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ISOSPECTRAL PROSPERTY OF DOUBLE DARBOUX TRANSFORMATION

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Abstract

The isospectral property of the algebro-geometric double Darboux transformation is studied by an algebraic method. This is established by verifying that the algebro-geometric double Darboux transformation of the n-th algebro-geometric potential solves the evolution equation of the higher order KdV type. Applying these results, the second Darboux-Lamé potential, which can be regarded as the 3-elliptic solitons, is explicitly constructed by an elementary method.

1. Introduction

In this paper, we study the isospectral property of the algebro-geometric double Darboux transformation of the second order ordinary differential operator in the complex domain

$$H(u) = -\frac{\partial^2}{\partial x^2} + u(x), \quad x \in \mathbb{C},$$

where $u(x)$ is the n-th algebro-geometric potential which satisfies a kind of degenerate condition concerned with the multiplicity of the discrete spectrum obtained in [10].

In [9] and [10], some of the fundamental properties of the spectrum of the algebro-geometric Darboux transformation of $H(u)$ have been clarified. Above all, in [9], the spectral discriminant $\Delta(\lambda; u)$ and the M-eigenfunction $M(x, \lambda; u)^{1/2}$ are constructed by an algebraic method, and, in [10], it is shown that if $\Delta(\lambda; u)$ has the multiple root λ_j, one can reduce the multiplicity of λ_j by the algebro-geometric Darboux transformation

$$u^*_{\lambda_j}(x) = u(x) - 2\frac{\partial^2}{\partial x^2} \log M(x, \lambda_j; u)^{1/2}.$$

In the present paper, firstly, we define the algebro-geometric double Darboux transformation $u^{**}_{\lambda_j}(x, \xi)$ of $u(x)$ by

$$u^{**}_{\lambda_j}(x, \xi) = u^*_{\lambda_j}(x) - 2\frac{\partial^2}{\partial x^2} \log F_{\lambda_j}(x, \xi),$$

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where $F_{\lambda_j}(x,\xi)$ is the 1-parameter family of the specified eigenfunction of $H(u^*_{\lambda_j})$ corresponding to the multiple spectrum λ_j of $H(u)$. Secondly, we show that $H\big(u^{**}_{\lambda_j}(x,\xi)\big)$ is the 1-parameter family of the isospectral differential operator. We prove this by showing that $u^{**}_{\lambda_j}(x,\xi)$ solves the evolution equation of the higher order KdV type. Lastly, applying these results to the second Lamé equation

$$-\frac{\partial^2}{\partial x^2} f(x,\lambda) + 6\wp(x,\tau) f(x,\lambda) = \lambda f(x,\lambda),$$

we construct the second Darboux-Lamé potential

$$u^{**}_{2,0}(x,\xi,\tau_*) = \frac{6\wp(x,\tau_*)\big(\phi_0(\xi)^2 - 3\xi\phi_0(\xi)\wp'(x,\tau_*) + (27/4)g_3(\tau_*)\xi^2\big)}{(\phi_0(\xi) + (3/2)\xi\wp'(x,\tau_*))^2} \tag{1}$$

for $\tau = \tau_*$ such that $J(\tau_*) = 0$, where $\wp(x,\tau)$ is the Weierstrass elliptic function with the periods 1 and τ, $\Im\tau > 0$ and $J(\tau)$ is the elliptic modular function. In [11], the isomonodromic property of the second Darboux-Lamé equation

$$-\frac{\partial^2}{\partial x^2} f(x) + u^{**}_{2,0}(x,\xi,\tau_*) f(x) = 0 \tag{2}$$

is studied. Moreover, in [12], applying these results, the various kinds of the exact solutions of the elliptic Calogero system are constructed. See [3], [4], [5], [6] and [7] for another approach to the algebro-geometric potential and the double Darboux transformation, which is mentioned as the double commutation method in them.

The contents of this paper are as follows. In §2, the necessary materials are summarized. In §3, the algebro-geometric double Darboux transformation is defined. In §4, the isospectral property of the algebro-geometric double Darboux transformation is discussed. In §5, the second Darboux-Lamé potential (1) is constructed.

2. Preliminaries

In this section, the necessary materials are summarized. We refer the reader to [9], [10] and [13] for more precise information.

The Λ-operator associated with the differential operator $H(u)$ is the formal pseudo differential operator defined by

$$\Lambda(u) = \left(\frac{\partial}{\partial x}\right)^{-1}\left(\frac{1}{2}u' + u\frac{\partial}{\partial x} - \frac{1}{4}\frac{\partial^3}{\partial x^3}\right).$$

The functions $Z_n(u)$, $n \in \mathbb{N}$ defined by the recursion relation

$$Z_0(u) \equiv 1, \quad Z_n(u) = \Lambda(u) Z_{n-1}(u), \quad n = 1, 2, \ldots \tag{3}$$

are the differential polynomials in $u(x)$. We call these differential polynomials the KdV polynomials.

Let $V(u)$ be the linear span of all KdV polynomials over $\mathbb{C}$. If $\dim V(u) = n+1$, then $u(x)$ is called the n-th algebro-geometric potential and we write $\operatorname{rank} u(x) = n$. In [9], it is shown that if $u(x)$ is the n-th algebro-geometric potential, then there uniquely exist the polynomials $a_j(\lambda; u)$, $j = 0, 1, \ldots, n$ in the spectral parameter λ of degree $n - j + 1$ such that

$$Z_{n+1}(u-\lambda) = \sum_{j=0}^{n} a_j(\lambda; u) Z_j(u-\lambda). \tag{4}$$

Suppose that $u(x)$ is the n-th algebro-geometric potential. The M-function $M(x, \lambda; u)$ associated with $u(x)$ is the differential polynomial defined by

$$M(x, \lambda; u) = Z_n(u-\lambda) - \sum_{j=1}^{n} a_j(\lambda; u) Z_{j-1}(u-\lambda),$$

where $a_j(\lambda; u)$, $j = 1, \ldots, n$ are the polynomials defined by (4).

