A STRUCTURE THEOREM OF COMPACT COMPLEX PARALLELIZABLE PSEUDO-KÄHLER SOLVMANIFOLDS

Dedicated to Professor Yusuke Sakane on his 60th birthday

TAKUMI YAMADA

(Received July 27, 2005, revised January 6, 2006)

Abstract

In this paper, we prove that the Mostow fibration of a compact complex parallelizable pseudo-Kähler solvmanifold is a complex torus bundle over a complex torus.

Introduction

A complex manifold $X^n$ of complex dimension $n$ is called complex parallelizable if there exist $n$ holomorphic vector fields which are linearly independent at each point. Wang [13] proved that a compact complex parallelizable manifold is of the form $G/\Gamma$, where $G$ is a complex Lie group and $\Gamma$ is a discrete subgroup of $G$. Wang also proved that if a compact complex parallelizable manifold $X$ admits a Kähler structure, then $X$ is a complex torus. On the other hand, Matsushima [10] proved that a compact homogeneous Kähler manifold is biholomorphic to a product of a homogeneous rational manifold and a complex torus. By a homogeneous Kähler manifold we mean a Kähler manifold on which the group of holomorphic isometric transformations acts transitively. Borel-Remmert [2] generalized the result of Matsushima to compact Kähler manifolds on which the group of holomorphic transformations acts transitively. Dorfmeister-Guan [3] proved that a compact homogeneous pseudo-Kähler manifold is also biholomorphic to a product of a homogeneous rational manifold and a complex torus. As for compact pseudo-Kähler manifolds on which the group of holomorphic transformations acts transitively, there exist non-toral compact complex parallelizable pseudo-Kähler solvmanifolds. In particular, we see that a compact non-homogeneous pseudo-Kähler manifold is not biholomorphic to a product of a homogeneous rational manifold and a complex torus in general (cf. [17]). It is therefore important to study compact complex parallelizable pseudo-Kähler solvmanifolds. In this paper we prove the following structure theorem, which is our main theorem:

2000 Mathematics Subject Classification. Primary 53C15; Secondary 53D05, 32M10.
Partly supported by JSPS Research Fellowships for Young Scientists.
Theorem 1.6 Let $X = G/\Gamma$ be a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure. Then the Mostow fibration of $X$ is a complex torus bundle over a complex torus.

We also investigate the Dolbeault cohomology groups of a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure.

The author would like to express his deep appreciation to Professor Yusuke Sakane and Professor Ryushi Goto for valuable advice and helpful discussion and encouragement during completion of this paper.

1. Proof of main theorem

In this section we prove our main theorem.

DEFINITION 1.1. Let $G$ be a Lie group. A discrete subgroup $\Gamma$ of $G$ is called a lattice if $G/\Gamma$ has a finite invariant measure.

If $G$ is a solvable Lie group, then a discrete subgroup $\Gamma$ of $G$ is a lattice if and only if $\Gamma$ is a discrete co-compact subgroup of $G$.

Let $\mathcal{O}_X = \mathcal{O}$ be the sheaf of holomorphic functions on a complex manifold $X$. We denote the Hodge number of $X$ by $h^{p,q}(X)$, i.e., $h^{p,q}(X) = \dim H^{p,q}(\overline{X})$. Let $G$ be a connected complex Lie group, $\Gamma$ a lattice of $G$, $N$ the maximal connected normal nilpotent subgroup. Let $G = S \cdot R$ be a Levi decomposition, where $S$ is a semi-simple part, and $R$ is the radical. We denote derived Lie subgroups of $G$, $N$ and $R$ by $G', N'$ and $R'$ respectively. Winkelmann has proven

Theorem 1.2 ([14]). Let $G, \Gamma, N, S, R, G', N'$ and $R'$ be as above. Let $A = [S, R] \cdot N'$. Furthermore let $W$ denote the maximal linear subspace of the Lie algebra Lie$(R'/A)$ of $R'/A$ such that $\text{Ad}(\gamma)|_W$, where $\text{Ad}$ is the adjoint representation of $G$, is a semisimple linear endomorphism with only real eigenvalues for each $\gamma \in \Gamma$. Then

$$\dim H^1(G/\Gamma, \mathcal{O}) \leq \dim G/G' + \dim H^1(G/R\Gamma, \mathcal{O}) + \dim W.$$ 

DEFINITION 1.3. Let $X$ be a complex manifold. A real $(1,1)$-form $\omega$ of $X$ is called a pseudo-Kähler structure if $\omega$ is a non-degenerate closed form.

