^0$ the sets of the partitions and p-regular partitions of n respectively. The *dominance order* $\trianglelefteq$ on $P(n)$ is defined as follows: given $\lambda, \mu \in P(n)$, $\lambda \trianglelefteq \mu$ if and only if $\sum_{1\le i\le j} \lambda_i \le \sum_{1\le i\le j} \mu_i$ for all $j \ge 1$.

The (i, j)-*hook* of the Young diagram $[\lambda]$ consists of the (i, j)-node along with the $\lambda_i - j$ nodes to the right of it (called the *arm* of the hook) and the $\lambda'_j - i$ nodes below it. The *length* of the (i, j)-hook of λ is $h_{ij}(\lambda) := \lambda_i + \lambda'_j + 1 - i - j$. An (i, j)-*rim hook* is a connected part of the rim of $[\lambda]$ of length $h_{ij}(\lambda)$ beginning at the node (λ'_j, j). We also call the integer $\lambda_i - j$ the *arm length* of the node (i, j). Moreover, a hook of $[\lambda]$ is called a *pillar* if its arm length is zero.

Let (b, c) is a node of $[\lambda]$ and suppose that $a < b$. We let $\lambda(a, b, c)$ be the partition of n obtained from λ by unwrapping the (b, c)-rim hook of $[\lambda]$ and wrapping the nodes back with the lowest nodes in the added rim hook lying on row a (if the resulting partition fails to be a non-increasing sequence of integers, $\lambda(a, b, c)$ is not defined). We occasionally write $\lambda(a, b, c, g)$ if the highest node in the added rim hook lies in row g. We call here each $\lambda(a, b, c)$ a *branch* of λ and set

$$\Gamma_\lambda := \{\lambda(a, b, c)\,;\, \nu_p(h_{ac}(\lambda)) \neq \nu_p(h_{bc}(\lambda))\}, \qquad \Gamma_\lambda^0 := \Gamma_\lambda \cap P(n)^0,$$

where $\nu_p(m)$ denotes the largest integer e such that p^e divides the integer m.

A branch $\mu = \lambda(a, b, c)$ is called a *pillar type branch* if the rim hook which has been removed and the rim hook which has been added are both pillars. Suppose that $\mu = \lambda(a, b, c)$ is a pillar type branch and put $d := \lambda_a + 1$, $q := h_{bc}(\lambda)$. Then μ is obtained by unwrapping the pillar of q nodes from column c and wrapping it back on column d (with the lowest node on row a). Hence we sometimes write $\mu = \lambda(c \mid d, q)$ for simplicity. For $\lambda \in P(n)$, let $\mathrm{SC}(S^\lambda)$ be the set of simple constituents of the Specht module S^λ.

REMARK. Let $\lambda \in P(n)^0$. Then if $\mu = \lambda(a, b, c)$ is a pillar type branch of $[\lambda]$, we have $h_{bc}(\lambda) \le p - 1$. Hence μ lies in Γ_λ if and only if $h_{ac}(\lambda)$ is divisible by p.

Now we list below some results for later use.

Theorem 2.1 ([2], [3]). *Let $\lambda \in P(n)^0$. Then S^λ is simple if and only if $\nu_p(h_{ac}(\lambda)) = \nu_p(h_{bc}(\lambda))$ for all $a, b, c \geq 1$.*

Theorem 2.2 (Carter and Payne [1]). *Suppose that $\alpha := \lambda(c \mid d, q)$ be a pillar type branch of λ and let a be the row index of $[\lambda]$ such that $d = \lambda_a + 1$. Put $e := \nu_p(h_{ac}(\lambda))$. If $p^e > q$, we have*

$$\mathrm{Hom}_G(S^\alpha, S^\lambda) \neq 0.$$

In paricular, it follows that $D^\alpha \in \mathrm{SC}(S^\lambda)$ if α is p-regular.

REMARK. The above statement is slightly different from the corresponding theorem in [1], but can be deduced easily from it. In fact, if λ and α are the same as above then with the languages in [1], λ' is obtained from α' by raising q nodes from row d to row c, whence we have $\mathrm{Hom}_G\left(S^{\lambda'}, S^{\alpha'}\right) \neq 0$. The rest of the proof will be done by routine arguments, using that $S^{\lambda'}$ is isomorphic to the L-dual of $S^\lambda \otimes S^{(1^n)}$ ([2, Theorem 8.15]).