REMARK. In [10], the notation "$F(x, \lambda; u)$" is used for the M-function $M(x, \lambda; u)$.

The spectral discriminant $\Delta(\lambda; u)$ is defined by

$$\Delta(\lambda; u) = M_x(x, \lambda; u)^2 - 2M(x, \lambda; u) M_{xx}(x, \lambda; u) + 4(u(x) - \lambda) M(x, \lambda; u)^2, \tag{5}$$

which is the polynomial of degree $2n+1$ in λ with constant coefficients. The set

$$\operatorname{Spec} H(u) = \{\lambda \mid \Delta(\lambda; u) = 0\} \subset \mathbb{C}$$

is called the Λ-spectrum of the operator $H(u)$.

The set $\operatorname{Spec} H(u)$ corresponds to the discrete spectrum of the operator $H(u)$. In [9], it is shown that if $\lambda_j \in \operatorname{Spec} H(u)$, then $M(x, \lambda_j; u)^{1/2}$ satisfies the eigenvalue problem

$$(H(u) - \lambda_j) M(x, \lambda_j; u)^{1/2} = 0. \tag{6}$$

We call $M(x, \lambda_j; u)^{1/2}$ the M-eigenfunction.

3. The algebro-geometric Darboux transformation

In this section, we explain the method of Darboux transformation, which was originated by G. Darboux [1], and define the algebro-geometric (double) Darboux transformation.

For $f(x, \lambda) \in \ker(H(u) - \lambda) \setminus \{0\}$, the Darboux transformation is the operator $H(u^*)$ with the potential $u^*(x)$ defined by

$$u^*(x) = u(x) - 2 \frac{\partial^2}{\partial x^2} \log f(x, \lambda).$$

We sometimes call the potential $u^*(x)$ itself the Darboux transformation.

The most essential fact related to the Darboux transformation is the following lemma.

Lemma 1 (Darboux [1]). *Define the function* $g(x, \alpha, \beta, \lambda)$ *by*

$$g(x, \alpha, \beta, \lambda) = \frac{1}{f(x, \lambda)} \left(\alpha + \beta \int f(x, \lambda)^2 \, dx \right),$$

then

$$(H(u^*) - \lambda) g(x, \alpha, \beta, \lambda) = 0$$

holds for arbitrary $\alpha, \beta \in \mathbb{C}$.

This is called Darboux's lemma. Moreover we have the following lemma.

Lemma 2 ([13]). *Suppose* $f(x, \lambda) \in \ker(H(u) - \lambda) \setminus \{0\}$, *and let*

$$B_\lambda^{(\pm)} = \pm \frac{\partial}{\partial x} + 2q(x, \lambda),$$

where

$$q(x, \lambda) = \frac{\partial}{\partial x} \log f(x, \lambda) = \frac{f'(x, \lambda)}{f(x, \lambda)},$$

then the identity

$$B_\lambda^{(-)} Z_k(u) = B_\lambda^{(+)} Z_k(u^*). \tag{7}$$

holds.

We call this formula (7) the fundamental identity of the Darboux transformation.

Now we define the algebro-geometric Darboux transformation (the ADT) and the algebro-geometric double Darboux transformation (the ADDT).

DEFINITION 1. The Darboux transformation of the algebro-geometric potential $u(x)$ by the corresponding M-eigenfunction $M(x, \lambda_j; u)^{1/2}$, $\lambda_j \in \operatorname{Spec} H(u)$

$$u^*_{\lambda_j}(x) = u(x) - 2\frac{\partial^2}{\partial x^2} \log M(x, \lambda_j; u)^{1/2} = u(x) - \frac{\partial^2}{\partial x^2} \log M(x, \lambda_j; u) \tag{8}$$

is called the algebro-geometric Darboux transformation (the ADT as the abbreviation).

By (6) and Lemma 1, we immediately have the following lemma.

Lemma 3 (The eigenfunction of $H\big(u^*_{\lambda_j}\big)$). *Let*

$$\widehat{M}(x, \lambda_j; u) = \int M(x, \lambda_j; u)\, dx \tag{9}$$

and fix the integration constant arbitrarily, then the function $F_{\lambda_j}(x, \xi)$ *defined by*

$$F_{\lambda_j}(x, \xi) = \frac{\phi_{\lambda_j}(\xi) + \xi \widehat{M}(x, \lambda_j; u)}{M(x, \lambda_j; u)^{1/2}} \tag{10}$$

is the 1*-parameter family of the eigenfunction of* $H\big(u^*_{\lambda_j}\big)$ *associated with the eigenvalue* λ_j*, i.e.,*

$$\big(H\big(u^*_{\lambda_j}\big) - \lambda_j\big) F_{\lambda_j}(x, \xi) = 0,$$

where $\phi_{\lambda_j}(\xi)$ *is an arbitrary function which depends only on* ξ*.*

By this lemma, we can define the algebro-geometric double Darboux transformation as follows.

DEFINITION 2. The algebro-geometric double Darboux transformation (the ADDT as the abbreviation) $H\big(u^{**}_{\lambda_j}(x, \xi)\big)$ is defined as the Darboux transformation of $H\big(u^*_{\lambda_j}(x)\big)$ by the eigenfunction $F_{\lambda_j}(x, \xi)$ defined by (10), i.e., the operator with the potential $u^{**}_{\lambda_j}(x, \xi)$ defined by

$$\begin{aligned} u^{**}_{\lambda_j}(x, \xi) &= u^*_{\lambda_j}(x) - 2\frac{\partial^2}{\partial x^2} \log F_{\lambda_j}(x, \xi) \\ &= u(x) - 2\frac{\partial^2}{\partial x^2} \log\big(\phi_{\lambda_j}(\xi) + \xi \widehat{M}(x, \lambda_j; u)\big). \end{aligned} \tag{11}$$

If $\phi_{\lambda_j}(\xi) \equiv 0$ then the ADDT $u^{**}_{\lambda_j}(x, \xi)$ does not depend on ξ. Hence, in what follows, we assume that $\phi_{\lambda_j}(\xi)$ does not vanish identically.