In the case of a compact complex parallelizable manifold, we have shown the following in the paper [17]:


Theorem 1.4. Let $X^n = G/\Gamma$ be a compact complex parallelizable manifold which admits a pseudo-Kähler structure. Then

$$h^{p,q}(X) \geq \binom{n}{p} \cdot \binom{n}{q}.$$ 

Corollary 1.5. Let $(G/\Gamma, \omega)$ be an $n$-dimensional compact complex parallelizable pseudo-Kähler manifold such that $\Gamma$ is a lattice of $G$. If $h^{0,1}(G/\Gamma) = \dim H^0(G/\Gamma, d\mathcal{O})$, then $G/\Gamma$ is a complex torus.

Proof. Let $g$ be the Lie algebra of $G$, $I$ the complex structure of $g$, and $g^+ = \{ X \in g^C \mid IX = \sqrt{-1}X \}$. We identify $g^+$ with the set of all right invariant holomorphic vector fields of $G$. Let $H^q(g^+)$ be the $q$th Lie algebra cohomology group of $g^+$. Since $H^0(G/\Gamma, d\mathcal{O}) \cong H^1(g^+)$ and $h^{0,1}(G/\Gamma) \geq n$, we see that $\dim H^1(g^+) = n$. Hence $g^+$ is abelian. 

Let $G/\Gamma$ be a compact complex parallelizable solvmanifold, i.e., $G$ is a simply connected complex solvable Lie group and $\Gamma$ is a lattice of $G$. Mostow proved that $\Gamma_N = N \cap \Gamma$ is a lattice of the maximal normal nilpotent Lie subgroup $N$ of $G$. A fibration $N/\Gamma_N \to G/\Gamma \to G/N\Gamma$ is called the Mostow fibration of $G/\Gamma$.

Theorem 1.6. Let $X^n = G/\Gamma$ be a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure. Then the Mostow fibration of $G/\Gamma$ is a complex torus bundle over a complex torus.

Proof. We use the notation of Theorem 1.2. Since $G$ is solvable, we see that $G = R, S = \{ e \}$ and $A = N'$. By Theorems 1.2 and 1.4, we see

$$n \leq \dim H^{0,1}(G/\Gamma) \leq \dim G/G' + \dim W \leq \dim G - \dim G' + \dim G'/N',$$

which implies $\dim N' = 0$. 

By the proof of our main theorem, we have

Corollary 1.7. If a compact complex parallelizable solvmanifold $X^n$ admits a pseudo-Kähler structure, then $h^{0,1}(X) = n$.

Corollary 1.8. If a compact complex parallelizable solvmanifold $G/\Gamma$ admits a pseudo-Kähler structure, then $N$ must be abelian and in particular the Lie algebra $g$ must satisfy $D^{(2)}g = 0$.

Proof. Since $n \supset [g, g]$, we have our corollary.
REMARKS 1.9. (i) There exists a complex solvable Lie group $G$ which has lattices $\Gamma_1, \Gamma_2$ such that $G/\Gamma_1$ has a pseudo-Kähler structure, while $G/\Gamma_2$ has no pseudo-Kähler structures (see [17]).