Theorem 2.3 ([4, Theorem 6]). *Let λ, μ be partitions of n with λ p-regular. Suppose that there is a number k $(1 \leq k \leq \lambda_1, \mu_1)$ such that the subdiagrams consisting of the first k columns of $[\lambda]$ and $[\mu]$ are the same and that each has m nodes. Let $\left[\widehat{\lambda}\right]$ ($[\widehat{\mu}]$ resp.) be the subdiagram to the right of column k of $[\lambda]$ ($[\mu]$ resp.). Then the composition multiplicity of D^λ in S^μ as S_n-modules equals the composition multiplicity of $D^{\hat{\lambda}}$ in $S^{\hat{\mu}}$ as S_{n-m}-modules.*

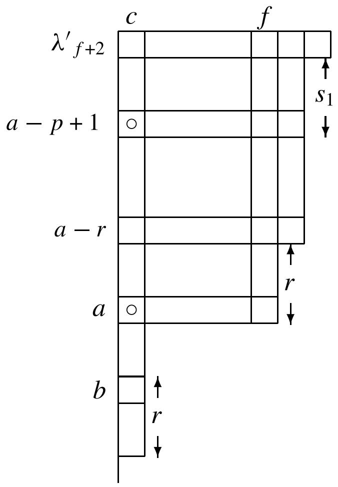
Proposition 2.4 (Jantzen-Schaper, cf. [6, Corollary 1]). *Let $\lambda \in P(n)$ and let μ be a minimal element of Γ_λ with respect to the dominance order. If μ is p-regular, $D^\mu \in \mathrm{SC}(S^\lambda)$.*

Proposition 2.5 ([6, Proposition 3]). *Let $\lambda \in P(n)^0$ and let $[\mu]$ be the diagram to the right of the first column of $[\lambda]$. If S^μ is simple, Γ_λ has no p-singular partition.*

3. Finding simple constituents of Specht modules

We shall show a short proof to Theorem 2 of [6] and a result on simple constituents of the Specht modules. First we show

Lemma 3.1. *Let $\lambda \in P(n)^0$. If there is a pillar type branch $\mu = \lambda(a, b, c) \in \Gamma_\lambda$ such that μ is p-regular on column $c - 1$, there is a pillar type branch $\widetilde{\lambda}$ in Γ^0_λ.*



Proof. We put $r := h_{bc}(\lambda)$ ($\leq p-1$) and $f := \lambda_a$. Note that $h_{ac}(\lambda)$ is a multiple of p since $\mu \in \Gamma_\lambda$. We may assume that μ is p-singular, so μ is p-singular on column $f+1$ by the assumption. (In the above diagram a circle in a node indicates that the hook length at the node is divisible by p.)

Namely $a - \lambda'_{f+2} \geq p$, so $a - p + 1 > \lambda'_{f+2}$. Put $s_1 := a - p + 1 - \lambda'_{f+2}$ (≥ 1). Then $r - s_1 = (p-1) - (a-r) + \lambda'_{f+2} = (p-1) - (\lambda'_{f+1} - \lambda'_{f+2}) \geq 0$, so $r \geq s_1$. Now let $\mu(1) = \lambda(c \mid f+2, s_1)$, which lies in Γ_λ since $h_{a-p+1,c}(\lambda)$ is divisible by p. Note that $\mu(1)$ is p-regular on column $c-1$. If $\mu(1)$ is p-regular, we may take $\mu(1)$ as $\widetilde{\lambda}$. Hence we may assume that $\mu(1)$ is p-singular, so $\lambda'_{f+2} \neq 0$ and $\mu(1)$ is p-singular on column $f+2$. Namely $(a-p+1) - \lambda'_{f+3} \geq p$, so $a - 2p + 2 > \lambda'_{f+3}$. Put $s_2 := a - 2p + 2 - \lambda'_{f+3}$ (≥ 1). Then $s_1 - s_2 = (p-1) - (\lambda'_{f+2} - \lambda'_{f+3}) \geq 0$, so $s_1 \geq s_2$. Now let $\mu(2) = \lambda(c \mid f+3, s_2)$, which lies in Γ_λ since $h_{a-2p+2,c}(\lambda)$ is divisible by p. Note that $\mu(2)$ is also p-regular on column $c-1$. By repeating similar arguments we finally obtain a p-regular pillar type branch $\mu(i)$ for some i, completing the proof of the lemma. □

Lemma 3.2. *Let $\lambda \in P(n)^0$. If there is a branch $\mu = \lambda(a, b, c) \in \Gamma_\lambda$ with $c \geq 2$ such that μ is p-singular on column $c-1$, there is a pillar type branch $\widetilde{\lambda}$ in Γ^0_λ.*