On the other hand, as for the rank of the ADT of the n-th algebro-geometric potential, we have the bounds

$$n - 1 \le \operatorname{rank} u^*_{\lambda_j}(x) \le n + 1$$

for all $\lambda_j \in \operatorname{Spec} H(u)$ [13].

It is well known that $\operatorname{rank} u^*_{\lambda_j}(x) \ge n$ holds generically, while the degenerate case $\operatorname{rank} u^*_{\lambda_j}(x) = n - 1$ occurs very exceptionally. For our purpose, this exceptional case is important. It is shown in [10] that if $u(x)$ is the n-th algebro-geometric potential, then the ADT $u^*_{\lambda_j}(x)$ is the $(n - 1)$-th algebro-geometric potential if and only if $\operatorname{Spec}_m H(u) \ne \emptyset$ and $\lambda_j \in \operatorname{Spec}_m H(u)$, where

$$\operatorname{Spec}_m H(u) = \{\lambda_j \mid \text{The multiple roots of } \Delta(\lambda; u)\},$$

which we call the multiple spectrum of $H(u)$.

4. Isospectral property of the ADDT

In this section, we clarify the isospectral property of the algebro-geometric double Darboux transformation $H\big(u^{**}_{\lambda_j}(x,\xi)\big)$.

Suppose that $v(x,\eta)$ is analytic with respect to the complex variables x and η. We say that the 1-parameter family of the differential operator $H(v(x,\eta))$ is isospectral if and only if $H(v(x,\eta))$ satisfies the Lax equation

$$\frac{d}{d\eta}H(v(x,\eta)) = [A(x,\eta), H(v(x,\eta))], \tag{12}$$

where $[A,B] = AB - BA$ is the commutator, and $A(x,\eta)$ is the odd order ordinary differential operator with respect to the variable x with a parameter η. We show this by verifying that the resulted potential $u^{**}_{\lambda_j}(x,\xi)$ solves the evolution equation of higher order non-stationary KdV type, and this evolution equation can be expressed as the Lax equation (12).

The double Darboux transformation method, or the double commutation method has been studied extensively by the functional analytic method by many authors. See e.g. [3], [4], [5], [6], and [7]. Different from these works, in the present one, we study its isospectral property by an algebraic method.

Now we prove that the potential $u^{**}_{\lambda_j}(x,\xi)$ solves the evolution equation of the higher order KdV type with the time variable ξ. Suppose $\lambda_j \in \mathrm{Spec}_m H(u)$, and define the functions $Q_{\lambda_j}(x,\xi)$ and $N_{\lambda_j}(x,\xi)$ by

$$\begin{aligned} Q_{\lambda_j}(x,\xi) &= \frac{\partial}{\partial x}\log F_{\lambda_j}(x,\xi), \\ N_{\lambda_j}(x,\xi) &= Z_n\big(u^{**}_{\lambda_j}(x,\xi)-\lambda_j\big) - \sum_{k=1}^{n} a_k(\lambda_j;u)Z_{k-1}\big(u^{**}_{\lambda_j}(x,\xi)-\lambda_j\big). \end{aligned}$$

Then we have the following lemma.

Lemma 4. *$N_{\lambda_j}(x,\xi)F_{\lambda_j}(x,\xi)^2$ is independent of x.*

Proof. Define the first order differential operators $B^{(\pm)}_{\lambda_j}(\xi)$ by

$$B^{(\pm)}_{\lambda_j}(\xi) = \pm\frac{\partial}{\partial x} + 2Q_{\lambda_j}(x,\xi),$$

then, by Lemma 2, we have

$$B^{(-)}_{\lambda_j}(\xi)Z_k\big(u^{*}_{\lambda_j}(x)-\lambda_j\big) = B^{(+)}_{\lambda_j}(\xi)Z_k\big(u^{**}_{\lambda_j}(x,\xi)-\lambda_j\big)$$

for all $k \geq 0$. Since $\lambda_j \in \mathrm{Spec}_m H(u)$, as mentioned in §3, rank $u^*_{\lambda_j}(x) = n-1$ holds. Moreover, by the argument similar to that in [10, p.968], we have immediately

$$Z_n\big(u^*_{\lambda_j}(x)-\lambda_j\big) - \sum_{k=1}^{n} a_k(\lambda_j;u) Z_{k-1}\big(u^*_{\lambda_j}(x)-\lambda_j\big) = 0.$$

Therefore, we have

$$\begin{aligned} &B^{(+)}_{\lambda_j}(\xi) N_{\lambda_j}(x,\xi) \\ &= B^{(+)}_{\lambda_j}(\xi)\left(Z_n\big(u^{**}_{\lambda_j}(x,\xi)-\lambda_j\big) - \sum_{k=1}^{n} a_k(\lambda_j;u) Z_{k-1}\big(u^{**}_{\lambda_j}(x,\xi)-\lambda_j\big)\right) \\ &= B^{(-)}_{\lambda_j}(\xi)\left(Z_n\big(u^{*}_{\lambda_j}(x)-\lambda_j\big) - \sum_{k=1}^{n} a_k(\lambda_j;u) Z_{k-1}\big(u^{*}_{\lambda_j}(x)-\lambda_j\big)\right) = 0. \end{aligned}$$