(ii) It is well known that a simply connected complex solvable Lie group $G$ is biholomorphic to $\mathbb{C}^n$. Moreover if its Lie algebra $\mathfrak{g}$ has a Chevalley decomposition, then there exists a good system of coordinates $(z_1, \ldots, z_n)$ of $G$ which satisfies the following:

(a) The Lie group $G$ is isomorphic to $(\mathbb{C}^n, \ast)$ as a complex Lie group, where the multiplication $\ast$ of $\mathbb{C}^n$ is given by

$$(z_1, \ldots, z_n) \ast (y_1, \ldots, y_n) = (z_1 + y_1, \ldots, z_r + y_r, F_{r+1}\cdot y)z_{r+1} + y_{r+1} + F_{r+1}(z, y), \ldots, F_{nn}(y)z_n + y_n + F_n(z, y))$$

for $z = (z_1, \ldots, z_n), y = (y_1, \ldots, y_n) \in \mathbb{C}^n$, where $r = \dim \mathfrak{g} - \dim[\mathfrak{g}, \mathfrak{g}]$. $F_i(z) = \exp(-\sum_{j=1}^k C^i_{ji} z_j)$, where $k = \dim G/N\Gamma$ and $C^i_{ji}$ are constant, and $F_i(z, y) = F_i(z_1, \ldots, z_{k-1}, y_1, \ldots, y_n)$ is a holomorphic function with respect to $(z_1, \ldots, z_{k-1}, y_1, \ldots, y_n)$ for each $\lambda$.

(b) Let $\Gamma$ be a lattice of $G$. Using the above system of coordinates $(z_1, \ldots, z_n)$ of $G$, we see that any element of $H^{0,0}_{\bar{\partial}}(G/\Gamma)$ has a representative of the following form:

$$\psi = \sum_{\lambda=1}^k c_\lambda d\bar{z}_\lambda + \sum_{\lambda=k+1}^{k+r(N/\Gamma_N)} f_i(z_1, \ldots, z_k) d\bar{z}_\lambda,$$

where $r(N/\Gamma_N) = \dim H^{0,0}_{\bar{\partial}}(N/\Gamma_N), c_\lambda$ are constant and $f_i(z_1, \ldots, z_k)$ are holomorphic in $z_1, \ldots, z_k$.

We say that a complex solvable Lie algebra $\mathfrak{g}$ has a Chevalley decomposition if $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ as a vector space, where $\mathfrak{a}$ is a commutative subalgebra and $\mathfrak{n}$ is the maximal nilpotent ideal. For further details see [11].

Using the above system of coordinates, we give another proof of our main theorem for the case where the Lie algebra $\mathfrak{g}$ of $G$ has a Chevalley decomposition (see Section 3).

2. The structure of the sheaf $R^1\pi_*\mathcal{O}_{G/\Gamma}$

For a holomorphic map $f : X \to Y$ between complex spaces, there exists a Leray spectral sequence for the sheaf $\mathcal{O}$. The respective lower term sequence yields the following:

$$0 \to H^1(Y, R^0 f_*\mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to H^0(Y, R^1 f_*\mathcal{O}_X) \to H^2(Y, R^0 f_*\mathcal{O}_X),$$

where $R^q f_*\mathcal{O}_X$ is the higher direct image sheaf. If $f$ is connected and proper, then $R^0 f_*\mathcal{O}_X = \mathcal{O}_Y$. 
In this section we prove the following:

**Proposition 2.1.** Let $X^n = G/\Gamma$ be a compact complex parallelizable pseudo-Kähler solvmanifold and $\pi : G/\Gamma \to G/\mathbb{N}\Gamma$ the Mostow fibration. Then $R^1\pi_*\mathcal{O}_X$ is the sheaf of sections of a trivial holomorphic vector bundle.

To prove this proposition we follow a part of the proof of Theorem 1.2 due to Winkelmann ([14]).

Let $G$ be a connected complex Lie group and $\Gamma$ a lattice of $G$. Let $V$ be a complex vector space, $\rho : G \to \text{GL}(V)$ an antiholomorphic representation and $V_1$ the set of all $v \in V$ which are invariant under $\rho(G')$, where $G'$ is the derived Lie subgroup of $G$. We denote by $V_0$ the subspace spanned by all vectors $v \in V_1$ such that $v$ is an eigenvector with a real eigenvalue for every $\rho(\gamma) (\gamma \in \Gamma)$.