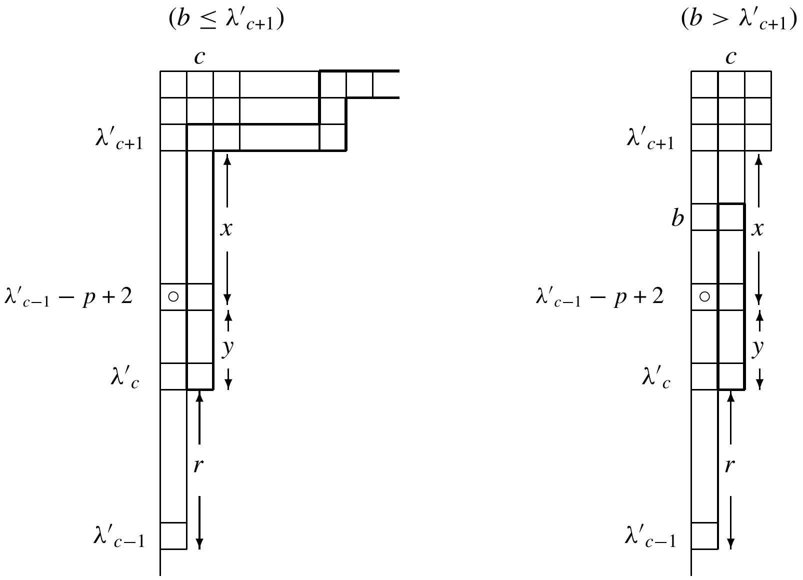
Proof. We may assume that μ is chosen so that c is the smallest and put $r := \lambda'_{c-1} - \lambda'_c$. Note that the (b, c)-rim hook of $[\lambda]$ is a pillar if and only if $b > \lambda'_{c+1}$.

CASE I. $r \leq p-2$.

As μ is p-singular on column $c-1$, $\lambda'_{c-1} - \lambda'_{c+1} \geq p-1$. Put $x := (\lambda'_{c-1} - p + 2) - \lambda'_{c+1}$ and $y := \lambda'_c - (\lambda'_{c-1} - p + 2)$, so $x \geq 1$ and $r + y = p - 2$.

SUBCASE (i) $x + y \leq p - 2$.

We have that $x \leq r$ from $x + y \leq p - 2 = r + y$. Now let $\gamma = \lambda(c-1 \mid c+1, x) \in \Gamma_\lambda$. If γ is p-regular, we may take γ as $\widetilde{\lambda}$. Hence we may assume that γ is p-singular.



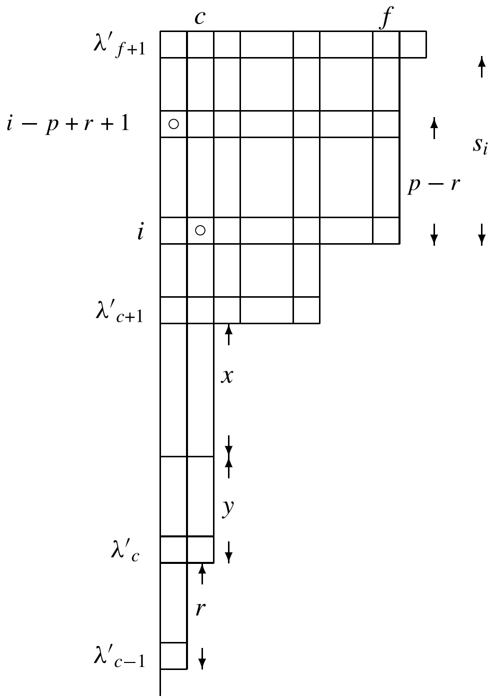
Then by the minimality of c, γ must be p-regular on column $c-2$ and there is a pillar type branch $\widetilde{\lambda} \in \Gamma_\lambda^0$ by Lemma 3.1, as asserted. (In the above diagramas the boldface rim hooks will be removed to make μ.)

SUBCASE (ii) $x + y = p - 1$.