In other words, by the definition of the operator $B^{(+)}_j(\xi)$, we have

$$B^{(+)}_{\lambda_j}(\xi) N_{\lambda_j}(x,\xi) = N_{\lambda_j x}(x,\xi) + \frac{2F_{\lambda_j x}(x,\xi) N_{\lambda_j}(x,\xi)}{F_{\lambda_j}(x,\xi)} = 0.$$

This implies

$$\frac{\partial}{\partial x} \log N_{\lambda_j}(x,\xi) F_{\lambda_j}(x,\xi)^2 = 0.$$

This completes the proof. □

By Definition 2, since $\phi_{\lambda_j}(\xi)$ does not identically vanish, we have immediately

$$u^{**}_{\lambda_j}(x,\xi) = u(x) - 2\frac{\partial^2}{\partial x^2} \log\left(1 + \frac{\xi}{\phi_{\lambda_j}(\xi)} \widehat{M}(x,\lambda_j;u)\right).$$

Since $N_{\lambda_j}(x,\xi)$ is the differential polynomial in $u^{**}_{\lambda_j}(x,\xi)$, it turns out that the function $N_{\lambda_j}(x,\xi)$ is the rational function of the new parameter $\eta = \xi/\phi_{\lambda_j}(\xi)$. Similarly, note that

$$G_{\lambda_j}(x,\eta) = \frac{1}{\phi_{\lambda_j}(\xi)} F_{\lambda_j}(x,\xi) = \frac{1+\eta \widehat{M}(x,\lambda_j;u)}{M(x,\lambda_j;u)^{1/2}}$$

is the eigenfunction of the ADT $H(u^*_{\lambda_j})$ corresponding to the eigenvalue λ_j, which is the linear function of η. On the other hand, we have

$$\frac{1}{\phi_{\lambda_j}(\xi)^2} N_{\lambda_j}(x,\xi) F_{\lambda_j}(x,\xi)^2 = \frac{\big(1+\eta\widehat{M}(x,\lambda_j;u)\big)^2 N_{\lambda_j}(x,\xi)}{M(x,\lambda_j : u)}.$$

The left hand side of the above is independent of x and rational in η. Hence we denote it $h_{\lambda_j}(\eta)$, i.e.,

$$h_{\lambda_j}(\eta) = \frac{1}{\phi_{\lambda_j}(\xi)^2} N_{\lambda_j}(x, \xi) F_{\lambda_j}(x, \xi)^2.$$

Since the ADT $u^*_{\lambda_j}(x)$ does not depend on η, both $G_{\lambda_j}(x, \eta)$ and $(\partial/\partial\eta) G_{\lambda_j}(x, \eta)$ solve the eigenvalue problem

$$\left(H\left(u^*_{\lambda_j}\right) - \lambda_j\right) f(x) = \left(-\frac{\partial^2}{\partial x^2} + (u^*_{\lambda_j}(x) - \lambda_j)\right) f(x) = 0.$$

Moreover one easily verifies

$$W\left[G_{\lambda_j}, \frac{\partial}{\partial\eta} G_{\lambda_j}\right] = 1,$$

where $W[f, g] = fg' - f'g$ is the Wronskian. Now we compute the derivative $(\partial/\partial\eta) u^{**}_{\lambda_j}$. Since

$$u^{**}_{\lambda_j}(x, \xi) = u^*_{\lambda_j}(x) - 2\frac{\partial^2}{\partial x^2} \log G_{\lambda_j}(x, \eta),$$

and, as mentioned above, $u^*_{\lambda_j}(x)$ does not depend on η, we have

$$\begin{aligned}\frac{\partial}{\partial\eta} u^{**}_{\lambda_j}(x, \xi) &= -2\frac{\partial^3}{\partial^2 x \partial\eta} \log G_{\lambda_j}(x, \eta) = -2\frac{\partial^2}{\partial x^2} \frac{G_{\lambda_j \eta}(x, \eta)}{G_{\lambda_j}(x, \eta)} \\ &= -2\frac{\partial}{\partial x} \frac{W\left[G_{\lambda_j}, G_{\lambda_j \eta}\right]}{G_{\lambda_j}(x, \eta)^2} = \frac{4G_{\lambda_j x}(x, \eta)}{G_{\lambda_j}(x, \eta)^3} = 4\phi_{\lambda_j}(\xi)^2 \frac{F_{\lambda_j x}(x, \xi)}{F_{\lambda_j}(x, \xi)^3}.\end{aligned}$$

On the other hand, one verifies

$$\frac{\partial}{\partial x} N_{\lambda_j}(x, \xi) = -2h_{\lambda_j}(\eta)\phi_{\lambda_j}(\xi)^2 \frac{F_{\lambda_j x}(x, \xi)}{F_{\lambda_j}(x, \xi)^3} = -\frac{h_{\lambda_j}(\eta)}{2} \frac{\partial}{\partial\eta} u^{**}_{\lambda_j}(x, \xi).$$

Hence, by the definition of $N_{\lambda_j}(x, \xi)$, we have

$$\begin{aligned}&-\frac{h_{\lambda_j}(\eta)}{2} \frac{\partial}{\partial\eta} u^{**}_{\lambda_j}(x, \xi) \\ &= \frac{\partial}{\partial x}\left(Z_n\left(u^{**}_{\lambda_j}(x, \xi) - \lambda_j\right) - \sum_{k=1}^{n} a_k(\lambda_j; u) Z_{k-1}\left(u^{**}_{\lambda_j}(x, \xi) - \lambda_j\right)\right).\end{aligned} \tag{13}$$