Let $E, E_0$ be flat vector bundles over $X = G/\Gamma$ which are induced by $\rho|_{\Gamma}$ on $V$, $V_0$ respectively, i.e., $E = G \times V / \sim$, where $(g, v) \sim (g', v')$ if and only if $(g', v') = (gy^{-1}, \rho(\gamma)v)$. Note that $E, E_0$ are holomorphic vector bundles.

**Proposition 2.2.** The flat vector bundle $E_0$ is a holomorphically trivial vector bundle and $\Gamma(X, E) = \Gamma(X, E_0) \cong V_0$.

Proof. See [14], Propositions 7.9.1 and 7.9.2. □

Let $\pi : X \to B$ be a holomorphic fiber bundle with an $n$-dimensional complex torus $T^n_C$ as typical fiber. Let $V = \Omega^1(T^n_C)$ denote the vector space of holomorphic 1-forms on $T^n_C$. Note that $T^n_C$ is a compact Kähler manifold. Let $\mathfrak{U} = \{U_i\}$ be a trivializing open cover of $B$ such that $X$ is given by transition functions $\phi_{ij} : U_i \cap U_j \to \text{Aut}(T^n_C)$, where $\text{Aut}(T^n_C)$ is the automorphism group of $T^n_C$. We denote by $\text{Aut}^0(T^n_C)$ the identity component of $\text{Aut}(T^n_C)$.

**Lemma 2.3.** Under the above assumptions $R^1\pi_*\mathcal{O}_X$ is a locally free coherent sheaf of $B$ isomorphic to the sheaf of sections of the flat vector bundle $E$ given by transition functions $\varphi_{ij} = \xi \circ \phi_{ij} : U_i \cap U_j \to \text{GL}(V)$, where $1 \to \text{Aut}^0(T^n_C) \to \text{Aut}(T^n_C) \to \text{GL}(V)$ is exact.

Proof. See [14], CLAIM 8.4.5. □

We apply this lemma to a complex parallelizable manifold $G/\Gamma$. Let $K$ be a normal abelian complex Lie subgroup of $G$ and $\mathfrak{k}$ its Lie algebra. Assume that $K/K \cap \Gamma$ is compact. Denote the natural projection map $X = G/\Gamma \to B = G/K\Gamma$ by $\pi$. 
Proposition 2.4. The sheaf $R^1\pi_\star\mathcal{O}_X$ is the sheaf of sections of the flat vector bundle $E$ of rank $\dim K$ over $B$ induced by the representation $\rho: \Gamma \to GL(k^s)$ given by $\gamma \mapsto \text{Ad}^{\gamma}(\gamma)$.

Proof. See [14], Proposition 8.4.6.

Moreover, if $G/K$ is abelian, we have

Proposition 2.5 ([14]). If $G/K$ is abelian, then

$$\dim H^1(G/\Gamma, \mathcal{O}) \leq \dim G/K + \dim U,$$

where $U$ denotes the maximal linear subspace of $\mathfrak{k}$ such that $\text{Ad}(\gamma)|_U$ is a semisimple linear endomorphism with only real eigenvalues for every $\gamma$ in $\Gamma$.

Proof. Let us consider the following term of the Leray spectral sequence for $\pi: G/\Gamma \to G/\Gamma \Gamma$. Then we have the following:

$$0 \to H^1(G/\Gamma \Gamma, \mathcal{O}) \to H^1(G/\Gamma, \mathcal{O}) \to H^0(G/\Gamma \Gamma, R^1\pi_\star\mathcal{O}).$$

Since $G/\Gamma \Gamma \cong (G/K)/(\Gamma \Gamma/K)$ and $G/K$ is abelian, by Propositions 2.2 and 2.4, we see

$$\dim H^1(G/\Gamma, \mathcal{O}) \leq \dim H^1(G/\Gamma \Gamma, \mathcal{O}) + \dim U = \dim G/K + \dim U.$$

Hence we have our proposition.