We have $x = r + 1$. As $\mu \in \Gamma_\lambda$, either $h_{ac}(\lambda)$ or $h_{bc}(\lambda)$ is divisible by p. Let $i = a$ or $i = b$ according as $h_{ac}(\lambda)$ is divisible by p or not. Let furthermore $f = \lambda_i$ and $s_i = i - \lambda'_{f+1}$. Then we see that $i \leq \lambda'_{c+1}$ since $h_{ic}(\lambda)$ is divisible by p. Since $s_i \leq p-1$, we can make the pillar type branch $\gamma = \lambda(c \mid f+1, s_i) \in \Gamma_\lambda$. If $s_i + r < p$, γ is p-regular on column $c-1$ and the assertion follows by Lemma 3.1. Now suppose that $s_i + r \geq p$ and put $t_i = s_i - (p - r - 1)$, so $t_i \geq 1$. Also $\lambda'_{f+1} = i - s_i < i - (p - r - 1) = i - p + r + 1$. Note that $t_i \leq r$ since $r - t_i = p - 1 - s_i \geq 0$. Hence we can make the pillar type branch $\delta = \lambda(c - 1 \mid f + 1, t_i)$, which lies in Γ_λ since $h_{i-p+r+1,c-1}(\lambda) = (r+1) + h_{ic}(\lambda) + (p - r - 1) = h_{ic}(\lambda) + p$ is divisible by p. By the minimality of c, δ is p-regular on column $c-2$ and so there is a pillar type branch $\widetilde{\lambda} \in \Gamma_\lambda^0$ by Lemma 3.1, as asserted.

CASE II. $r = p - 1$.

We use the same notation as in subcase (ii). Then $h_{i,c-1}(\lambda)$ is divisible by p, since $h_{i,c-1}(\lambda) = h_{ic}(\lambda) + p$. In the diagram below, we have $r = p - 1$, so we can make the pillar type branch $\gamma = \lambda(c - 1 \mid f + 1, s_i) \in \Gamma_\lambda$. By the minimality of c, γ is p-regular on column $c-2$ and the assertion follows by Lemma 3.1. This completes the proof of the lemma.

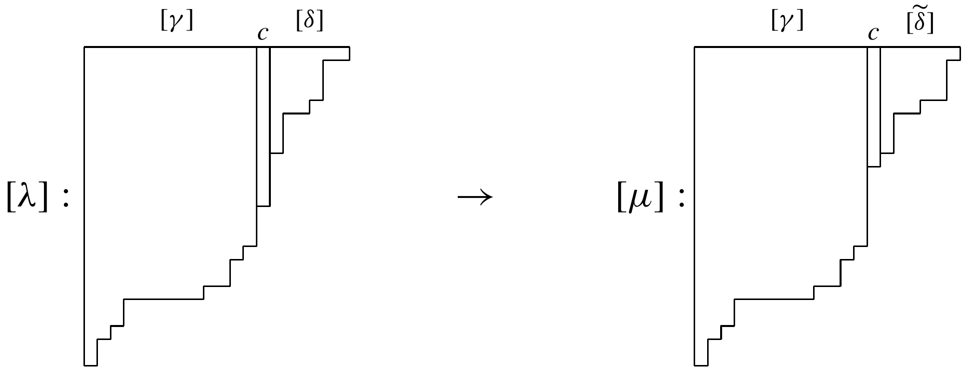


□

Now we are ready to give an alternative proof of the following theorem:

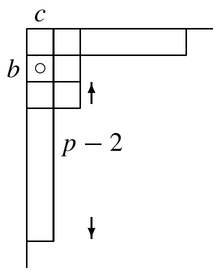
Theorem 3.3 ([6, Theorem 2]). *Let λ be a p-regular partition of n. If S^λ is reducible, there is a p-regular branch $\widetilde{\lambda} \in \Gamma_\lambda$ such that $D^{\tilde{\lambda}} \in \mathrm{SC}(S^\lambda)$.*

Proof. Since S^λ is reducible, there is a column number c such that $\nu_p(h_{a,c}(\lambda)) \neq \nu_p(h_{b,c}(\lambda))$ for some a, b with $1 \leq a, b \leq \lambda'_c$. Let c be the largest number that satisfies the condition. Let $[\delta]$ be the subdiagram of $[\lambda]$ with column c as the first column, $[\gamma]$ the remaining diagram and write $\lambda = (\gamma, \delta)$. Then every branch in Γ_δ is p-regular by Proposition 2.5. Hence, if $\widetilde{\delta}$ is a minimal element of Γ_δ with respect to the dominance order, $D^{\tilde{\delta}} \in \mathrm{SC}(S^\delta)$ by a direct consequence of the Jantzen-Schaper theorem (see Proposition 2.4). Put $\mu := \left(\gamma, \widetilde{\delta}\right) \in \Gamma_\lambda$. If μ is p-singular on column $c - 1$, then c must be greater than 1 and there is a pillar type branch $\widetilde{\lambda} \in \Gamma^0_\lambda$ by Lemma 3.2. Thus we have $D^{\tilde{\lambda}} \in \mathrm{SC}(S^\lambda)$ by the Carter and Payne theorem (see Theorem 2.2). So we may assume that μ is p-regular on column $c - 1$. Then $\mu \in \Gamma^0_\lambda$ and we have $D^\mu \in \mathrm{SC}(S^\lambda)$ by Theorem 2.3. This completes the proof of the theorem.