On the other hand, define the $(2k+1)$-th order differential operator $A_k(x, \lambda_j; \eta)$ by

$$\begin{aligned}&A_k(x, \lambda_j; \eta) \\ &= \frac{1}{2}\sum_{l=0}^{k}\left(Z_l\left(u^{**}_{\lambda_j}(x, \xi) - \lambda_j\right)\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial}{\partial x} Z_l\left(u^{**}_{\lambda_j}(x, \xi) - \lambda_j\right)\right) H\left(u^{**}_{\lambda_j}(x, \xi) - \lambda_j\right)^{k-l},\end{aligned}$$

then we have

$$\frac{\partial}{\partial x} Z_k\left(u^{**}_{\lambda_j}(x,\xi) - \lambda_j\right) = \left[A_k(x,\lambda_j;\eta), H\left(u^{**}_{\lambda_j}(x,\xi) - \lambda_j\right)\right].$$

(cf. [9, pp.411–412] and its references.) Hence we have

$$\frac{\partial}{\partial \eta} u^{**}_{\lambda_j}(x,\xi) - \left[\widehat{A}(x,\lambda_j;\eta), H\left(u^{**}_{\lambda_j}(x,\xi)\right) - \lambda_j\right] = 0,$$

where

$$\widehat{A}(x,\lambda_j;\eta) = -\frac{2}{h_{\lambda_j}(\eta)}\left(A_n(x,\lambda_j;\eta) - \sum_{k=1}^{n} a_k(\lambda_j;u) A_{k-1}(x,\lambda_j;\eta)\right).$$

Since $\left[\widehat{A}, \lambda_j\right] = 0$ and

$$\frac{\partial}{\partial \eta} u^{**}_{\lambda_j}(x,\xi) = \frac{d}{d\eta} H\left(u^{**}_{\lambda_j}(x,\xi)\right),$$

hold obviously, we finally obtain the Lax representation

$$\frac{d}{d\eta} H\left(u^{**}_{\lambda_j}(x,\xi)\right) = \left[\widehat{A}(x,\lambda;\eta), H\left(u^{**}_{\lambda_j}(x,\xi)\right)\right].$$

This shows that the evolution equation (13) is of Lax type. Hence, we have the following theorem.

Theorem 1. *The* 1*-parameter family of the ordinary differential operator* $H\left(u^{**}_{\lambda_j}(x,\xi)\right)$ *is isospectral.*

As for the phase space of the isospectral flow $u^{**}_{\lambda_j}(x,\xi)$, we have the following theorem.

Theorem 2. *The linear relation*

$$Z_{n+1}\left(u^{**}_{\lambda_j}(x,\xi) - \lambda_j\right) - \sum_{k=0}^{n} a_k(\lambda_j;u) Z_k\left(u^{**}_{\lambda_j}(x,\xi) - \lambda_j\right) = 0$$

holds for arbitrary ξ.

Proof. Since

$$N_{\lambda_j}(x,\xi) = \frac{h_{\lambda_j}(\eta)\phi_{\lambda_j}(\xi)^2}{F_{\lambda_j}(x,\xi)^2},$$

one can express the derivatives $(\partial/\partial x)^k N_{\lambda_j}$, $k = 1, 2, 3$ in terms of F_{λ_j} and its derivative, i.e.,

$$N_{\lambda_j x} = -2h_{\lambda_j}\phi_{\lambda_j}^2 \frac{F_{\lambda_j x}}{F\lambda_j^3}, \quad N_{\lambda_j xx} = -2h_{\lambda_j}\phi_{\lambda_j}^2 \frac{F_{\lambda_j xx}F_{\lambda_j} - 3F_{\lambda_j x}^2}{F_{\lambda_j}^4}$$

and

$$N_{\lambda_j xxx} = -2h_{\lambda_j}\phi_{\lambda_j}^2 \frac{F_{\lambda_j xxx}F_{\lambda_j}^2 - 9F_{\lambda_j xx}F_{\lambda_j x}F_{\lambda_j} + 12F_{\lambda_j x}^3}{F_{\lambda_j}^5}.$$

Moreover, by Lemma 1, it turns out that $1/F_j$ solves the differential equation

$$\left(-\frac{\partial^2}{\partial x^2} + \left(u_{\lambda_j}^{**}(x,\xi) - \lambda_j\right)\right) f(x) = 0.$$

Hence we have

$$u_{\lambda_j}^{**}(x,\xi) - \lambda_j = \frac{-F_{\lambda_j xx}F\lambda_j + 2F_{\lambda_j x}^2}{F_{\lambda_j}^2}$$

and

$$\frac{\partial}{\partial x}u_{\lambda_j}^{**}(x,\xi) = \frac{-F_{\lambda_j xxx}F_{\lambda_j}^2 + 5F_{\lambda_j xx}F_{\lambda_j x}F_{\lambda_j} - 4F_{\lambda_j x}^3}{F_{\lambda_j}^3}.$$

By straightforward calculation using these expressions, one easily verifies

$$\frac{\partial}{\partial x}\Lambda\left(u_{\lambda_j}^{**} - \lambda_j\right)N_{\lambda_j}(x,\xi) = \frac{1}{2}\frac{\partial u_{\lambda_j}^{**}}{\partial x}N_{\lambda_j} + \left(u_{\lambda_j}^{**} - \lambda_j\right)\frac{\partial N_{\lambda_j}}{\partial x} - \frac{1}{4}\frac{\partial^3 N_{\lambda_j}}{\partial x^3} = 0.$$

On the other hand, we have

$$\begin{aligned}
&\frac{\partial}{\partial x}\Lambda\left(u_{\lambda_j}^{**} - \lambda_j\right)N_{\lambda_j}(x,\xi) \\
&= \frac{\partial}{\partial x}\Lambda\left(u_{\lambda_j}^{**} - \lambda_j\right)\left(Z_n\left(u_{\lambda_j}^{**} - \lambda_j\right) - \sum_{k=1}^{n} a_k(\lambda_j;u)Z_{k-1}\left(u_{\lambda_j}^{**} - \lambda_j\right)\right) \\
&= \frac{\partial}{\partial x}\left(Z_{n+1}\left(u_{\lambda_j}^{**} - \lambda_j\right) - \sum_{k=1}^{n} a_k(\lambda_j;u)Z_k\left(u_{\lambda_j}^{**} - \lambda_j\right)\right).
\end{aligned}$$