Proof of Proposition 2.1. In Section 1, we have seen that if a compact complex parallelizable solvmanifold $X^n = G/\Gamma$ admits a pseudo-Kähler structure, then the maximal normal nilpotent Lie subgroup $N$ of $G$ is abelian and $h^{0,1} = n$. Thus let us consider the Mostow fibration $\pi: G/\Gamma \to G/N \Gamma$. Since $G/N$ is abelian and $G/\Gamma$ admits a pseudo-Kähler structure, we have $W = n$ by Proposition 2.5, where $W$ is the maximal linear subspace of $n$ such that $\text{Ad}(\gamma)|_W$ is a semisimple endomorphism with only real eigenvalues for every $\gamma \in \Gamma$. By Propositions 2.2 and 2.4, this means that the flat vector bundle $E$ induced by the representation $\rho|_{\Gamma}: \Gamma \to GL(n^s)$ given by $\gamma \mapsto \text{Ad}^{\gamma}(\gamma)$ is trivial as a holomorphic vector bundle.

3. Dolbeault cohomology of compact complex parallelizable pseudo-Kähler solvmanifolds

In this section we consider the Dolbeault cohomology groups of compact complex parallelizable pseudo-Kähler solvmanifolds.
Let $G$ be a complex Lie group and $\mathfrak{g}$ its Lie algebra. Let $I$ denote the complex structure of $\mathfrak{g}$, and $\mathfrak{g}^+$ (resp. $\mathfrak{g}^-$) denote the vector space of the $+\sqrt{-1}$ (resp. $-\sqrt{-1}$) eigenvectors of the complex structure $I$ respectively. Then we have $\mathfrak{g}^C = \mathfrak{g}^+ \oplus \mathfrak{g}^-$. In this section, we identify $\mathfrak{g}^+$ with the set of all right invariant holomorphic vector fields of $G$. Recall that $H_\partial^{p,q}(G/\Gamma) \cong H_\partial^{0,q}(G/\Gamma) \otimes \wedge^p(\mathfrak{g}^+)^s$ for a compact complex parallelizable manifold $G/\Gamma$. Sakane [12] has proved that if $G$ is a complex nilpotent Lie group, then $H_\partial^{p,q}(G/\Gamma) \cong H^q(\mathfrak{g}^-) \otimes \wedge^p(\mathfrak{g}^+)^s$, where $H^q(\mathfrak{g}^-)$ is the $q$th Lie algebra cohomology group of $\mathfrak{g}^-$. Let $F \to X \to B$ be a holomorphic fiber bundle such that $X, B, F$ are connected and $F$ is compact. Then $\bigcup_{b \in B} H_\partial^{p,q}(F_b)$ is the total space of a differentiable vector bundle over $B$. This bundle is denoted by $H^{p,q}(F)$ and $\mathfrak{H}_3(F)$ is the direct sum of $H^{p,q}(F)$. If every connected component of the structure group of $\pi : X \to B$ acts trivially on $\mathfrak{H}_3(F)$, then the vector bundle is a holomorphic vector bundle. Thus if the fiber $F$ is a compact Kähler manifold, then $\mathfrak{H}_3(F)$ is a holomorphic vector bundle.

**Theorem 3.1** ([8]). Let $\xi = (X, B, F, \pi)$ be a holomorphic fiber bundle, where $X, B, F$ are connected and $F$ is compact. Assume that every connected component of the structure group of $\xi$ acts trivially on $\mathfrak{H}_3(F)$, i.e., $\mathfrak{H}_3(F)$ is a holomorphic vector bundle. Then there exists a spectral sequence $(E_r, d_r)$, $(r \geq 0)$, with the following properties:

(i) $E_r$ is $4$-graded, by the fiber-degree, the base-degree and the type. Let $p,qE_{s,t}^s$ be the subspace of elements of $E_r$ of type $(p, q)$, fiber-degree $s$ and base-degree $t$. We have $p,qE_{s,t}^s = 0$ if $p+q \neq s+t$ or if one of $p, q, s, t$ is negative. The differential $d_r$ maps $p,qE_{s,t}^s$ into $p,qE_{s+1,t-1}^{s+1}$. 