□

Now a node (b, c) is called a *$\langle p, 1\rangle$-point* of $[\lambda]$ with arm length one, if $h_{bc}(\lambda) = p$ and $h_{\lambda'_c, c}(\lambda) = 1$.



Proposition 3.4. *Suppose $p > 2$ and that S^λ is not simple. Let λ be a $(p-1)$-regular partition of n. Then*

(1) *If $[\lambda]$ has no $\langle p, 1\rangle$-point with arm length one, we have $\Gamma_\lambda^0 = \Gamma_\lambda$. Hence $D^\mu \in \mathrm{SC}(S^\lambda)$ for any minimal element μ of Γ_λ with respect to the dominance order.*

(2) *If $[\lambda]$ has a $\langle p, 1\rangle$-point with arm length one, there is a pillar type branch $\mu = \lambda(c \mid d, q)$ such that $D^\mu \in \mathrm{SC}(S^\lambda)$ for some c, d, q with $q \leq p-2$.*

Proof. (1) The second half follows immediately from the first half and Proposition 2.4. So we need only prove the first half. Suppose the contrary and take a p-singular branch, say $\mu = \lambda(a, b, c, g)$, from Γ_λ.

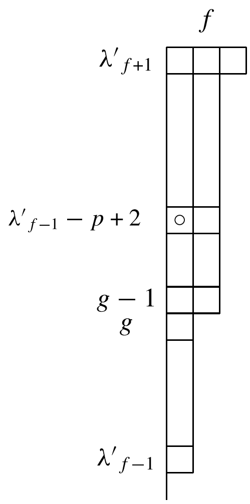
CASE I. μ is p-singular on column $c - 1$ (hence $c \geq 2$).

Since λ is $(p-1)$-regular, it follows that $\lambda'_{c-1} - p + 2 \leq \lambda'_c$ and so $(\lambda'_{c-1} - p + 2, c-1)$ is a $\langle p, 1\rangle$-point of $[\lambda]$ with arm length one, being contrary to the assumption.

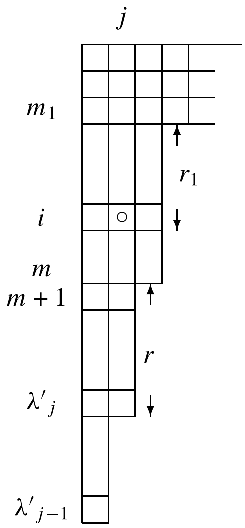
CASE II. μ is p-singular on column λ_{g-1} (hence $g \geq 2$).

As λ is $(p-1)$-regular, we find easily that $\lambda_{g-1} = \lambda_g + 1$. Let $f = \lambda_{g-1}$.

((1), Case II)



(2)



Then $\lambda'_{f-1} - \lambda'_{f+1} \geq p - 1$, and the node $(\lambda'_{f-1} - p + 2, f - 1)$ is a $\langle p, 1\rangle$-point of $[\lambda]$ with arm length one, being contrary to the assumption. This completes the proof of (1).

(2) Let (i, j) be a $\langle p, 1\rangle$-point of $[\lambda]$ with arm length one and $m := \lambda'_{j+1}$. Then $i \leq m < \lambda'_j = i + p - 2$ and $\lambda_m - 1 = \lambda_{m+1}$.

Now we assume that the above (i, j) is chosen so that j is the smallest. Let $m_1 := \lambda'_{j+2}$ and $r := h_{m+1,j}(\lambda) = i + p - 2 - m$. Then $m_1 < i$ since the node (i, j) has arm length one. Let $r_1 := i - m_1$. Then $r - r_1 = (p - 2) - (m - m_1) \geq 0$, so $r_1 \leq r$. Therefore we can make the pillar type branch $\mu = \lambda(j \mid j + 2, r_1) \in \Gamma_\lambda$. If μ is p-singular on column $j - 1$, then j must be greater than 1 and $(\lambda'_{j-1} - (p - 2), j - 1)$ is a $\langle p, 1\rangle$-point of $[\lambda]$ with arm length one, contradicting the minimality of j. Hence μ is p-regular on column $j - 1$ and by Lemma 3.1, there is a pillar type branch in Γ^0_λ, whence the assertion follows by the Carter and Payne theorem. □

ACKNOWLEDGEMENT. The authors thank the referee for his/her careful reading of the manuscript.

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