Hence we have

$$Z_{n+1}\left(u_{\lambda_j}^{**} - \lambda_j\right) - \sum_{k=1}^{n} a_k(\lambda_j;u)Z_k\left(u_{\lambda_j}^{**} - \lambda_j\right) = c(\xi),$$

where $c(\xi)$ depends on only ξ. In addition, applying Lemma 2 twice, one easily verifies $c(\xi) = a_0(\lambda_j)$. This completes the proof. □

By Theorem 2, if $\lambda_j \in \mathrm{Spec}_m H(u)$, the ADDT induces the isospectral flow described by the evolution equation of the higher order KdV type (13) in the phase space

$$\mathbb{M} = \left\{ u(x) \,\middle|\, Z_{n+1}(u(x) - \lambda_j) - \sum_{k=0}^{n} a_k(\lambda_j; u) Z_k(u(x) - \lambda_j) = 0 \right\}.$$

However, the isospectral property discussed above is formal one. Hence, next we consider how the spectral discriminant $\Delta\big(\lambda; u^{**}_{\lambda_j}(\,\cdot\,, \xi)\big)$ depends on the deformation parameter ξ.

In [10, p.959, Theorem], it is shown that if $\lambda_j \in \mathrm{Spec}_m H(u)$, then we have

$$\mathrm{rank}\, u^{*}_{\lambda_j}(x) = n - 1, \quad \Delta\big(\lambda; u^{*}_{\lambda_j}\big) = \frac{\Delta(\lambda; u)}{(\lambda - \lambda_j)^2}. \tag{14}$$

Therefore, we have

$$n - 2 \le \mathrm{rank}\, u^{**}_{\lambda_j}(x, \xi) \le n$$

for arbitrary $\xi \in \mathbb{C} \cup \{\infty\}$. If $\mathrm{rank}\, u^{**}_{\lambda_j}(x, \xi) = n - 2$, then, by (14), we have

$$\Delta\big(\lambda; u^{**}_{\lambda_j}(\,\cdot\,, \xi)\big) = \frac{\Delta(\lambda; u)}{(\lambda - \lambda_j)^4}.$$

Next, if $\mathrm{rank}\, u^{**}_{\lambda_j}(x, \xi) = n - 1$, then, by (14), we have

$$\Delta\big(\lambda; u^{**}_{\lambda_j}(\,\cdot\,, \xi)\big) = \frac{\Delta(\lambda; u)}{(\lambda - \lambda_j)^2}.$$

Finally, if $\mathrm{rank}\, u^{**}_{\lambda_j}(x, \xi) = n$, by Theorem 2, we have

$$\Delta\big(\lambda; u^{**}_{\lambda_j}(\,\cdot\,, \xi)\big) = \Delta(\lambda; u).$$

Thus, in all cases, we have the following theorem.

Theorem 3. *The spectral discriminant* $\Delta\big(\lambda; u^{**}_{\lambda_j}(\,\cdot\,, \xi)\big)$ *is independent of* ξ.

5. The second Darboux-Lamé equation

In this section, we apply the results obtained in the above to the n-th Lamé operator

$$H(u_n(x, \tau)) = -\frac{\partial^2}{\partial x^2} + u_n(x, \tau), \quad n \in \mathbb{N},$$

where $u_n(x,\tau) = n(n+1)\wp(x,\tau)$ is the n-th Lamé potential. It is well known that the n-th Lamé potential $u_n(x,\tau)$ is the n-th algebro-geometric potential. Let $R(P(\lambda), Q(\lambda))$ be the resultant of polynomials $P(\lambda)$ and $Q(\lambda)$, and define $D(\tau; u_n)$ by

$$D(\tau; u_n) = R\left(\Delta(\lambda; u_n), \frac{d}{d\lambda}\Delta(\lambda; u_n)\right).$$

Then $D(\tau; u_n)$ is the meromorphic function of $\tau \in H^+$, where $H^+ \subset \mathbb{C}$ is the upper half plane. Let

$$\Theta_n = \{\tau \mid D(\tau; u_n) = 0\} \subset H^+,$$

which we call the lattice of degenerate periods associated with the n-th Lamé potential $u_n(x,\tau)$. If $\tau \in \Theta_n$, then $\mathrm{Spec}_m\, H(u_n(x,\tau)) \neq \emptyset$ follows.

In [10, p.976, §5], it is shown that

$$\Delta(\lambda; u_1) = -4\lambda^3 + g_2(\tau)\lambda - g_3(\tau),$$

and, since $g_2(\tau)^2 - 27g_3(\tau)^2 \neq 0$, $\Theta_1 = \emptyset$ follows. Hence, suppose that $\tau_* \in \Theta_n$, $n \geq 2$, and $\lambda_j \in \mathrm{Spec}_m\, H(u_n(x,\tau_*))$. Let $M_n(x,\lambda_j,\tau_*)$ be the M-function corresponding to the n-th Lamé-Ince potential $u_n(x,\tau_*)$, i.e.,

$$M_n(x,\lambda,\tau_*) = M(x,\lambda; n(n+1)\wp(x,\tau_*)).$$

By (11) in Definition 2, the ADDT of $u_n(x,\tau_*)$ is expressed as

$$u^{**}_{n,\lambda_j}(x,\xi) = u_n(x,\tau_*) - 2\frac{\partial^2}{\partial x^2}\log\big(\phi_{\lambda_j}(\xi) + \xi\widehat{M}_n(x,\lambda_j,\tau_*)\big),$$

where $\widehat{M}_n(x,\lambda_j,\tau_*)$ is defined by (9) for $M = M_n$. We call the 1-parameter family of the ordinary differential equation

$$\frac{\partial^2}{\partial x^2}f(x) - \left(u^{**}_{n,\lambda_j}(x,\xi) - \lambda_j\right)f(x) = 0$$

the n-th Darboux-Lamé equation of degenerate type.