(ii) If $p+q = s+t$, then we have $p,qE_{s,t}^s \cong \sum_{i \geq 0} H_\partial^{i,s-i}(B, \mathfrak{H}_{p-i, q-s+i}(F))$.

(iii) The spectral sequence converges to $\mathfrak{H}_3(X) = \bigoplus_{p,q} H_\partial^{p,q}(X)$.

We put $p,qE_r = \sum_{s,t \geq 0} p,qE_{s,t}^s$. We call the above spectral sequence the Borel’s spectral sequence.

**Remark 3.2.** If $F$ is a complex torus, then the vector bundle $\mathfrak{H}_3^{0,1}(F) \to B$ is isomorphic to the holomorphic vector bundle $E$ considered in Section 2 (see Lemma 2.3).

By Theorem 1.4 and the Borel’s spectral sequence, we have

**Proposition 3.3.** Let $X = G/\Gamma$ be a compact complex parallelizable manifold which admits a pseudo-Kähler structure and $F \to X \to B$ a holomorphic fiber bun-
dle such that $F, B$ are complex tori. If $H_3(F) \to B$ is trivial as a holomorphic vector bundle, then

$$h^{p,q}(X) = \binom{n}{p} \cdot \binom{n}{q}.$$ 

Proof. By our assumption we see that the Borel’s spectral sequence satisfies

$$p,q \ E_2 \cong \sum_{i \geq 0} H_3^{2i-j}(B) \otimes H_3^{p-j,q+i}(F).$$

Thus by the relation $\dim h^{p,q} \leq \dim h^{p,q} E_2$, we see that $h^{p,q}(X) \leq \binom{n}{p} \cdot \binom{n}{q}$. Thus we have our proposition by Theorem 1.4.

If $H_0^1(T^n_C) \to B$, where $T^n_C$ is an $n$-dimensional complex torus, admits global holomorphic sections $\sigma_1, \ldots, \sigma_n$ which are linearly independent at each point, then $H_0^1(T^n_C) \to B$ is trivial as a holomorphic vector bundle. Indeed, consider $\sigma_j = \sigma_{j_1} \wedge \cdots \wedge \sigma_{j_k}$ (Note that $h^{0,q}(T^n_C) = \binom{n}{q}$). Thus by Proposition 2.1 and Lemma 2.3 we have

**Corollary 3.4.** If a compact complex parallelizable solvmanifold $X^n = G/\Gamma$ admits a pseudo-Kähler structure, then

$$h^{p,q}(X) = \binom{n}{p} \cdot \binom{n}{q}.$$ 

Proof. By our assumption, we see that $H_0^{0,q}(N/\Gamma_N) \to G/N\Gamma$ is trivial as a holomorphic vector bundle.

Let $(X^n, \omega)$ be a compact pseudo-Kähler manifold. We say that $(X^n, \omega)$ has the hard Lefschetz property with respect to the Dolbeault cohomology if for any $p+q \leq n$, the homomorphism

$$L^{n-p,q} : H_3^{p,q}(X) \to H_3^{n-q,n-p}(X), \quad L^{n-p,q}(\alpha) = [\alpha \wedge \omega^{n-p-q}]$$

is an isomorphism.

**Corollary 3.5.** Let $(G/\Gamma, \omega)$ be an $n$-dimensional compact complex parallelizable pseudo-Kähler solvmanifold. Then $(G/\Gamma, \omega)$ has the hard Lefschetz property with respect to the Dolbeault cohomology.

Proof. Put $\bar{\tau}_i = i(X^+_i)\omega$, where $[X^+_1, \ldots, X^+_n]$ is a basis of $g^+$. We denote the dual basis of $g^*$ by $[\omega^+_1, \ldots, \omega^+_n]$. Then $\omega$ can be written as $\omega = \sum_{i=1}^n \bar{\tau}_i \wedge \omega^+_i$. In particular, we see that $\bar{\tau}_i$ are non-exact $\bar{\partial}$-closed. We also see that $\alpha = \sum_{j,k} a_{jk} \bar{\tau}_j \wedge \omega^+_k$. 

is non-exact $\bar{\partial}$-closed, where $a_{JK} \in \mathbb{C}$, $\bar{\tau}_J = \bar{\tau}_{j_1} \wedge \cdots \wedge \bar{\tau}_{j_q}$ for $J = (j_1, \ldots, j_q)$ and $\omega^+_K = \omega^+_{k_1} \wedge \cdots \wedge \omega^+_{k_p}$ for $K = (k_1, \ldots, k_p)$. Thus by Corollary 3.4 for each Dolbeault cohomology class we can choose a representative of the form $\alpha = \sum_{JK} a_{JK} \bar{\tau}_J \wedge \omega^+_K$. Hence $(G/\Gamma, \omega)$ has the hard Lefschetz property.

By the proof of Corollary 3.5, we see that if an $n$-dimensional compact complex parallelizable solvmanifold $G/\Gamma$ admits a pseudo-Kähler structure, then $H_{\bar{\partial}}(G/\Gamma) = \bigoplus_{p,q} H^{p,q}_{\bar{\partial}}(G/\Gamma)$ is isomorphic to the cohomology ring $H_{\bar{\partial}}(T^\ast_G)$.

**Remark 3.6.** Mathieu’s theorem of the Dolbeault cohomology on a compact pseudo-Kähler manifold $(X, \omega)$ also holds (see [18], [19]), i.e., the following two assertions are equivalent: (a) every Dolbeault cohomology class contains a $\bar{\partial}$-harmonic representative. (b) $(X, \omega)$ has the hard Lefschetz property with respect to the Dolbeault cohomology. We define $\bar{\partial}: \Omega^{p,q}(X) \to \Omega^{p-1,q}(X)$ by $\bar{\partial} = (-1)^{p+q} \bar{\partial}^*$, where $\Omega^{p,q}(X)$ is the set of all differential $(p,q)$-forms on $X$. A form $\alpha$ is called a $\bar{\partial}$-harmonic form if it satisfies $\bar{\partial}^* \alpha = 0$, where $\ast: \Omega^{p,q}(X^n) \to \Omega^{n-q,n-p}(X^n)$ is defined as an analogy of the star operator for a compact Riemannian manifold. In the above case, for each Dolbeault cohomology class, we can choose a $\bar{\partial}$-harmonic representative of the form $\alpha = \sum_{JK} a_{JK} \bar{\tau}_J \wedge \omega^+_K$.

Let $(G/\Gamma, \omega)$ be an $n$-dimensional compact complex parallelizable pseudo-Kähler solvmanifold. We now give another proof of our main theorem for the case where the Lie algebra $\mathfrak{g}$ of $G$ has a Chevalley decomposition. We use a system of coordinates of Remarks 1.9 and the notation of the proof of Corollary 3.5. Then the pseudo-Kähler structure $\omega$ on $G/\Gamma$ can be written as follows:

$$\omega = \sum_{i=1}^{n} \bar{\tau}_i \wedge \omega^+_i.$$ 

Note that $\bar{\tau}_i$, $\omega^+_i$ are $\bar{\partial}$-closed. By Remarks 1.9, $\bar{\tau}_i$ are expressed by

$$\bar{\tau}_i = \psi_i + \bar{\partial} \gamma_i,$$

where $\psi_i = \sum_{\lambda=1}^{k} c^{i}_{\lambda} d \bar{z}_\lambda + \sum_{\lambda=k+1}^{k+r(F)} f^{i}_{\lambda}(z) d \bar{z}_\lambda$, $F = N/\Gamma_N$, $c^{i}_{\lambda}$ are constant and $f^{i}_{\lambda}$ are holomorphic. Hence we can write