Now we consider the second Darboux-Lamé equation of degenerate type. First, we compute the M-function $M_2(x,\lambda,\tau)$. The following formulas are well known.

$$\wp'(x,\tau)^2 = 4\wp(x,\tau)^3 - g_2(\tau)\wp(x,\tau) - g_3(\tau) \tag{15}$$

$$\wp''(x,\tau) = 6\wp(x,\tau)^2 - \frac{1}{2}g_2(\tau) \tag{16}$$

$$\wp'''(x,\tau) = 12\wp(x,\tau)\wp'(x,\tau)$$

$$\wp^{(\mathrm{IV})}(x,\tau) = 120\wp(x,\tau)^3 - 18g_2(\tau)\wp(x,\tau) - 12g_3(\tau) \tag{17}$$

On the otherhand, by the recursion relation (3), one has the following KdV polynomials.

(18) $$Z_0(u) = 1$$

(19) $$Z_1(u) = \frac{1}{2}u$$

(20) $$Z_2(u) = -\frac{1}{8}u'' + \frac{3}{8}u^2$$

(21) $$Z_3(u) = \frac{1}{32}u^{(IV)} - \frac{5}{16}uu'' - \frac{5}{32}u'^2 + \frac{5}{16}u^3$$

Calculating the righthand side of (21) for $u = 6\wp$, we have immediately

(22) $$Z_3(6\wp) = \frac{3}{16}\wp^{(IV)} - \frac{45}{4}\wp\wp'' - \frac{45}{8}\wp'^2 + \frac{135}{2}\wp^3.$$

Eliminate $\wp^{(IV)}$, $\wp''$, and $\wp'^2$ from the righthand side of (22) using (15), (16) and (17). Then we have

$$Z_3(6\wp) = \frac{63}{8}g_2\wp + \frac{27}{8}g_3.$$

Similarly, by (18), (19) and (20), one verifies

$$Z_2(6\wp) = 9\wp^2 + \frac{3}{8}g_2, \quad Z_1(6\wp) = 3\wp, \quad Z_0(6\wp) = 1.$$

Therefore, we have the linear relation

$$Z_3(6\wp) = \frac{21}{8}g_2 Z_1(6\wp) + \frac{27}{8}g_3 Z_0(6\wp).$$

This implies

(23) $$a_2(0; 6\wp) = 0, \quad a_1(0; 6\wp) = \frac{21}{8}g_2(\tau), \quad a_0(0; 6\wp) = \frac{27}{8}g_3(\tau).$$

Hence, by [10, Lemma 2, pp.962–963], we have the following simple expression of the M-function $M_2(x, \lambda, \tau)$;

$$M_2(x, \lambda, \tau) = Z_2(6\wp) + p_1(\lambda)Z_1(6\wp) + p_0(\lambda)Z_0(6\wp),$$

where

$$p_1(\lambda) = \lambda, \quad p_0(\lambda) = \lambda^2 - \frac{21}{8}g_2.$$

Therefore, we have

$$M_2(x, \lambda, \tau) = 9\wp(x, \tau)^2 + 3\lambda\wp(x, \tau) + \lambda^2 - \frac{9}{4}g_2(\tau).$$

Next we calculate $\Delta(\lambda; u_2)$ according to the definition (5). Note that the righthand side of (5) does not depend on x. Hence it suffices to evaluate it at $x = a$ such that $\wp(a) = 0$. By (15) and (16), we have

$$\wp'(a,\tau)^2 = -g_3(\tau), \quad \wp''(a,\tau) = \frac{1}{2}g_2(\tau).$$

By direct calculation, one verifies

$$M_{2x}(a,\lambda,\tau)^2 = -9g_3(\tau)\lambda^2,$$
$$M_2(a,\lambda,\tau)M_{2x}(a,\lambda,\tau) = \left(\lambda^2 - \frac{9}{4}g_2(\tau)\right)\left(-\frac{3}{2}g_2(\tau)\lambda - 18g_3(\tau)\right),$$

and

$$M(a,\lambda,\tau)^2 = \left(\lambda^2 - \frac{9}{4}g_2(\tau)\right)^2.$$

Hence we have

$$\Delta(\lambda; u_2) = -4\lambda^5 + 21g_2(\tau)\lambda^3 + 27g_3(\tau)\lambda^2 - 27g_2(\tau)^2\lambda - 81g_2(\tau)g_3(\tau). \tag{24}$$

One can easily factorize the righthand side of (24) as

$$\Delta(\lambda; u_2) = -4\left(\lambda^2 - 3g_2(\tau)\right)\left(\lambda^3 - \frac{9}{4}g_2(\tau)\lambda - \frac{27}{4}g_3(\tau)\right).$$

Hence $\mathrm{Spec}_m\, H(u_2) \neq \emptyset$ holds if and only if $g_2(\tau) = 0$. Note that $g_2(\tau) = 0$ holds if and only if $J(\tau) = g_2(\tau)^3/(g_2(\tau)^3 - 27g_3(\tau)^2) = 0$. Since $g_2(e^{2\pi i/3}) = 0$, the modular invariance of $J(\tau)$ implies

$$\Theta_2 = \left\{\tau \;\middle|\; \tau = \frac{ae^{2\pi i/3} + b}{ce^{2\pi i/3} + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\right\} \subset H^+.$$