$$\omega = \sum_{i=1}^{n} \psi_i \wedge \omega^+_i + \bar{\partial} \theta.$$ 

Put $\omega' = \sum_{i=1}^{n} \psi_i \wedge \omega^+_i$. Then a volume form $\omega^n$ is expressed by $\omega^n = \omega'^n + \bar{\partial} \Omega$, where $\Omega \in \Omega^{n,n-1}(X)$. Since $\int_{G/\Gamma} \omega'^n = \int_{G/\Gamma} \omega^n + \int_{G/\Gamma} \bar{\partial} \Omega = \int_{G/\Gamma} \omega'^n$, we see that $r(F) = \dim H^{0,1}_{\bar{\partial}}(F) = \dim F = n - k$. Since $F$ is a compact complex parallelizable
nilmanifold, we see \( n - k = \dim H^0(\Omega_N) = \dim H^1(n^-) \) by Sakane’s theorem. Thus \( n^- \) is abelian, which implies that \( F \) is a complex torus.

4. Examples

**Example 4.1** ([11]). Define a multiplication \(*\) of \( \mathbb{C}^{n+2m} \) by

\[
(z_1, \ldots, z_n, w_1, w_2, \ldots, w_{2m-1}, w_{2m}) * (z'_1, \ldots, z'_n, w'_1, w'_2, \ldots, w'_{2m-1}, w'_{2m}) = (z_1 + z'_1, \ldots, z_n + z'_n, \ldots, e^{-\sum a_i z_i} w'_{2k-1} + w_{2k-1}, e^{\sum a_i z'_i} w'_{2k} + w_{2k}, \ldots),
\]

where \( a_i^k \) are integers. The solvable Lie group \( G = (\mathbb{C}^{n+2m}, *) \) has a lattice \( \Gamma \) such that \( G/\Gamma \) has a pseudo-Kähler structure. Indeed, for a suitable lattice \( \Gamma \) of \( G \),

\[
\omega = \sqrt{-1} \sum_{k=1}^{n} d\bar{z}_k \wedge d\bar{z}_k + \sum_{k=1}^{m} (d\bar{w}_k + d\bar{w}_k) + d\bar{w}_k + \ldots
\]

is a pseudo-Kähler structure on \( G/\Gamma \) (for details, see [17]). By Corollary 3.4, we see \( h^{p,q}(G/\Gamma) = \binom{n+2m}{p} \cdot \binom{n+2m}{q} \).

**Example 4.2** (cf. [4]). Let us consider the following solvable Lie group:

\[
G = \left\{ \begin{pmatrix} e^{z_1} & 0 & z_2 e^{z_1} & 0 & 0 & 0 & w_1 \\
0 & e^{-z_1} & 0 & z_2 e^{-z_1} & 0 & 0 & w_2 \\
0 & 0 & e^{z_1} & 0 & 0 & 0 & w_3 \\
0 & 0 & 0 & e^{-z_1} & 0 & 0 & w_4 \\
0 & 0 & 0 & 0 & 1 & 0 & z_2 \\
0 & 0 & 0 & 0 & 0 & 1 & z_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \ \mid z_1, z_2, w_1, w_2, w_3, w_4 \in \mathbb{C} \right\}
\]

The Lie algebra \( \mathfrak{g} \) of \( G \) is given by

\[\mathfrak{g} = \text{span}_\mathbb{C}\{Z_1, Z_2, W_1, W_2, W_3, W_4\}\]

with

\[
[Z_1, W_{2k-1}] = W_{2k-1}, \quad [Z_1, W_{2k}] = -W_{2k}, \quad [Z_2, W_3] = W_1, \quad [Z_2, W_4] = W_2
\]

for \( k = 1, 2 \). The solvable Lie group \( G \) admits a lattice \( \Gamma \) (see [16]). Since the maximal nilpotent ideal \( n \) is not abelian, we see that for any lattice \( \Gamma \), \( G/\Gamma \) has no pseudo-Kähler structures by Corollary 1.8.
References