Suppose that $\tau_* \in \Theta_2$, then we have

$$M_2(x,\lambda,\tau_*) = \lambda^2 + 3\wp(x,\tau_*)\lambda + 9\wp(x,\tau_*)^2.$$

Since $g_2(\tau_*) = 0$,

$$\Delta(\lambda; u_2) = -4\lambda^2\left(\lambda^3 - \frac{27}{4}g_3(\tau_*)\right) \tag{25}$$

holds. Hence $\mathrm{Spec}_m\, H(u_2(x,\tau_*)) = \{0\}$ and

$$M_2(x,0,\tau_*)^{1/2} = 3\wp(x,\tau_*) \in \ker H(u_2)$$

follow. Therefore, by (8), the ADT $u_{2,0}^*(x)$ is given by

$$u_{2,0}^*(x) = u_2(x) - \frac{\partial^2}{\partial x^2} \log M_2(x, 0, \tau_*) = 2\wp(x, \tau_*) - \frac{2g_3(\tau_*)}{\wp(x, \tau_*)^2}$$

and, by Lemma 1, we have

$$\frac{1}{3\wp(x, \tau_*)} \in \ker H\big(u_{2,0}^*\big).$$

On the other hand, we have

$$\widehat{M}_2(x, 0, \tau_*) = \int 9\wp(x, \tau_*)^2\, dx = \frac{3}{2}\wp'(x, \tau_*). \tag{26}$$

Hence, by direct calculation, we obtain the explicit expression of the eigenfunction $F_0(x, \xi)$ defined by (10) for $\lambda_j = 0$;

$$F_0(x, \xi) = \frac{\phi_0(\xi) + (3/2)\xi\wp'(x, \tau_*)}{3\wp(x, \tau_*)},$$

where $\phi_0(\xi)$ is an arbitrary analytic function of ξ. Therefore, by (11) and (26), we obtain the explicit expression

$$\begin{aligned} u_{2,0}^{**}(x, \xi, \tau_*) &= u_2(x) - 2\frac{\partial^2}{\partial x^2} \log\left(\phi_0(\xi) + \frac{3}{2}\xi\wp'(x, \tau)\right) \\ &= \frac{6\wp(x, \tau_*)\big(\phi_0(\xi)^2 - 3\xi\phi_0(\xi)\wp'(x, \tau) + (27/4)g_3(\tau_*)\xi^2\big)}{(\phi_0(\xi) + (3/2)\xi\wp'(x, \tau_*))^2}, \end{aligned}$$

which is the second Darboux-Lamé potential (1). Note that

$$u_{2,0}^{**}(x, 0, \tau_*) = 6\wp(x, \tau_*) = u_2(x)$$

holds. Hence, by Theorem 1, the 2-nd Darboux-Lamé equation (2) is the isospectral family of the differential equations on the torus $E_{\tau_*} = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau_*$, and, by Theorem 3 and (25),

$$\operatorname{Spec} H\big(u_{2,0}^{**}(\,\cdot\,, \xi)\big) = \operatorname{Spec} H(u_2) = \{0, 3 \cdot 2^{-2/3} g_3(\tau_*)^{1/3}\}$$

follows. By (13) and (23), it turns out that $v(x, \eta) = u_{2,0}^{**}(x, \xi, \tau_*)$ solves the evolution equation

$$-\frac{h_0(\eta)}{2}\frac{\partial}{\partial \eta} v(x, \eta) = -\frac{1}{8}\frac{\partial^3}{\partial x^3} v(x, \eta) + \frac{3}{4} v(x, \eta)\frac{\partial}{\partial x} v(x, \eta) + \frac{27}{16} g_3(\tau_*)\frac{\partial}{\partial x} v(x, \eta), \tag{27}$$

where $\eta = \xi/\phi_0(\xi)$. The equation (27) coincides with the original KdV equation after slight change of variable. Actually, by the straightforward calculation, one can verifies that if we define the complex time variable t by the indefinite integral

$$t = t(\eta) = \frac{1}{4}\int \frac{1}{h_0(\eta)}\,d\eta$$

such that $t(0) = 0$ and the function $V(x,t)$ by

$$V(x,t) = u^{**}_{2,0}(x,\xi,\tau_*) + \frac{27}{12}g_3(\tau_*),$$

then, $V(x,t)$ solves the KdV equation

$$\frac{\partial V}{\partial t} = \frac{\partial^3 V}{\partial x^3} - 6V\frac{\partial V}{\partial x}$$

with the initial condition

$$V(x,0) = 6\wp(x,\tau_*).$$

Although the calculation of $h_0(\eta)$ is very complicated, if we use the computer algebra system, we can compute it explicitly as

$$h_0(\eta) = 1 + \frac{1}{2}(135c - 144c^2)\eta^2 + \frac{2187}{2}(c^2 - c^4)\eta^4,$$

where $c = 2^{-2/3}g_3(\tau_*)^{1/3}$.

The solution $V(x,t)$ has been constructed previously in [2] by the method of the algebraic geometry. On the other hand, in [11], the isomonodromic property of the second Darboux-Lamé equation (2) is studied for

$$\phi_0(\xi) = \left(-\frac{27}{4}g_3(\tau_*)\xi^2 + c\right)^{1/2}, \quad c \neq 0.$$

Moreover, in [12], several kinds of the exact solutions of the third elliptic Calogero system

$$\frac{d^2\beta_j}{dt^2} = -6\sum_{\substack{k=1\\k\neq j}}^{3}\wp'(\beta_j - \beta_k,\tau_*), \quad \beta_j(0) = 0, \quad j = 1,2,3$$

are explicitly constructed using the pole expansion of the solution $V(x,t)$ constructed above and the specific covering map of the elliptic curve over the projective line.

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